ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN-POLONIA
VOL. LI. 1, 12 SECTIO A 1997

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The Density of a Three-parameter Random Variable with the Gamma Distribution as the Density of a Finite or Infinite Product

ABSTRACT. In 1960-1970, many papers appeared in which the density of finite products of continuous random variables with distributions of the types gamma, beta, normal, Bessel and others was defined. The aim of this paper is to show how to present a three-parameter density function of a random variable with the gamma (generalized) distribution as a density of a finite or infinite product of independent random variables X_k, where $k \in \{1, \dots, n\}$ or $k \in \mathbb{N}$, which we write

$$X \stackrel{st}{=} \prod_{k=1}^{n} X_k^* \quad \text{or} \quad X \stackrel{st}{=} \prod_{k=1}^{\infty} X_k .$$

The presentation of a random variable, with the gamma distribution, in the form of an infinite product of random variables, with the same distributions was used by Lu and Richards in 1993, to define square of the Vandermonde determinant with random elements.

1. Introduction. We consider three-parameter (generalized) gamma distribution of a random variable X with the density function

$$(1)\quad f(x \mid \Theta, \lambda, p) = \frac{\lambda}{\Theta^{p/\lambda}\Gamma(p/\lambda)} x^{p-1} \exp\left(-\frac{x^{\lambda}}{\Theta}\right), \quad x \geq 0, p, \lambda, \Theta > 0,$$

Key words and phrases. Mellin transform, Knar formula, Marcinkiewicz-Zygmund theorem, stochastic equality, convergence with probability 1.

where $\Theta^{1/\lambda}$ is a parameter of scale, and p, λ are parameters of shape.

In particular, we consider

1. the Weibull distribution, if $\lambda = p > 0$,
2. the Rayleigh distribution, if $\lambda = p = 2$,
3. the Maxwell distribution, if $\lambda = 2, p = 3$,
4. the exponential distribution, if $\lambda = 1, p = 1$,
5. the Erlang distribution, if $\lambda = 1, p \in \mathrm{N}$,
6. the chi-square distribution, if $\lambda = 1, \Theta = 2, p = \frac{1}{2}n$,
7. the gamma distribution, if $\lambda = 1, p \neq 1$,
8. the normal distribution, restricted at $x = 0$, for $x \geq 0$ if $p = 1$, $\lambda = 2, \Theta = 2\sigma^2$.

In a similar way as in [11], we use the Mellin transform $M_X(s)$ of the form

$$M_X(s) = EX^s,$$

where s is a complex variable, which gives

$$M_X(s) = \Theta^{s/\lambda}\Gamma\left(\frac{p+s}{\lambda}\right)\Gamma^{-1}\left(\frac{p}{\lambda}\right), \quad \mathrm{Re}\, s > -p. \tag{2}$$

The gamma function can be replaced as a finite or infinite product, [2, 8.322-8.335].

2. The case of a finite product. Applying the formula (8.335) in [2]:

$$\Gamma(nx) = (2\pi)^{(1-n)/2}\, n^{nx-0.5}\prod_{k=1}^{n}\Gamma\left(x+\frac{k-1}{n}\right) \tag{3}$$

to the function gamma in (2), we can rewrite (2) after some calculation as

$$M_X(s) = \prod_{k=1}^{n}\frac{(\Theta n)^{s/(n\lambda)}\Gamma\left(\frac{p+s}{n\lambda}+\frac{k-1}{n}\right)}{\Gamma\left(\frac{p}{n\lambda}+\frac{k-1}{n}\right)}. \tag{4}$$

We now apply to each factor $g_k(s)$ of the finite product $\prod_{k=1}^{n} g_k(s)$, the inverse of the Mellin transform. The result will be denoted by

$$f_k(x \mid \Theta, \lambda, p) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s-1} g_k(s)\, ds, \quad \mathrm{Re}\, s > -p. \tag{5}$$

Since $c = \mathrm{Re}\, s > -p$, we can assume that c = 0 and we obtain

$$\begin{aligned} f_k(x \mid \Theta, \lambda, p) &= \frac{n\lambda}{(n\Theta)^{\frac{p}{n\lambda}+\frac{k-1}{n}}\,\Gamma(\frac{p}{n\lambda}+\frac{k-1}{n})} \\ &\times x^{p+(k-1)\lambda-1}\exp\left(-\frac{x^{n\lambda}}{n\Theta}\right), \quad k = 1, 2, \dots, n. \end{aligned} \tag{6}$$

It is easy to check that, for each $k = 1, 2, .., n$, $\int_0^\infty f_k(x|\Theta, \lambda, p) = 1$. Therefore, by the nonnegativity of the integrand, we conclude that the formula (6) defines the probability density function of the random variable X_k on the interval $[0, \infty)$, and consequently, we conclude that each factor $g_k(s), k = 1, 2, .., n$ of the product $\prod_{k=1}^n g_k(s)$ is the Mellin transform of the random variable X_k with density (6).

Since the finite product of the Mellin transforms of indepedent random variables is equal to the Mellin transform of the product of these random variables, we obtain

$$(7) \qquad \prod_{k=1}^{n} M_{X_k}(s) = M_{\prod_{k=1}^n X_k}(s)$$

and, by (2) and (4), we have

$$(8) \qquad M_X(s) = \prod_{k=1}^{n} M_{X_k}(s)$$

thus, from (7) and (8) follows

$$M_X(s) = M_{\prod_{k=1}^n X_k}(s) .$$

If we now apply to both sides of the last equation the inverse of the Mellin transform, we obtain the following stochastic equality

$$X \stackrel{st}{=} \prod_{k=1}^{n} X_k$$

which implies

Theorem 1. *The density of the three-parmeter (generalized) gamma distribution of a random variable X is eqal to the density of a product $\prod_{k=1}^n X_k$ of indepedent and nonnegative random variables $X_k, k = 1, 2, .., n$ with generalized gamma distributions too, defined by (6), and with a common parameter of scale $\sqrt[n\lambda]{n\Theta}$.*

Since $f_k(x|\Theta, p, \lambda)$ as defined by (6) is the density function of the random variable X_k, and the Mellin transform of X_k is the factor $g_k(s)$ of the product (4), we can apply the formula (3) for each $n \geq 2$, for example $n = n_1$, to the both gamma functions in (4) and we can replace $(\Theta n)^{s/(n\lambda)}$ by $(\Theta n)^{s/(nn_1\lambda)}$. Then we can rewrite (4) as

$$(9) \qquad M_X(s) = \prod_{k=1}^{n} \prod_{k_1=1}^{n_1} \frac{(\Theta n)^{s/(nn_1\lambda)} n_1^{s/(nn_1\lambda)} \Gamma\left(\frac{p+s}{nn_1\lambda} + \frac{k-1}{nn_1} + \frac{k_1-1}{n_1}\right)}{\Gamma\left(\frac{p}{nn_1\lambda} + \frac{k-1}{nn_1} + \frac{k_1-1}{n_1}\right)} .$$

Applying to each factor of the above product the inverse of the Mellin transform we obtain

$$f_{k,k_1}(x \mid \Theta, \lambda, p) = \frac{nn_1\lambda}{(nn_1\Theta)^{\frac{p}{nn_1\lambda}+\frac{k-1}{nn_1}+\frac{k_1-1}{n_1}}\Gamma(\frac{p}{nn_1\lambda}+\frac{k-1}{nn_1}+\frac{k_1-1}{n_1})}$$

(10)
$$\times\ x^{p+(k-1)\lambda+n(k_1-1)\lambda-1}\exp\left(-\frac{x^{nn_1\lambda}}{nn_1\Theta}\right),\quad k=1,2,\ldots,n,\quad k_1=1,2,\ldots,n_1.$$

We denote by X_{k,k_1} the random variable with the density function $f_{k,k_1}(x|\Theta,\lambda,p)$ and the result can be written as

$$X \stackrel{st}{=} \prod_{k=1}^{n} X_k = \prod_{k=1}^{n}\prod_{k_1=1}^{n_1} X_{k,k_1}.$$

Repeating the previous method arbitrarily many times and changing parameters of scale, parameters of shape and taking a corresponding constant we obtain that the integrals of the density functions defined by (1), (6), (10) and analogously, on the interval $[0,\infty)$ are equal to 1.

3. The case of an infinite product. Applying the Knar formula [2,8.324] in [2]:

$$\Gamma(x+1) = 4^x \prod_{k=1}^{\infty}\left[\Gamma\left(\frac{1}{2}+\frac{x}{2^k}\right)\Gamma^{-1}\left(\frac{1}{2}\right)\right],\qquad \operatorname{Re} x > -1$$

to the function gamma in (2), we can rewrite (2) after some calculation as

(11)
$$M_X(s) = \prod_{k=1}^{\infty}\frac{(4\Theta)^{s/(2^k\lambda)}\Gamma\left(\frac{1}{2}+\frac{p-\lambda+s}{2^k\lambda}\right)}{\Gamma\left(\frac{1}{2}+\frac{p-\lambda}{2^k\lambda}\right)},\qquad \operatorname{Re} s > -p.$$

We now use, to some factor $g_k(s)$ of the infinite product $\prod_{k=1}^{\infty} g_k(s)$, the inverse of the Mellin transform. By $\operatorname{Re} s > -p$, we can assume $c=0$ and we obtain

(12)
$$f_k(x \mid \Theta,\lambda,p) = \frac{2^k\lambda}{(4\Theta)^{\frac{1}{2}+\frac{p-\lambda}{2^k\lambda}}\Gamma(\frac{1}{2}+\frac{p-\lambda}{2^k\lambda})}\, x^{(2^{k-1}-1)\lambda+p-1}\exp\left(-\frac{x^{2^k\lambda}}{4\Theta}\right).$$

It is easy to check that, for each $k\in \mathrm{N}$, the condition $\int_0^\infty f_k(x|\theta,\lambda,p)dx = 1$ holds. Therefore, by $f_k \geq 0$, we conclude that the formula (12) has defined the probability density function of the random variable X_k on the interval $[0,\infty)$, and consequently, we conclude that each factor $g_k(s), k \in \mathrm{N}$ of the product $\prod_{k=1}^{\infty} g_k(s)$ is the Mellin transform of the random variable X_k with density (12).

At last we shall prove that $\prod_{k=1}^{\infty} g_k(s)$ is the Mellin transform of $\prod_{k=1}^{\infty} X_k$, where

$$g_k(s) = \frac{(4\Theta)^{s/(2^k\lambda)}\Gamma\left(\frac{1}{2}+\frac{p-1+s}{2^k\lambda}\right)}{\Gamma\left(\frac{1}{2}+\frac{p-1}{2^k\lambda}\right)}.$$

Since $M_X(it) = \phi_{ln\, X}(t)$, where ϕ is the characteristic function, we have also

$$M_X(it) = \prod_{k=1}^{\infty} \frac{(4\Theta)^{it/(2^k\lambda)}\Gamma\left(\frac{1}{2}+\frac{p-1+s}{2^k\lambda}\right)}{\Gamma\left(\frac{1}{2}+\frac{p-1}{2^k\lambda}\right)} = \prod_{k=1}^{\infty} \phi_{ln\, X_k}(t) \tag{13}$$

$$= \phi_{\sum_{k=1}^{\infty} ln X_k}(t) = \phi_{\ln \prod_{k=1}^{\infty} X_k}(t).$$

Therefore we conclude that

$$X \overset{st}{=} \prod_{k=1}^{\infty} X_k, \tag{14}$$

where X_k are random variables, whose distributions are determined by formula (12).

We have the following

Theorem 2. *The density of a three-parameter gamma distribution of a random variable X which is of the form (1) is equal to the density of the infinite product $\prod_{k=1}^{\infty} X_k$ of indepedent and nonnegative random variables $X_k, k \in \mathrm{N}$ with distributions of the gamma gamma defined by (12) with the scale parameter $(4\Theta)^{1/(\lambda 2^k)}$, respectively.*

In this case we apply the Knar formula to the both gamma functions in (11) and each factor of the product $\prod_{k=1}^{\infty}\prod_{k_1=1}^{\infty} g_{k,k_1}(s)$. Then we have the decomposition of the random variable X

$$X \overset{st}{=} \prod_{k=1}^{\infty}\prod_{k_1=1}^{\infty} X_{k,k_1}. \tag{15}$$

This method can be repeated to the succesive infinite decompositions.

By the equality (14) it follows that

$$\ln X \overset{st}{=} \sum_{k=1}^{\infty} \ln X_k.$$

We shall prove

Theorem 3. *The series of random variables* $\sum_{k=1}^{\infty} \ln X_k$ *converges with probability* 1 *to* $\ln X$.

Proof. We use the Marcinkiewicz-Zygmund theorem [6]: if, for a sequence of indepedent random variables $Z_k, k \in \mathrm{N}$, both series

$$\sum_{k=1}^{\infty} EZ_k\,, \qquad \sum_{k=1}^{\infty} \mathrm{Var}\ Z_k$$

converge, then the series $\sum_{k=1}^{\infty} Z_k$ converges with probability 1 to the random variable Z, which means $\sum_{k=1}^{\infty} Z_k = Z$ and

$$EZ = \sum_{k=1}^{\infty} EZ_k\,, \qquad \mathrm{Var}\ Z = \sum_{k=1}^{\infty} \mathrm{Var}\ Z_k\,.$$

For this purpose we compute $E \ln X_k$ using the formula, [2, 4.352.1]:

$$\int_0^{\infty} \ln x\, x^{\nu-1} e^{-\mu x} dx = \frac{1}{\mu^{\nu}} \Gamma(\nu)[\Psi(\nu) - \ln \mu]\,, \quad \mathrm{Re}\,\mu > 0,\ \mathrm{Re}\,\nu > 0\,.$$

Then $\sum_{k=1}^{\infty} E \ln X_k$, after some calculation, can be rewritten as

$$\sum_{k=1}^{\infty} E \ln X_k = \frac{1}{\lambda} \sum_{k=1}^{\infty} \frac{1}{2^k} \left[\Psi\left(\frac{1}{2} - \frac{1}{2^k}\left(1 - \frac{p}{\lambda}\right)\right) + \ln(4\Theta) \right],$$

where $\Psi(x) = \frac{d}{dx} \ln \Gamma(x)$, [2, 8.360] is the Euler psi-function, increasing for $x > 0$. Then

(i) for $0 < p/\lambda \leq 1$,

$$-\infty < \frac{1}{\lambda}\left[\Psi(0) + \ln(4\Theta)\right] < \sum_{k=1}^{\infty} E \ln X_k \leq \frac{1}{\lambda}\left[\Psi\left(\frac{1}{2}\right) + \ln(4\Theta)\right],$$

(ii) for $p/\lambda > 1$,

$$\frac{1}{\lambda}\left[\Psi\left(\frac{1}{2}\right) + \ln(4\Theta)\right] < \sum_{k=1}^{\infty} E\ \ln X_k < \frac{1}{\lambda}\left[\Psi\left(\frac{p}{2\lambda}\right) + \ln(4\Theta)\right].$$

Next we apply the formula (4.358.2) in [2]:

$$(16) \quad \int_0^{\infty} x^{\nu-1} \exp(-\mu x)(\ln x)^2 dx = \frac{\Gamma(\nu)}{\mu^{\nu}} \left\{ [\Psi(\nu) - \ln \mu]^2 + \zeta(2, \nu - 1) \right\},$$

$\operatorname{Re}\mu > 0$, $\operatorname{Re}\nu > 0$ and, therefore

$$\sum_{k=1}^{\infty} E(\ln X_k)^2 = \sum_{k=1}^{\infty} \frac{1}{(2^k\lambda)^2}\left[\left\{\psi\left[\frac{1}{2} - \frac{1}{2^k}\left(1 - \frac{p}{\lambda}\right)\right] + \ln(4\Theta)\right\}^2 + \zeta\left[2, \frac{1}{2} - \frac{1}{2^k}\left(1 - \frac{p}{\lambda}\right) - 1\right]\right] \tag{17}$$

where $\zeta(z, q) = \sum_{n=0}^{\infty} 1/(n+q)^z$, $q \neq -n$, $\operatorname{Re} z > 1$, is a two-argument Riemann function.

With respect to the function ζ we shall consider three cases

(a) if $0 < p/\lambda < 1$ then $-1 < q < -1/2$, thus $\zeta(2, q) = \sum_{n=0}^{\infty} 1/(n+q)^2$,

(b) if $1 \leq p/\lambda < 2$ then $-1/2 \leq q < 0$, thus $\zeta(2, q) = \sum_{n=0}^{\infty} 1/(n+q)^2$,

(c) if $p/\lambda \geq 2$ then $q \geq 0$, thus $\zeta(2, q) \leq \sum_{n=0}^{\infty} 1/n^2 = \zeta(2)$.

The estimation from the above will be the following

(a) if $0 < p/\lambda < 1$ then

$$\sum_{k=1}^{\infty} E(\ln X_k)^2 < \frac{1}{3\lambda^2}\left\{\left[\Psi\left(\frac{1}{2}\right) + \ln(4\Theta)\right]^2 + \sum_{n=0}^{\infty}(n+q)^{-2}\right\}$$

(b) if $1 \leq p/\lambda < 2$, then

$$\sum_{k=1}^{\infty} E(\ln X_k)^2 < \frac{1}{3\lambda^2}\left\{[\Psi(1) + \ln(4\Theta)]^2 + \sum_{n=0}^{\infty}(n+q)^{-2}\right\}$$

(c) if $p/\lambda \geq 2$ then

$$\sum_{k=1}^{\infty} E(\ln X_k)^2 < \frac{1}{3\lambda^2}\left\{\left[\Psi\left(\frac{p}{2\lambda}\right) + \ln(4\Theta)\right]^2 + \zeta(2)\right\}$$

and as $\zeta(2)$ we put $\pi^2/6$.

Combining (i) with (a), (ii) with (b) and (ii) with (c) we conclude that in all cases the inequalites

$$-\infty < \sum_{k=1}^{\infty} E(\ln X_k) < \infty\,, \qquad \sum_{k=1}^{\infty} \operatorname{Var}(\ln X_k) < \infty\,, \tag{18}$$

hold and that by the Marcinkiewicz-Zygmund theorem the series $\sum_{k=1}^{\infty} \ln X_k$ converges with probability 1 and

$$\sum_{k=1}^{\infty} \ln X_k = \ln X\,, \qquad \sum_{k=1}^{\infty} E(\ln X_k) = E(\ln X)\,, \qquad \sum_{k=1}^{\infty} \operatorname{Var}(\ln X_k) = \operatorname{Var} \ln X\,. \tag{19}$$

Now we set the distribution $\ln X = Y$. Hence $x = e^y$ and therefore

$$(20)\quad f_Y(y\,|\,\theta,\lambda,p) = \frac{\lambda}{\Theta^{p/\lambda}\Gamma(p/\lambda)}\exp(py)\,[-\exp(\lambda y)/\Theta]\ ,\ -\infty < y < \infty\,.$$

This is the doubly exponential distribution, which has, for example, the reduced range R^*, [4.14.95; 14.92]. The density function and the distribution function of the above distribution was tabulated by E. J. Gumbel, [3].

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received January 22, 1997