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PIOTR KOWALSKI and ZDZISŁAW RYCHLIK (Lublin)

On the Weak Law of Large Numbers for Randomly Indexed Partial Sums for Arrays

Dedicated to Professor Dominik Szynal on the occasion of his 60th birthday

ABSTRACT. We present a general weak law of large numbers for randomly indexed partial sums for arrays. We consider the case where no assumption concerning the interdependence between the summation random indices and partial sums is made.

1. Introduction. Let $\{X_{ni},\ i \geq 1,\ n \geq 1\}$ be an array of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Let us put

$$S_{nm} = \sum_{i=1}^{m} X_{ni},$$
$$\mathcal{F}_{n0} = \{\varnothing, \Omega\},$$
$$\mathcal{F}_{nm} = \sigma\{X_{ni},\ 1 \leq i \leq m\},\ n \geq 1,\ m \geq 1.$$

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Let $\{N_n,\ n \geq 1\}$ be a sequence of positive integer-valued random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$.

Let f be a nondecreasing function such that $f(x) > 0$, for $x \geq 0$, and

$$\lim_{x \to \infty} f(x) = +\infty. \tag{1}$$

Let $\{k_n,\ n \geq 1\}$ be a sequence of positive integers such that $k_n \to \infty$ as $n \to \infty$.

Define

$$a_{nm} = \sum_{i=1}^{m} E\left[X_{ni} I\left(|X_{ni}| \leq f(k_n)\right) | \mathcal{F}_{n,i-1}\right]. \tag{2}$$

In the present paper we present sufficient conditions under which

$$\left(S_{nN_n} - a_{nN_n}\right)/N_n \xrightarrow{P} 0 \ \text{ as } n \to \infty \tag{3}$$

and

$$\left(S_{nN_n} - a_{nN_n}\right)/b_n \xrightarrow{P} 0 \ \text{ as } n \to \infty, \tag{4}$$

where $\{b_n,\ n \geq 1\}$ is a sequence of positive numbers such that $b_n \to \infty$ as $n \to \infty$. Here, and in what follows, $\xrightarrow{P}$ denotes the convergence in probability. We consider the case where no assumption concerning the interdependence between the summation indices N_n and the sequence $\{S_{nm},\ m \geq 1\}$ is made.

Weak law of large numbers for sequences of nonidentically distributed random variables has been intensively studied in several papers. Pyke and Root (1968), Chatterji (1969), Chow (1971), Gut (1974), Klass and Teicher (1977), and Rosalsky and Teicher (1981) generalized weak law of large numbers for sequences of independent and identically distributed random variables. Chandra (1989) introduced so-called Cesàro uniform integrability condition. Under this condition Chandra (1989), Chandra and Bose (1993), and Gut (1992) have studied L_p-convergence of several types of sequences of random variables. Recently Hong and Oh (1995) introduced another condition which relaxes Cesàro uniform integrability one and, under this condition, studied the weak law of large numbers for arrays. Hong (1996) has also presented the weak law of large numbers for randomly indexed partial sums for arrays. His result has the following form. If

$$N_n/f(n) \xrightarrow{P} \lambda \ \text{ as } n \to \infty, \tag{5}$$

where $0 < \lambda < \infty$ is a constant, then under some additional assumptions on the random variables $\{X_{ni},\ i \geq 1,\ n \geq 1\}$

$$\left(S_{nN_{k_n}} - a_{nN_{k_n}}\right)/N_{k_n} \xrightarrow{P} 0 \quad \text{as } n \to \infty.$$

In this paper we extend the result of Hong (1996). Our basic assumptions on $\{N_n, n \geq 1\}$ are much weaker than (5) and include the case when, in (5), λ is a positive random variable. We also present sufficient conditions under which (4) holds, too. Since the sequence $\{b_n,\ n \geq 1\}$ does not depend on chance, (4) is usually more useful than (3) in applications. The assumptions concerning the random variables $\{X_{ni}, i \geq 1,\ n \geq 1\}$, follow closely those of Gut (1992), Hong and Oh (1995) and Hong (1996). In the proofs of our results we also use some ideas of these authors.

Here we would also like to mention that almost sure convergence of sequences with random indices have been studied by Rychlik and Zygo (1991). On the other hand, the complete convergence of sequences of randomly indexed partial sums has been intensively studied by Szynal (1972), Csörgő and Révész (1981), see pages 252–254, Gut (1983, corrections 1985), Csörgő and Rychlik (1985).

2. Main results.

Theorem 1. *Let $\{X_{ni},\ i \geq 1,\ n \geq 1\}$ be an array of random variables. Let $\{N_n,\ n \geq 1\}$ be a sequence of positive integer-valued random variables. If there exists a non-random sequence of positive integers $\{k_n,\ n \geq 1\}$ such that $k_n \to \infty$ as $n \to \infty$,*

$$P\left(N_n > k_n\right) \to 0 \text{ as } n \to \infty, \tag{6}$$

and, for some f satisfying (1),

$$\sum_{i=1}^{k_n} P\left(|X_{ni}| > f(k_n)\right) \to 0 \quad \text{as } n \to \infty, \tag{7}$$

and

$$\left(f(k_n)\right)^{-2} \sum_{i=1}^{k_n} EX_{ni}^2 I\left(|X_{ni}| \leq f(k_n)\right) \to 0 \quad \text{as } n \to \infty, \tag{8}$$

then

$$\left(S_{nN_n} - a_{nN_n}\right)/f(k_n) \xrightarrow{P} 0 \quad \text{as } n \to \infty. \tag{9}$$

If, in addition, there exists a constant C_0 such that $0 < C_0 < \infty$ and

$$P(N_n < C_0 f(k_n)) \to 0 \text{ as } n \to \infty, \tag{10}$$

then

$$(S_{nN_n} - a_{nN_n})/N_n \xrightarrow{P} 0 \text{ as } n \to \infty. \tag{11}$$

Remark 1. Let us observe that if (9) holds and

$$N_n/f(k_n) \xrightarrow{D} \lambda \text{ as } n \to \infty, \tag{12}$$

where λ is a random variable such that $P(0 < \lambda < \infty) = 1$, then by Corollary 2 in Chow and Teicher (1988), p. 254, (11) holds, too. Here, and in what follows, $\xrightarrow{D}$ denotes the convergence in distribution.

Remark 1 can also be taken into account in Theorems 2 and 3, since the convergence in distribution is weaker than the convergence in probability.

Let f be a nondecreasing function such that $f(x) > 0$, for $x \geq 0$,

$$x^{-1} f(x) \text{ is nonincreasing as } x \to \infty, \tag{13}$$

$$x^{-1} f^2(x) \to +\infty \text{ as } x \to \infty \tag{14}$$

and

$$\int_0^t x^{-1} f(x) df(x) = O\left(t^{-1} f^2(t)\right) \text{ as } t \to \infty. \tag{15}$$

Theorem 2. *Let $\{X_{ni},\ i \geq 1,\ n \geq 1\}$ be an array of random variables. Let $\{N_n,\ n \geq 1\}$ be a sequence of positive integer-valued random variables such that for some non-random sequence of integers $\{k_n,\ n \geq 1\}$, $k_n \to \infty$ as $n \to \infty$, (6) holds. If for some f satisfying (13), (14) and (15)*

$$k_n^{-1} \sum_{i=1}^{k_n} a P(|X_{ni}| > f(a)) \to 0 \text{ as } a \to \infty \tag{16}$$

uniformly in n, then (9) holds. If, in addition, either (10) or (12) holds, then (11) holds, too.

Remark 2. Let us observe that if for some non-random sequence $\{l_n,\ n \geq 1\}$, $l_n \to \infty$ as $n \to \infty$, and for some f satisfying (13) and (14)

$$N_n/f(l_n) \xrightarrow{P} c \ \text{ as } n \to \infty, \tag{17}$$

where $0 < c = \text{const.} < \infty$, then (6) and (10) hold e. g. with $k_n = 2\,[cf(l_n)]$. Here, and in what follows, $[x] =$ the largest integer $\leq x$. Furthermore, in this case, $f(k_n) = f\,(2\,[cf(l_n)]) \leq 2cf(1)f(l_n)$ and this inequality can be taken into account in (9) and (10).

Theorem 3. *Let $\{X_{ni},\ i \geq 1,\ n \geq 1\}$ be an array of random variables. Let $\{N_n,\ n \geq 1\}$ be a sequence of positive integer-valued random variables. If for some f satisfying (13), (14) and (15)*

$$m^{-1} \sum_{i=1}^{m} aP\,(|X_{ni}| > f(a)) \to 0 \ \text{ as } a \to \infty, \tag{18}$$

uniformly in m and n, and for some sequence of positive integers $\{k_n,\ n \geq 1\}$, $k_n \to \infty$, as $n \to \infty$,

$$N_n/f(k_n) \xrightarrow{P} \lambda \ \text{ as } n \to \infty, \tag{19}$$

where λ is a random variable such that $P(0 < \lambda < \infty) = 1$, then

$$(S_{nN_n} - a_{nN_n})\,/\,f(k_n) \xrightarrow{P} 0 \ \text{ as } n \to \infty \tag{20}$$

and

$$(S_{nN_n} - a_{nN_n})\,/N_n \xrightarrow{P} 0 \ \text{ as } n \to \infty. \tag{21}$$

Theorem 4. *Let $\{X_n,\ n \geq 1\}$ be a sequence of independent and identically distributed random variables such that for some $1 \leq p < 2$*

$$nP\,(|X_1|^p > n) \to 0 \ \text{ as } n \to \infty. \tag{22}$$

If $\{N_n,\ n \geq 1\}$ is a sequence of positive integer-valued random variables such that

$$N_n/n^{1/p} \xrightarrow{P} \lambda \ \text{ as } n \to \infty, \tag{23}$$

where λ is a random variable such that $P(0 < \lambda < \infty) = 1$, then

$$(24) \qquad \left(S_{N_n}/n^{1/p}\right) - EX_1 I\left(|X_1|^p \le n\right) \to 0 \ \text{ as } n \to \infty$$

and

$$(25) \qquad \left(S_{N_n}/N_n\right) - EX_1 I\left(|X_1|^p \le n\right) \to 0 \ \text{ as } n \to \infty,$$

where $S_n = X_1 + \cdots + X_n$, $n \ge 1$.

3. Proofs.

Proof of Theorem 1. Let us put

$$X'_{ni} = X_{ni} I\left(|X_{ni}| \le f(k_n)\right),\ i \ge 1,\ n \ge 1,$$

$$S'_{nm} = \sum_{i=1}^{m} X'_{ni},\ n \ge 1,\ m \ge 1.$$

Then, by (6) and (7), we get

$$(26) \qquad P\left(S_{nN_n} \ne S'_{nN_n}\right) \le P\left(N_n > k_n\right) + P\left(\bigcup_{i=1}^{k_n} [X_{ni} \ne X'_{ni}]\right)$$

$$\le P\left(N_n > k_n\right) + \sum_{i=1}^{k_n} P\left(|X_{ni}| > f(k_n)\right) \to 0 \ \text{ as } n \to \infty.$$

Thus it is enough to prove

$$(27) \qquad \left(S'_{nN_n} - a_{nN_n}\right)/f(k_n) \xrightarrow{P} 0 \ \text{ as } n \to \infty.$$

For an arbitrary $\varepsilon > 0$ we define

$$B_j^n = \left[\left|S'_{nj} - \sum_{i=1}^{j} E\left(X'_{ni} | \mathcal{F}_{n,i-1}\right)\right| > \varepsilon f(k_n)\right],$$

$$D_n = \bigcup_{j=1}^{k_n} B_j^n .$$

Since, for every $n \geq 1$, $X'_{ni} - E(X'_{ni}|\mathcal{F}_{n,i-1})$, $1 \leq i \leq k_n$, is a martingale difference sequence, we have, by the Hájek-Rényi inequality, Chow and Teicher (1988), Theorem 7.4.8,

(28)

$$\begin{aligned} P(D_n) &= P\left(\max_{1\leq j\leq k_n}\left|S'_{nj} - \sum_{i=1}^{j} E(X'_{ni}|\mathcal{F}_{n,i-1})\right| > \varepsilon f(k_n)\right) \\ &\leq \varepsilon^{-2} f^{-2}(k_n) \sum_{i=1}^{k_n} E(X'_{ni} - E(X'_{ni}|\mathcal{F}_{n,i-1}))^2 \\ &\leq (\varepsilon f(k_n))^{-2} \sum_{i=1}^{k_n} E(X'_{ni})^2 \\ &= (\varepsilon f(k_n))^{-2} \sum_{i=1}^{k_n} EX^2_{ni} I(|X_{ni}| \leq f(k_n)). \end{aligned}$$

Furthermore

$$\begin{aligned} P\left(B^n_{N_n}\right) &\leq P\left(B^n_{N_n} \cap [N_n \leq k_n]\right) + P(N_n > k_n) \\ &\leq P(D_n) + P(N_n > k_n). \end{aligned} \tag{29}$$

Thus, by (6), (28), (8) and (29), we obtain (27). Hence, by (26) and (27), (9) holds. On the other hand, by (9) and (10), we get (11).

Proof of Theorem 2. We show that (13), (14), (15) and (16) imply (7) and (8). Let us observe that, by (16),

$$\sum_{i=1}^{k_n} P(|X_{ni}| > f(k_n)) \leq k_n^{-1} \sum_{i=1}^{k_n} k_n P(|X_{ni}| > f(k_n)) \to 0 \text{ as } n \to \infty,$$

so that (7) holds.

On the other hand

(30)

$$\begin{aligned} &\sum_{i=1}^{k_n} EX^2_{ni} I(|X_{ni}| \leq f(k_n)) \\ &= \sum_{i=1}^{k_n} \sum_{j=1}^{k_n} EX^2_{ni} I(f(j-1) < |X_{ni}| \leq f(j)) \\ &\leq \sum_{i=1}^{k_n} \sum_{j=1}^{k_n} f^2(j) \{P(|X_{ni}| > f(j-1)) - P(|X_{ni}| > f(j))\} \end{aligned}$$

$$= \sum_{i=1}^{k_n} \Big\{ f^2(1) P(|X_{ni}| > f(0)) - f^2(k_n) P(|X_{ni}| > f(k_n))$$
$$+ \sum_{j=1}^{k_n-1} \left[f^2(j+1) - f^2(j) \right] P(|X_{ni}| > f(j)) \Big\}$$
$$\leq f^2(1) \sum_{i=1}^{k_n} P(|X_{ni}| > f(0))$$
$$+ \sum_{i=1}^{k_n} \sum_{j=1}^{k_n} \left[f^2(j+1) - f^2(j) \right] P(|X_{ni}| > f(j)).$$

Now, by (14),

$$(31) \qquad (f(k_n))^{-2} \sum_{i=1}^{k_n} P(|X_{ni}| > f(0)) \to 0 \text{ as } n \to \infty.$$

Set, for every $n \geq 1$ and $j \geq 1$,

$$a_n(j) = k_n^{-1} \sum_{i=1}^{k_n} j P(|X_{ni}| > f(j)).$$

Then, by (16),

$$(32) \qquad \sup_n a_n(j) \to 0 \text{ as } j \to \infty.$$

Furthermore, by (13), (14) and (15),

$$(33) \qquad \sum_{j=1}^{k_n} \left[f^2(j+1) - f^2(j) \right] / j \leq 2 \sum_{j=1}^{k_n} j^{-1} f(j+1) (f(j+1) - f(j))$$
$$\leq 8 \sum_{j=1}^{k_n} (j+1)^{-1} f(j) (f(j+1) - f(j))$$
$$\leq 8 \int_0^{k_n+1} x^{-1} f(x) df(x) = O\left(k_n^{-1} (f(k_n))^2\right) \text{ as } n \to \infty.$$

Thus, by (33), (32), (14) and Toeplitz Lemma [Ash (1972), Lemma 7.1.2], we get

$$(34) \qquad (f(k_n))^{-2} \sum_{i=1}^{k_n} \sum_{j=1}^{k_n} \left[f^2(j+1) - f^2(j) \right] P(|X_{ni}| > f(j))$$

$$\leq k_n(f(k_n))^{-2}\sum_{j=1}^{k_n} j^{-1}\left[f^2(j+1)-f^2(j)\right]\left(\sup_n a_n(j)\right) \to 0 \text{ as } n\to\infty.$$

Now (8) easily follows from (30), (31) and (34). Thus the proof of Theorem 2 is completed.

Proof of Theorem 3. Taking into account (19) and Remark 1 it is enough to prove that (20) holds.

Let $\varepsilon > 0$ and $\delta > 0$ be given. Then, by (19), there exists n_0 such that for every $n \geq n_0$

$$P(|N_n - \lambda f(k_n)| \geq \delta f(k_n)) < \varepsilon. \tag{35}$$

Let us choose now $0 < a < b < \infty$ so that

$$P(a \leq \lambda \leq b) > 1-\varepsilon. \tag{36}$$

Thus, by (35), (36), (18) and (13), for every $n \geq n_0$ we get

$$\begin{aligned} P\left(S_{nN_n} \neq S'_{nN_n}\right) &\leq 2\varepsilon + P\Big(\left[S_{nN_n} \neq S'_{nN_n}\right] \cap [a\leq\lambda\leq b] \\ &\quad \cap [(\lambda-\delta)f(k_n) \leq N_n \leq (\lambda+\delta)f(k_n)]\Big) \\ &\leq 2\varepsilon + \sum_{i=1}^{l_n} P(|X_{ni}| > f(k_n)) \\ &= 2\varepsilon + (l_n/k_n)\left\{l_n^{-1}\sum_{i=1}^{l_n} k_n P(|X_{ni}| > f(k_n))\right\} \to 2\varepsilon \text{ as } n\to\infty, \end{aligned}$$

where $l_n = [(b+\delta)f(k_n)]$, $n \geq 1$.

Since $\varepsilon > 0$ can be chosen arbitrarily small it is enough to prove that (27) holds, of course with $\{k_n,\ n\geq 1\}$ and f from Theorem 3.

For an arbitrary $\eta > 0$ we define, similarly as in the proof of Theorem 1,

$$H_j^n = \left[|S'_{nj} - a_{nj}| > \eta f(k_n)\right], \quad G_n = \bigcup_{j=1}^{l_n} H_j^n.$$

Hence, again by (35), (36) and (28) with $\{l_n,\ n\geq 1\}$ instead of $\{k_n,\ n\geq 1\}$, for every $n \geq n_0$, we obtain

$$\begin{aligned} P\left(H_{N_n}^n\right) &\leq 2\varepsilon + P(G_n) \\ &\leq 2\varepsilon + (\eta f(k_n))^{-2}\sum_{i=1}^{l_n} EX_{ni}^2 I(|X_{ni} \leq f(k_n)). \end{aligned} \tag{37}$$

Now, step by step as in (30)–(34) with necessary changes, we get

(38)
$$\sum_{i=1}^{l_n} EX_{ni}^2 I\left(|X_{ni} \le f(k_n)\right) \le f^2(1)\sum_{i=1}^{l_n} P\left(|X_{ni}| > f(0)\right)$$

$$+\sum_{i=1}^{l_n}\sum_{j=1}^{k_n}\left(f^2(j+1)-f^2(j)\right)P\left(|X_{ni}| > f(j)\right)$$

$$\le l_n f^2(1) + \sum_{j=1}^{k_n}\left\{\left(f^2(j+1)-f^2(j)\right)/j\right\}\left\{\sum_{i=1}^{l_n} jP\left(|X_{ni}| > f(j)\right)\right\}$$

$$= l_n f^2(1) + l_n \sum_{j=1}^{k_n}\left\{\left(f^2(j+1)-f^2(j)\right)/j\right\}\left\{l_n^{-1}\sum_{i=1}^{l_n} jP\left(|X_{ni}| > f(j)\right)\right\}.$$

Furthermore, by (13), (14) and (15)

$$l_n \le (b+\delta)f(k_n) \le (b+\delta)f(1)k_n,$$

and (33) holds. Thus, by (18), (33), (38), Toeplitz Lemma [Ash (1972), Lemma 7.1.2] and (37), we get

$$P\left(H_{N_n}^n\right) \to 2\varepsilon \quad n \to \infty.$$

This completes the proof of Theorem 3.

Proof of Theorem 4. Theorem 4 is an immediate consequence of Theorem 3 obtained by choosing $f(x) = x^{1/p}$, $1 \le p < 2$, $k_n = n$, $n \ge 1$.

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Instytut Matematyki UMCS
pl. Marii Curie–Skłodowskiej 1
20–031 Lublin, Poland
e-mail
rychlik@golem.umcs.lublin.pl
kovalius@golem.umcs.lublin.pl

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