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RALF BADER

Fixed Point Theorems for Compositions of Set–Valued Maps with Single–Valued Maps

ABSTRACT. Some fixed point theorems are presented for a class of set-valued maps which seems to be interesting in view of applications in the theory of differential equations and inclusions. Our approach is based upon the topological fixed point theory initiated in [12] and developed in [1].

Introduction. The subject of the paper is to prove a number of fixed point theorems for a certain class of set-valued maps. This class of maps consists basically of the so-called Aronszajn-maps (i.e. set-valued maps with values being R_δ-sets) composed with single-valued maps. As it will become clear such types of maps provide great flexibility and are useful in applications in the field of differential equations and inclusions (see [11], [12]).

As the main tool we will apply the topological fixed point index theory as it is introduced in [12] and developed in [3], [1]. Clearly, any such theory requires that the maps under consideration satisfy a certain compactness condition (comp. [21]). Here we will assume conditions in terms of measures of noncompactness. As in the case of single-valued maps the use of measures of noncompactness unifies the treatment of compact maps (i.e. maps whose range lies in compact subset) and contraction mappings.

Finally let us note, that some of the results of the paper are known, even in a more general version. This is due to the fact that using methods of algebraic topology it is possible to construct a fixed point index theory for more general classes of maps, e.g. acyclic-valued and compositions of acyclic-valued maps (see [10], [13]). However, we would like to emphasize that the fixed point index used in this paper can be constructed by the natural and elementary technique of graph approximations and thus for our purpose the use of algebraic topology is not neccessary.

1. Preliminaries. Given topological spaces T_1, T_2, we consider set-valued maps $\varphi : T_1 \to T_2$, i.e. maps which assign to each $x \in T_1$ a non-empty set $\varphi(x) \subset T_2$. The graph Γ_φ of the set-valued map φ is the set

$$\Gamma_\varphi := \{(x,y) \in T_1 \times T_2 | y \in \varphi(x)\}.$$

A set-valued map φ is called upper semi-continuous (u.s.c.), if for each open $V \subset T_2$ the set $\varphi^{-1}(V) := \{x \in T_2 | \varphi(x) \subset V\}$ is open in T_1 and $\varphi(x)$ is compact for each $x \in T_1$. The fixed point set of a set-valued map φ, i.e. the set of all points such that $x \in \varphi(x)$, will be denoted by $\operatorname{Fix}(\varphi)$. In the sequel we use the convention that single-valued maps are denoted by Latin letters: $f, g, h, \ldots$, and for set-valued maps we use Greek letters: φ, ψ, η etc.

In the following we will impose compactness conditions on set-valued maps in terms of measures of noncompactness. Recall that a measure of noncompactness is a map m from the bounded subsets of a Banach space X into the positive real numbers $\mathbb{R}_0^+$ such that $m(A) = 0 \iff \overline{A}$ compact, $m(A_1 \cup A_2) = \max\{m(A_1), m(A_2)\}$, $m(A_1 + A_2) \le m(A_1) + m(A_2)$ and $m(\overline{\operatorname{conv}}\, A) = m(A)$ with bounded subsets A, A_1, A_2 of X; m is called positively homogeneous if $m(\lambda A) = |\lambda| m(A)$ for all $\lambda \in \mathbb{R}$. Important examples of positively homogeneous measures of noncompactness are the Kuratowski and the Hausdorff measure of noncompactness m_K and m_H respectively, where

$$m_K(A) := \inf\Big\{ r > 0 |\ \text{there are finitely many sets } A_1, \ldots, A_n \subset X \text{ with } \operatorname{diam} A_i < r \text{ and } A \subset \bigcup_{i=1}^n A_i \Big\},$$

$$m_H(A) := \inf\Big\{ r > 0 |\ \text{there are finitely many points } x_1, \ldots, x_n \in X \text{ with } A \subset \bigcup_{i=1}^n N_r(x_i) \Big\}$$

(given $r > 0$ and a subset A of a metric space M, by $N_\epsilon(A)$ we denote an open ϵ-neighborhood of A in M; clearly, for a point $x \in M$ we use $N_r(x)$ instead of $N_r(\{x\})$).

Now let $\varphi : X \supset D(\varphi) \to X$ be an u.s.c. set-valued map and m a measure of noncompactness on X. Then φ is called k-set-contraction (w.r.t. m), where $0 \leq k \leq 1$, if for each bounded subset $A \subset D(\varphi)$, the set $\varphi(A)$ is bounded and $m(\varphi(A)) \leq k\, m(A)$. The set-valued map φ is called condensing (w.r.t. m) if for each bounded set $A \subset D(\varphi)$ $\varphi(A)$ is bounded and $m(\varphi(A)) < m(A)$ provided that $m(A) \neq 0$.

The typical example of k-set-contractions are mappings of the form $\varphi_1 + \varphi_2$ where φ_1 is a k_1-set-contraction and φ_2 is a contraction of constant k_2 and $k = k_1 + k_2$. This will follow from Lemma 1 below. Recall that a map $\varphi : X \supset D(\varphi) \to X$ is called a contraction of constant k, $0 \leq k \leq 1$, provided $d_H(\varphi(x), \varphi(y)) \leq k\|x - y\|$ for all $x, y \in D(\varphi)$, where d_H denotes the Hausdorff metric and if $k = 1$, we say φ is a nonexpansive map.

Lemma 1. *Let $\varphi : X \to X$ be a contraction of constant k with compact values. Then φ is a k-set-contraction w.r.t. m_H.*

Proof. Let $A \subset X$ be bounded with $m_H(A) = r$. Then given $\epsilon > 0$, there are points $x_1, \ldots, x_n \in X$ such that $A \subset \bigcup_{i=1}^n N_{r+\epsilon}(x_i)$. Since φ is a contraction of constant k we see that $\varphi(A) \subset \bigcup_{i=1}^n \varphi(x_i) + N_{k(r+\epsilon)}(0)$ and thus $m_H(\varphi(A)) \leq m_H(N_{k(r+\epsilon)}(0)) \leq k(r + \epsilon)$. □

For a set-valued contractive map φ with contraction constant k defined on a proper subset D of X, we are not able to show that it is a k-set-contraction w.r.t. m_H (it seems to us that the proof given in [5, p. 113], is in error since there is no guarantee that the centers of the balls defining $m_H(D)$ lie in D). However it is not difficult to show that such contraction mappings are $2k$-set-contractions w.r.t. m_H. Notice also that the arguments used in the Lemma applied on m_K yield only that φ is a $2k$-set-contraction (independently whether φ is defined on a proper subset or not); but the factor 2 disappears in the single-valued case. We add in passing that a condensing map in general is not a k-set-contraction with $k < 1$ and, that a map φ which satisfies $d_H(\varphi(x), \varphi(y)) < \|x - y\|$ is not a condensing map (comp. [17, p. 222]).

2. The fixed point index. As the main tool in our consideration stands the topological fixed point index for the so-called decomposible set-valued maps as it is introduced in [12], [3] (see also [11]). In order to state this theory we need the following definitions.

Following [6] we say that a compact subset K of a metric space M is proximally ∞-connected if, for each $\epsilon > 0$, there is $0 < \delta \leq \epsilon$ such that the inclusion $N_\delta(K) \overset{i}{\subset} N_\epsilon(K)$ induces the trivial homomorphism on the homotopy groups, i.e. $i_n : \pi_n(N_\delta(K)) \to \pi_n(N_\epsilon(K))$ is zero for any $n \geq 0$ (we suppress the base points from the notations since they are not necessary).

As an example, recall that any R_δ-set (i.e. the intersection of a decreasing sequence of compact contractible sets, see [14]) lying in an ANR is proximally ∞-connected. Hence, compact contractible and compact convex subsets of a Banach space are proximally ∞-connected.

We say that a set-valued map $\varphi : T \to M$, where T is a topological space and M a metric space, is a J-map, if φ is u.s.c. and the set $\varphi(x)$ is a proximally ∞-connected subset of M. In [12] it is shown that a J-map $\varphi : Y \to M$, where Y is a compact ANR and M an arbitrary metric space admits arbitrary close graph approximations, i.e. given $\epsilon > 0$ there exists a continuous map $g : Y \to M$ such that $\Gamma_g \subset N_\epsilon(\Gamma_\varphi)$.

Now let X be a Banach space and U an open subset of X. By a decomposition we mean a diagram $U \xrightarrow{\varphi} M \xrightarrow{f} X$, where M is a (metric) ANR, φ is a J-map and f is a continuous map. A map Φ is called decomposable, if it has a decomposition, i.e. the diagram

$$\mathcal{D}(\Phi) : U \xrightarrow{\varphi} M \xrightarrow{f} X, \tag{1}$$

is such that $\Phi = f \circ \varphi$.

If another map $\Psi : U \to X$ has a decomposition

$$\mathcal{D}(\Psi) : U \xrightarrow{\psi} N \xrightarrow{g} X, \tag{2}$$

we say that the decompositions $\mathcal{D}(\Phi)$ and $\mathcal{D}(\Psi)$ are homotopic, provided $M = N$, there is a J-map $\eta : U \times [0,1] \to M$ and a continuous map $h : M \times [0,1] \to X$ such that $\eta(\cdot,0) = \varphi, \eta(\cdot,1) = \psi, h(\cdot,0) = f, h(\cdot,1) = g$. Observe that in this case we have a homotopy $H : U \times [0,1] \to X$ given by $H(x,t) := \bigcup_{y \in \eta(x,t)} h(y,t)$ since $H(\cdot,0) = \Phi$ and $H(\cdot,1) = \Psi$.

For the class of decomposable, condensing maps there is a topological fixed point index. This fixed point index can be obtained by employing the technique of arbitrary close graph approximations. Moreover, the construction uses methods similar to those in [17] in the case of single-valued condensing maps (see [1] for details).

Theorem 2. *Let X be a Banach space, $U \subset X$ open and let $\Phi : U \to X$ be a condensing map w.r.t. a positively homogeneous measure of noncompactness m. Let $\mathcal{D}(\Phi)$ (see (1)) be a decomposition of Φ and let* $\mathrm{Fix}(\Phi)$ *be a compact set. Then there exists a fixed point index* $\mathrm{Ind}_X(\mathcal{D}(\Phi), U) \in \mathbb{Z}$ *with the following properties:*

(i) (Additivity)
If U_1, U_2 are open disjoint subsets of U with $\mathrm{Fix}\,(\varphi) \subset U_1 \cup U_2$, *then*
$$\mathrm{Ind}_X(\mathcal{D}(\Phi), U) = \mathrm{Ind}_X(\mathcal{D}(\Phi)|_{U_1}, U_1) + \mathrm{Ind}_X(\mathcal{D}(\Phi)|_{U_2}, U_2),$$
where $\mathcal{D}(\Phi)|_{U_i}$ stands for the decomposition $U_i \xrightarrow{\varphi|_{U_i}} M \xrightarrow{f} X, i = 1, 2.$
Particularly, $\mathrm{Ind}_X(\mathcal{D}(\Phi), U) \neq 0$ *implies* $\mathrm{Fix}\,(\varphi) \neq \emptyset$.

(ii) (Homotopy)
If $\Psi : U \to X$ is a map with a decomposition $\mathcal{D}(\Psi)$ (see (2)) homotopic with $\mathcal{D}(\Phi)$ such that the homotopy $H : U \times [0,1] \to X$ is a condensing map, i.e. $m(H(A \times [0,1])) < m(A)$ for each bounded $A \subset U$ with $m(A) \neq 0$, and the set $\Sigma := \{(x,t) \in U \times [0,1] | x \in H(x,t)\}$ is compact, then $\mathrm{Ind}_X(\mathcal{D}(\Phi), U) = \mathrm{Ind}_X(\mathcal{D}(\Psi), U)$.

(iii) (Units)
If $c \in X$ and $\Phi(x) = c$ for each $x \in U$, then the decomposition $\mathcal{D} : U \xrightarrow{\Phi} X$ *has*
$$\mathrm{Ind}_X(\mathcal{D}, U) = \begin{cases} 1, & \text{if } c \in U, \\ 0, & \text{if } c \notin U. \end{cases}$$

(iv) (Oddness)
If $r > 0$ and $\Phi : \overline{N_r(0)} \to X$ is a condensing map w.r.t. m with $\mathrm{Fix}\,(\Phi) \cap \partial N_r(0) = \emptyset$, *having a decomposition* $\mathcal{D}(\Phi) : N_r(0) \xrightarrow{\varphi} \tilde{X} \xrightarrow{f} X$, *where φ, f are odd mappings (i.e. $-\varphi(x) = \varphi(-x)$ for every x) and $\tilde{X}$ is a Banach space, then* $\mathrm{Ind}_X(\mathcal{D}(\Phi), N_r(0)) \equiv 1\ (\mathit{mod}\ 2)$.

We will also use the following simple

Proposition 3. *Let Φ be as above with the decomposition $\mathcal{D}(\Phi)$ (see (1)) and let*
$$\mathcal{D}'(\Phi) : U \xrightarrow{\varphi'} M' \xrightarrow{f'} X$$
be a decomposition such that there exists a map $p : M \to M'$ such that the triangles

$$\begin{tikzcd} & M \arrow[dd, "p"] \\ U \arrow[ur, "\varphi"] \arrow[dr, "\varphi'"'] & \\ & M' \end{tikzcd} \qquad \begin{tikzcd} M \arrow[dd, "p"'] \arrow[dr, "f"] & \\ & X \\ M' \arrow[ur, "f'"'] & \end{tikzcd}$$

commute. Then $\mathrm{Ind}_X(\mathcal{D}(\Phi), U) = \mathrm{Ind}_X(\mathcal{D}'(\Phi), U)$. □

Remark 4. In the paper we consider the class of decomposable mappings for several reasons:

- The class is more general than J-maps: In [11, Example 6.2] it is shown that a map which takes the unit sphere $\mathbb{S}^1$ in $\mathbb{R}^2$ as its values has a decomposition. However, it is obvious that $\mathbb{S}^1$ is not proximally ∞-connected in $\mathbb{R}^2$ and thus this map is not a J-map.
- The class of J-maps is *not* closed under addition. As an example consider the subsets $A := N_{\frac{1}{2}}(1,0)$ and $B := \mathbb{S}^1 \setminus A$ in $\mathbb{R}^2$. Now $\overline{A}$ and B are proximally ∞-connected (since they are contractible), but $\overline{A}+B$ is homotopically aquivalent with $\mathbb{S}^1$ and thus it is not proximally ∞-connected in $\mathbb{R}^2$.
- Let $F : [0,a] \times X \to X$. Then we can consider the Poincaré map P_t associated with the differential equation $x' = F(t,x)$ and in many cases it can be shown that this map is decomposable. To see this, observe that $P_t = e_t \circ S$, where

$$S(x_0) := \{x : [0,a] \to X \,|\, x \text{ is a solution of the initial value problem } x' = F(t,x),\ x(0) = x_0\}$$

and e_t is the evaluation map in $t \in [0,a]$. Under various assumptions (e.g. the Banach space X may be finite or infinite dimensional; the map F may be continuous or a Carathéodory map or it may be an u.s.c. set-valued map) it can be shown that S takes R_δ-sets as its values and it follows that P_t is a decomposable map (see [19], [11] and the references given there).

3. Fixed point theorems coming from homotopy. Let X be a Banach space, U an open subset of X and m a positively homogeneous measure of noncompactness.

Lemma 5. *Let $\Phi, \Psi : \overline{U} \to X$ be condensing maps w.r.t. m having bounded range and let $\mathcal{D}(\Phi), \mathcal{D}(\Psi)$ (see (1), (2)) be decompositions of the maps $\Phi|_U, \Psi|_U$, respectively. Let the set-valued map*

$$\Lambda : \overline{U} \times [0,1] \to X,\ \Lambda(x,t) := (1-t)\Phi(x) + t\Psi(x)$$

be such that

$$x \notin \Lambda(x,t) \text{ for each } x \in \partial U,\ t \in [0,1]. \tag{3}$$

Then $\operatorname{Ind}_X(\mathcal{D}(\Phi), U) = \operatorname{Ind}_X(\mathcal{D}(\Psi), U)$.

Observe that Λ does not establish a homotopy of the decompositions $\mathcal{D}(\Phi)$ and $\mathcal{D}(\Psi)$ and thus Theorem (2 (ii)) can not be applied directly.

Proof (of Lemma 5). Consider the decompositions

$$\mathcal{D}_1 : U \xrightarrow{\varphi \times \psi} M \times N \xrightarrow{q_1 \circ (f \times g)} X, \qquad \mathcal{D}_2 : U \xrightarrow{\varphi \times \psi} M \times N \xrightarrow{q_2 \circ (f \times g)} X,$$

where $q_1, q_2 : X \times X \to X$, $q_1(x,y) := x, q_2(x,y) := y$. Now the single valued map $h : M \times N \times [0,1] \to X, h(x,y,t) := (1-t)f(x) + tg(y)$ shows the homotopy of $\mathcal{D}_1$ and $\mathcal{D}_2$ in the sense of decompositions and clearly

$$\Lambda(x,t) = \bigcup_{(y,\bar{y}) \in \varphi(x) \times \psi(x)} h(y, \bar{y}, t).$$

Hence, $\operatorname{Ind}_X(\mathcal{D}_1, U) = \operatorname{Ind}_X(\mathcal{D}_2, U)$ follows if we can show that Λ is a condensing map and that the set $\Sigma := \{(x,t) \in U \times [0,1] | x \in \Lambda(x,t)\}$ is compact.

Let $A \subset \overline{U}$ bounded with $m(A) \neq 0$. Since

$$\Lambda(A \times [0,1]) \subset \overline{\operatorname{conv}}\{\Phi(A) \cup \Psi(A)\}$$

we have

$$m(\Lambda(A \times [0,1])) \leq m(\Phi(A) \cup \Psi(A)) = \max\{m(\Phi(A)), m(\Psi(A))\} < m(A)$$

since Φ and Ψ are condensing. Thus Λ is condensing and, in order to show that Σ is compact, observe first that Σ is bounded, since H is bounded by assumption and therefore also the set $r(\Sigma)$ is bounded ($r : X \times [0,1] \to X$ denotes the projection). Now assume $m(r(\Sigma)) \neq 0$. Then, since $r(\Sigma) \subset \Lambda(\Sigma)$ and Λ is condensing, we have $m(r(\Sigma)) \leq m(\Lambda(\Sigma)) < m(r(\Sigma))$, i.e. a contradiction. Thus $\overline{r(\Sigma)}$ and also $\overline{\Sigma}$ is compact. Now (3) shows $\overline{\Sigma} = \Sigma$.

It remains to show, that $\operatorname{Ind}_X(\mathcal{D}(\Phi), U) = \operatorname{Ind}_X(\mathcal{D}_1, U)$ and $\operatorname{Ind}_X(\mathcal{D}(\Psi), U) = \operatorname{Ind}_X(\mathcal{D}_2, U)$. To this end, observe that there are maps $p_1 : M \times N \to M$, $p_2 : M \times N \to N$ given by $p_1(x,y) := x$, $p_2(x,y) := y$ such that $p_1 \circ (\varphi \times \psi) = \varphi$, $f \circ p_1 = q_1 \circ (f \times g)$ and $p_2 \circ (\varphi \times \psi) = \psi$, $g \circ p_2 = q_2 \circ (f \times g)$. Thus, the above equalities follow from Proposition 3. □

Theorem 6 (Nonlinear alternative). *Let U be an open subset of a Banach space X and let $z \in U$. Let $\Phi : \overline{U} \to X$ be a condensing map with bounded range having the decomposition (1) of $\Phi|_U$. Then either*

(A1) *Φ has a fixed point or*

(A2) *there exists $x_0 \in \partial U$ and $t_0 \in (0,1)$ such that $x_0 - z \in t_0(\Phi(x_0) - \{z\})$.*

Proof. Without loss of generality we may assume $\operatorname{Fix}(\Phi) \cap \partial U = \emptyset$ (otherwise (A1) holds). Define a map

$$\Lambda : \overline{U} \times [0,1] \to X, \ \Lambda(x,t) := (1-t)\{z\} + t\Phi(x).$$

Now assume that $x \notin \Lambda(x,t)$ for each $x \in \partial U$ and $t \in [0,1]$. Then consider $z : U \to X, x \mapsto z$ with $\mathcal{D} : U \xrightarrow{z} X$ and it follows from Lemma 5 and Theorem 2 (iii) that $\mathrm{Ind}_X(\mathcal{D}(\Phi), U) = \mathrm{Ind}_X(\mathcal{D}, U) = 1$, i.e. Φ has a fixed point and (A1) is valid.

If on the other hand there is $x \in \partial U$ and $t \in [0,1]$ such that $x \in \Lambda(x,t)$, then we have (A2) with $x_0 := x$ and $t_0 := t$. □

Corollary 7. *Let $\Phi : \overline{U} \to X$ be as above, $z \in U$ and assume one of the following conditions:*

(i) *U is convex and $\Phi(\partial U) \subset \overline{U}$.* (Rothe's condition)

(ii) *If $t_0(x-z) \in \Phi(x) - \{z\}$ for $x \in \partial U$ then $t_0 \leq 1$.* (Yamamuro's condition)

(iii) *$\|y-z\|^2 \leq \|x-z\|^2 + \|y-x\|^2$ for each $x \in \partial U, y \in \Phi(x)$.* (Altman's condition).

Then Φ has a fixed point.

Proof. The proof follows from the fact that each of the conditions implies that (A2) in the nonlinear alternative is wrong. □

Theorem 8 (Leray-Schauder alternative). *Let $\Phi : X \to X$ be a condensing map with a decomposition $\mathcal{D}(\Phi) : X \xrightarrow{\varphi} M \xrightarrow{f} X$. Then either the set*

$$G := \{x \in X | x \in t\Phi(x) \ \text{for} \ t \in (0,1)\}$$

is unbounded or Φ has a fixed point.

Proof. Let G be bounded. Then there is $R > 0$ such that $G \subset N_R(0) =: U$. Now the map $\Phi|_{\overline{U}}$ satisfies the conditins of the nonlinear alternative and there is no $x \in \partial U$ and $t \in (0,1)$ such that $x \in t\Phi(x)$. Thus (A2) is not fulfilled (with $z = 0$), and therefore Φ has a fixed point. □

4. A fixed point theorem coming from oddness. In what follows we say that $\varphi : X \to X$ is a homogeneous map if $\lambda\varphi(x) \subset \varphi(\lambda x)$ for every $x \in X$, $\lambda \in \mathbb{R}$.

We now extend to condensing decomposable mappings the result of Lasota and Opial [16] obtained by them for convex-valued maps. Results of this type have applications in boundary value problems (e.g. the Nicoletti boundary value problem, see also [20]) and optimal control problems.

Theorem 9. *Let $\Phi : X \to X$ be a condensing map w.r.t. m having a decomposition $\mathcal{D}(\Phi) : X \xrightarrow{\varphi} M \xrightarrow{f} X$ and let $\varphi_0 : X \to X$ be a condensing map w.r.t. m with compact convex values. Assume also that φ_0 is homogeneous and that $x \in \varphi_0(x)$ implies $x = 0$. Suppose that there exists $\alpha > 0$ such that*

$$(4) \qquad \Phi(x) \subset \varphi_0(x) + \overline{N}_\alpha(0) \quad \text{for each } x \in X.$$

Then Φ has a fixed point.

Proof. We may choose $\beta > 0$ such that $x \in \partial N_1(0)$ und $y \in \varphi_0(x)$ implies $\|x - y\| \geq \beta$. Indeed, if this were not the case, then one could select x_n in $\partial N_1(0)$ for every n such that $\|x_n - y_n\| < \frac{1}{n}$ for some $y_n \in \varphi_0(x_n)$. Now, $(x_n) \subset (y_n) + (x_n - y_n)$ and thus, in case $m((x_n)) \neq 0$, we obtain $m((x_n)) \leq m((y_n)) + m((x_n - y_n)) = m((y_n)) < m((x_n))$, since φ_0 is condensing. It follows that for some subsequence $x_{n_k} \longrightarrow x \in \partial N_1(0)$ and $x \in \varphi_0(x)$, which is a contradiction.

Now we choose $r \in \mathbb{R}$ with $r \cdot \beta > \alpha$. Then if $x \in \partial N_r(0)$ and $y \in \varphi_0(x)$, we have

$$(5) \qquad \|x - y\| = r \left\| \frac{1}{r} x - \frac{1}{r} y \right\| \geq r\,\beta > \alpha$$

since $y/r \in \varphi_0(x/r)$. Next define the homotopy $\Lambda : \overline{N}_r(0) \times [0,1] \to X$ by $\Lambda(x,t) := (1-t)\Phi(x) + t\varphi_0(x)$. Using (4) we see that $\Lambda(x,t) \subset \varphi_0(x) + \overline{N}_\alpha(0)$ for each $x \in \overline{N}_r(0)$, $t \in [0,1]$ and hence it follows from (5) that $x \notin \Lambda(x,t)$ provided $x \in \partial N_r(0)$, $t \in [0,1]$. Thus we may apply Lemma 5 and obtain $\mathrm{Ind}_X(\mathcal{D}(\Phi), N_r(0)) = \mathrm{Ind}_X(\mathcal{D}(\Phi_0), N_r(0))$, where $\mathcal{D}(\Phi_0) : N_r(0) \xrightarrow{\varphi_0} X \xrightarrow{id} X$. But, since φ_0 is an odd map, it follows from the oddness property 2 (iv) that $\mathrm{Ind}_X(\mathcal{D}(\Phi_0), N_r(0))$ is nonzero, so Φ has a fixed point. □

5. The case of 1-set-contractions. Clearly such maps need not to have any fixed points. However we have the following result.

Theorem 10. *Let U be an open subset of a Banach space X, $\Phi : \overline{U} \to X$ be bounded, 1-set-contractive w.r.t. a positively homogeneous measure of noncompactness m having the decomposition (1) of $\Phi|_U$ and such that the condition (ii) of* Corollary 7 *is valid. Then Φ has a fixed point if and only if*

($\star$) *for each sequence* (x_n) *in* $\overline{U}$ *such that there is* $y_n \in \Phi(x_n)$ *with* $x_n - y_n \longrightarrow 0$ *there exists a point* $x \in \overline{U}$ *such that* $x \in \Phi(x)$.

Proof. Consider the maps $\Phi_n : \overline{U} \to X, \Phi_n(x) := (1-t_n)\{z\} + t_n\Phi(x)$ where (t_n) is a squence of real numbers such that $t_n \longrightarrow 1$ and $t_n < 1$. Clearly, Φ_n is t_n-set-contraction w.r.t. m and the condition (ii) of Corollary 7 holds for Φ_n: If $x \in \partial U$ and t_0 is such that $t_0(x-z) \in \Phi_n(x) - \{z\}$, i.e. $t_0(x-z) \in (1-t_n)\{z\} + t_n\Phi(x) - \{z\}$ then $t_0(x-z) \in t_n(\Phi(x) - \{z\})$ and thus by assumption $t_0/t_n \leq 1$. It follows $t_0 \leq t_n < 1$. Moreover, observe that if we replace in the decomposition (1) of $\Phi|_U$ the map f by the map $f_n : M \to X, f_n(x) := (1-t_n)z + t_n f(x)$, we obtain a decomposition of $\Phi_n|_U$. Hence, Corollary 7 (ii) implies that for each $n \in \mathbb{N}$ there exists a point $x_n \in \overline{U}$ such that $x_n \in \Phi_n(x_n)$. But now, by definition of Φ_n, we obtain $y_n \in \Phi(x_n)$ such that $x_n = (1-t_n)z + t_n y_n$, i.e. $x_n - y_n = (1-t_n)(z-y_n)$ and it follows that $x_n - y_n \longrightarrow 0$. Thus assumption ($\star$) implies the existence of a fixed point in $\overline{U}$. □

If Φ is condensing it follows that $I - \Phi$ maps closed subsets onto closed subsets and therefore condition ($\star$) is fulfilled in this case (I denotes the identity on the Banach space X). To give an example of a mapping which is only a 1-set-contraction such that condition ($\star$) is fulfilled, we have the following Corollary 11.

Recall that a Banach space X satisfies the condition of Opial, if for each $x \in X$ and each sequence (x_n) with $x_n \rightharpoonup x$ ($\rightharpoonup$ denotes convergence in the weak topology), it follows that $\liminf \|x_n - y\| > \liminf \|x_n - x\|$ for $y \neq x$. All uniformly convex Banach spaces with weakly continuous duality mapping have this property; in particular, Hilbert spaces and l_p spaces with $p > 1$ have this property (see [15]).

Corollary 11. *Let* X *be a Banach space which satisfies the condition of Opial, and suppose* U *is an open subset of* X *such that* $\overline{U}$ *is compact in the weak topology. Assume that* $\varphi_1 : X \to X$ *is a nonexpansive* J*-map and* $\varphi_2 : \overline{U} \to X$ *is a strongly u.s.c.* J*-map (i.e.* $\varphi_2^{-1}(V)$ *is weakly open for each open subset* V *of* X *and* $\varphi_2(x)$ *is proximally* ∞*-connected in* X *for each* $x \in \overline{U}$*). If the mapping* $\Phi := \varphi_1 + \varphi_2 : \overline{U} \to X$ *satisfies condition (ii) of Corollary 7, then* Φ *has a fixed point.*

Proof. By Lemma 1 φ_1 is a 1-set-contraction w.r.t. m_H and, since φ_2 maps weakly compact sets onto compact sets it follows that $m_H(\varphi_2(A)) = 0$ for each $A \subset \overline{U}$. Thus Φ is a 1-set-contraction w.r.t. m_H. Next, we have a decomposition $\mathcal{D}(\Phi)$ of Φ given by $U \xrightarrow{\varphi_1 \times \varphi_2} X \times X \xrightarrow{\text{add}} X$, where $\text{add}(x,y) := x + y$.

We show that Φ also satisfies condition $(\star)$: To this end, let (x_n) be a sequence in $\overline{U}$ such that there is $y_n \in \Phi(x_n)$ with $x_n - y_n \longrightarrow 0$. Then there are $u_n \in \varphi_1(x_n)$ and $v_n \in \varphi_2(x_n)$ such that $y_n = u_n + v_n$. Since $\overline{U}$ is weakly compact, we may assume that $x_n \rightharpoonup x$ and, using the strong upper semi-continuity of φ_2, we see that there is $v \in X$, a subsequence still denoted (v_n) such that $v_n \longrightarrow v$ and therefore, $v \in \varphi_2(x)$. But now $x_n - u_n = x_n - y_n + v_n \longrightarrow v$, and since $x_n - u_n \in (I - \varphi_1)(x_n)$ it follows from a result of Lami Dozo ([15, Th. 3.1]) that $v \in (I - \varphi_1)(x)$. Thus $x \in \Phi(x)$. □

Remark 12. Theorem 10 and Corollary 11 are the generalizations from [9] where analogous results were obtained for maps with compact convex values. Observe, that in order to see that the map Φ in Corollary 11 is a 1-set-contraction, we have to assume that φ_1 is defined on X. If φ_1 is defined only on $\overline{U}$ the result is true in case φ_1 is a single-valued map or X is a Hilbert space and $U = N_R(0)$ (since in this case there exists a nonexpansive retraction $r : X \to \overline{N}_R(0)$ and we can apply Corollary 11 on $\varphi_1 \circ r$ instead of φ_1). In general it seems to be a difficult problem to prove Corollary 11 without assuming that φ_1 is defined on all of X.

6. A fixed point theorem in locally convex spaces. Starting from the paper [7] of Ky Fan in 1952 the problem of the existence of fixed points for set-valued maps in locally convex spaces was considered by many authors (see [18] and the refernces given there). Here we would like to give a version of the Ky Fan fixed point theorem for the mappings as considered in this paper.

Assume X is a locally convex topological vector space. By $\mathfrak{U}(0)$ we denote the fundamental system of neighborhoods of $0 \in X$.

Theorem 13. *Let C be a compact convex subset of X and let $\Phi : C \to C$. Let $\varphi : C \to M$ be J-map, $f : M \to C$ a continuous map where M is an ANR and let $\Phi = f \circ \varphi$. Then Φ has a fixed point.*

For the proof of this result we need the following

Lemma 14 (see [4. p. 89]). *Let X and C be as above and let $U \in \mathfrak{U}(0)$. Then there is a finite dimensional subspace X_U of X and a continuous map $P_U : C \to C \cap X_U$ such that $P_U(x) - x \in U$ for each $x \in C$.*

Proof of Theorem 13. Let $U \in \mathfrak{U}(0)$. Consider the map

$$\Phi_U : C \to C,\ \Phi_U(x) := (\Phi(x) + \overline{U}) \cap C.$$

If we can prove that each fixed point set $\mathrm{Fix}(\Phi_U)$ is non-empty and closed, then it will follow that also any finite intersection of these sets is non-empty. Hence, it follows from the finite intersection property of C that $\bigcap_{U\in\mathfrak{U}(0)} \mathrm{Fix}(\Phi_U) \neq \emptyset$. Clearly any point x_0 in this intersection satisfies $x_0 \in \Phi(x_0)$.

Now let $x \notin \mathrm{Fix}(\Phi_U)$ for some $U \in \mathfrak{U}(0)$. Then we have $x \notin \Phi(x) + \overline{U}$ and thus there exists $V_1 \in \mathfrak{U}(0)$ such that

$$(x + V_1) \cap (\Phi(x) + \overline{U} + V_1) = \emptyset. \tag{6}$$

(Here we use the fact that X as a topological linear space carries a uniform structure induced by the system $\mathfrak{U}(0)$.)

Since Φ is u.s.c. there is a set $V_2 \in \mathfrak{U}(0), V_2 \subset V_1$, such that $\Phi(y) \subset (\Phi(x) + V_1) \cap C$ for each $y \in (x + V_2) \cap C$. But for such y we have $\Phi_U(y) \subset \Phi(y) + \overline{U} \subset \Phi(x) + V_1 + \overline{U}$ and thus, by (6), $y \notin \Phi_U(y)$, i.e. $\mathrm{Fix}(\Phi_U)$ is closed.

In order to show that for each $U \in \mathfrak{U}(0)$ the fixed point set $\mathrm{Fix}(\Phi_U)$ is non-empty, we define a map $\Psi_U : C \cap X_U \to C \cap X_U$ by $\Psi_U(x) := P_U \circ \Phi(x)$, where X_U and P_U are as in the Lemma 14 w.r.t. U. Observe that for each $x \in C \cap X_U$ we have $\Psi_U(x) \subset \Phi_U(x)$, since, if $y \in \Psi_U(x)$, i.e. there is $z \in \Phi(x)$ such that $y = P_U(z)$, we have $y = z + (P_U(z) - z) \in \Phi(x) + U$. But the map Ψ_U has the decomposition

$$C \cap X_U \xrightarrow{\varphi|_{C\cap X_U}} M \xrightarrow{P_U \circ f} C \cap X_U$$

and thus (e.g. by Corollary 7 (i)) there exists a fixed point x_U of Ψ_U. It follows $x_U \in \Psi_U(x_U) \subset \Phi_U(x_U)$, i.e. $\mathrm{Fix}(\Phi_U) \neq \emptyset$. □

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Institut der Universität München
Theresienstraße 39
D–80333 München, Germany
e-mail: bader@rz.mathematik.uni-muenchen.de

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