ANNALES
UNIVERSITATIS MARIAE CURIE – SKŁODOWSKA
LUBLIN – POLONIA

VOL. LI.1, 6 SECTIO A 1997

LESŁAW GAJEK and ANDRZEJ OKOLEWSKI (Łódź)

Steffensen–type Inequalities for Order and Record Statistics

Dedicated to Professor Dominik Szynal
on the occasion of his 60th birthday

ABSTRACT. The Steffensen inequality is applied to derive quantile and moment bounds for the expectations of order and record statistics based on independent identically distributed random variables.

1. Introduction. Let $X, X_1, X_2, \ldots$ be i.i.d. random variables with a common distribution function F and the quantile function F^{-1}. Let $X_{i,n}$ denote the i-th order statistics from the sample $X_1, \ldots, X_n$. The k-th record statistics $Y_n^{(k)}$ from the sequence $X_1, X_2, \ldots$ are defined by

$$Y_n^{(k)} = X_{L_k(n),L_k(n)+k-1}, \quad n = 0, 1, 2, \ldots, \quad k = 1, 2, \ldots,$$

where $L_k(0) = 1$, $L_k(n+1) = \min\{j : X_{L_k(n),L_k(n)+k-1} < X_{j,j+k-1}\}$ for $n = 0, 1, 2, \ldots$ (c.f. [4]).

In Section 2 we present quantile lower and upper bounds for the α-th moment of the order statistics $X_{k,n}$. The bounds are derived by the use of Steffensen inequality (Lemma 1). In general we do not have to assume that the α-th moment of X exists in order to get meaningful bounds on $\mathrm{E}\, X_{k,n}^{\alpha}$.

It is worth pointing out that for some distributions our upper bounds are more precise than their counterparts obtained by Moriguti (1951) and Ludwig (1960)

$$(1) \qquad |\,\mathrm{E}\,X_{k,n}| \le \left(n \frac{\binom{2k-2}{k-1}\binom{2n-2k}{n-k}}{\binom{2n-1}{n}} - 1 \right)^{1/2}$$

who applied the Schwarz inequality (under the assumptions $\mathrm{E}\,X = 0$, $\mathrm{E}\,X^2 = 1$) instead of the Steffensen inequality. Our inequalities have the form similar to the upper bounds

$$(2) \qquad \mathrm{E}\,X_{k,n} \le \frac{n}{n+1-k} \int_{(k-1)/n}^{1} F^{-1}(t)dt,$$

obtained by Caraux and Gascuel (1992) and Rychlik (1992) via a different approach for possibly dependent identically distributed random variables. However for some values of k (close to n) our bounds are in general more precise than (2) for all distribution functions F. A summary of known bounds for order statistics is presented in [1]. In the case of restricted families of distributions Moriguti's result was improved by Gajek and Rychlik (1996 a, b). The method of projection developed by them is applicable for independent [8], as well as dependent [7] identically distributed random variables. In [5] Hölder type inequalities for order and record statistics were proved (in [6] some improvements were derived under restrictions on their distributions).

Section 2 contains also some moment bounds for order statistics based on i.i.d. r.v.'s with a bounded support (Proposition 3 and 4) and the bounds on $\mathrm{E}\,X_{k,n}^{\alpha}$ involving the quantiles of X only (Proposition 2).

In Section 3 we give analogous inequalities for k-th record statistics and compare them with the following bound of Grudzień and Szynal (1983)

$$(3) \qquad |\,\mathrm{E}\,Y_n^{(k)}| \le \left(\frac{k^{2n+2}}{(2k-1)^{2n+1}} \binom{2n}{n} - 1 \right)^{1/2}$$

which is the record analogue of (1).

2. Inequalities for order statistics. We shall frequently apply the following Steffensen inequalities (see Mitrinović (1972)).

Lemma 1. *Let f and g be integrable functions defined on $[a; b]$, f be nondecreasing and let $0 \le g(t) \le 1$ for $t \in [a; b]$. If we denote $\int_a^b g(t)dt$ by λ then the following inequalities hold*

$$\int_a^{a+\lambda} f(t)dt \le \int_a^b f(t)g(t)dt \le \int_{b-\lambda}^b f(t)dt. \tag{4}$$

The first equality is attained iff one of the following conditions is satisfied:
(a) $g(\cdot) = \mathbb{I}_{[a;\, a+\lambda]}(\cdot)$ a.e. on $[a; b]$;
(b) *f is equal to some constant a.e. on the set*

$$\{x \in [a;\, a+\lambda] : g(x) < 1\} \cup \{x \in [a+\lambda;\, b] : g(x) > 0\}.$$

The second equality is attained iff one of the following conditions is satisfied:
(a$'$) $g(\cdot) = \mathbb{I}_{[b-\lambda;\, b]}(\cdot)$ a.e. in $[a; b]$;
(b$'$) *f is equal to some constant a.e. on the set*

$$\{x \in [a;\, b-\lambda] : g(x) > 0\} \cup \{x \in [b-\lambda;\, b] : g(x) < 1\}.$$

Corollary 1. *If f is nonincreasing then all signs of inequalities in (4) are reverse.*

Proof. Apply Lemma 1 for the function $-f$. □

We can now formulate our first result.

Proposition 1. *Let $k, n \in \mathrm{N}$ be such that $k \le n$. Let $k_0 = [(n+1)/2]$. If $\alpha = 2i - 1$, $i = 1, 2, \ldots$ then*

$$\text{(i)} \qquad \mathrm{E}\,X_{k,n}^{\alpha} \ge \begin{cases} A(k,n) \int_0^{1/A(k,n)} \left[F^{-1}(t)\right]^{\alpha} dt & \text{for } k < k_0 \\ A(k_0,n) \int_0^{1/A(k_0,n)} \left[F^{-1}(t)\right]^{\alpha} dt & \text{for } k \ge k_0, \end{cases}$$

$$\text{(ii)} \qquad \mathrm{E}X_{k,n}^{\alpha} \le \begin{cases} A(k,n) \int_{1-1/A(k,n)}^{1} \left[F^{-1}(t)\right]^{\alpha} dt & \text{for } k \ge k_0 \\ A(k_0,n) \int_{1-1/A(k_0,n)}^{1} \left[F^{-1}(t)\right]^{\alpha} dt & \text{for } k < k_0, \end{cases}$$

where

$$A(k,n) = \begin{cases} n B_{k-1,n-1}\left(\frac{k-1}{n-1}\right) & \text{for } n > 1 \\ 1 & \text{for } n = k = 1, \end{cases} \tag{5}$$

and $B_{j,m}(x) = \binom{m}{j} x^j(1-x)^{m-j}$ *is the classical Bernstein polynomial. If* $\alpha = 2i$, $i = 1, 2, \ldots$, *then*

$$\text{(iii)} \qquad \mathrm{E}\, X_{k,n}^{\alpha} \le A(k,n) \left\{ \int_0^{\lambda_1} \left[F^{-1}(t)\right]^{\alpha} dt + \int_{1-\lambda_2}^{1} \left[F^{-1}(t)\right]^{\alpha} dt \right\}$$

$$\text{(iv)} \qquad \mathrm{E}\, X_{k,n}^{\alpha} \ge A(k,n) \left\{ \int_{z_0-\lambda_1}^{z_0+\lambda_2} \left[F^{-1}(t)\right]^{\alpha} dt \right\},$$

where $z_0 = F(0)$,

$$\lambda_1 = \frac{I_{z_0}(k, n-k+1)}{A(k,n)}, \qquad \lambda_2 = \frac{1 - I_{z_0}(k, n-k+1)}{A(k,n)}$$

and $I_z(a,b) = \int_0^z t^{a-1}(1-t)^{b-1}dt / \int_0^1 t^{a-1}(1-t)^{b-1}dt$, $z \in [0;1]$, $a, b > 0$, *denotes the normalized incomplete Beta-function. The equalities in* (i)–(iv) *are attained iff* $k = n = 1$ *or* F *has one atom only.*

Proof. For $n = 1$ (i)–(iv) are obvious. So let us consider the case $n \ge 2$. From David (1981) we have

$$\text{(6)} \qquad \mathrm{E}\, X_{k,n}^{\alpha} = \int_0^1 \left[F^{-1}(t)\right]^{\alpha} k \binom{n}{k} t^{k-1}(1-t)^{n-k} dt.$$

Let us define $g_1(t,k,n) = k\binom{n}{k} t^{k-1}(1-t)^{n-k}$ and

$$A(k,n) = \max\{g_1(t,k,n) : t \in [0;1]\}, \quad k, n \in \mathrm{N}, \ k \le n.$$

An easy computation shows that

$$A(k,n) = nB_{k-1,n-1}\left(\frac{k-1}{n-1}\right).$$

Applying Lemma 1 for $f(t) = \left[F^{-1}(t)\right]^{\alpha}$ and $g(t) = g_1(t,k,n)/A(k,n)$ we can get

$$\mathrm{E}\, X_{k,n}^{\alpha} \ge A(k,n) \int_0^{\lambda} \left[F^{-1}(t)\right]^{\alpha} dt$$

and

$$\mathrm{E}\, X_{k,n}^{\alpha} \le A(k,n) \int_{1-\lambda}^{1} \left[F^{-1}(t)\right]^{\alpha} dt,$$

where $\lambda = \int_0^1 g(t)dt = 1/A(k,n)$. The proof of (i) and (ii) will be completed whenever we show that

1) the function $h_1 : \mathbb{R}_+ \to \mathbb{R}$ defined by $h_1(t) = t\int_0^{1/t} \left[F^{-1}(t)\right]^\alpha dt$ is nonincreasing,
2) the function $h_2 : \mathbb{R}_+ \to \mathbb{R}$ defined by $h_2(t) = t\int_{1-1/t}^1 \left[F^{-1}(t)\right]^\alpha dt$ is nondecreasing,
3) $A(k,n) = A(n-k+1,n)$ for $1 \le k \le n$,
4) $A(k+1,n) \le A(k,n)$ for $k \in \mathrm{N}$ such that $2k \le n$.

It is easy to check that 1)–3) hold. The inequality 4) can be proved by considering the difference $R(k,n) = A(k,n) - A(k+1,n)$ and showing that $R(k,n) \ge 0$ for $k \le n-k$. To this end it suffices to rewrite $R(k,n)$ as

$$R(k,n) = \binom{n}{k}\frac{k^k(n-k)^{n-k}}{(n-1)^{n-1}}\left\{\left(1-\frac{1}{k}\right)^{k-1} - \left(1-\frac{1}{n-k}\right)^{(n-k)-1}\right\}$$

and to check that the function $r : (1;+\infty) \to \mathbb{R}$ defined by

$$r(x) = \left(1-\frac{1}{x}\right)^{x-1}$$

is decreasing. Finally observe that the equality in (i) may hold iff either $g(x) = 1$ a.e. on $[0;1]$, which is equivalent to $n = k = 1$, or $F^{-1}(t) =$ const on $(0;1)$, which means that F has one atom. The proof of (ii) uses respectively the right hand side inequality in (4) and the fact that the function $h_1(t) = t\int_{1-1/t}^1 \left[F^{-1}(t)\right]^\alpha dt$ is nondecreasing. In order to prove (iii) and (iv) we rewrite (6) as

$$\mathrm{E}\, X_{k,n}^\alpha = \int_0^{z_0} \left[F^{-1}(t)\right]^\alpha k\binom{n}{k} t^{k-1}(1-t)^{n-k}dt + \int_{z_0}^1 \left[F^{-1}(t)\right]^\alpha k\binom{n}{k} t^{k-1}(1-t)^{n-k}dt,$$

where $z_0 = F(0)$. Applying now Corollary 1 and Lemma 1 in the same way as in the first part of the proof we get

$$\mathrm{E}\, X_{k,n}^\alpha \le A(k,n)\int_0^{\lambda_1} \left[F^{-1}(t)\right]^\alpha dt + A(k,n)\int_{1-\lambda_2}^1 \left[F^{-1}(t)\right]^\alpha dt$$

and

$$\mathrm{E}\, X_{k,n}^\alpha \ge A(k,n)\int_{z_0-\lambda_1}^{z_0} \left[F^{-1}(t)\right]^\alpha dt + A(k,n)\int_{z_0}^{z_0+\lambda_2} \left[F^{-1}(t)\right]^\alpha dt,$$

where

$$\lambda_1 = \frac{1}{A(k,n)} \int_0^{z_0} k \binom{n}{k} t^{k-1}(1-t)^{n-k} dt = \frac{1}{A(k,n)} I_{z_0}(k, n-k+1),$$

$$\lambda_2 = \frac{1}{A(k,n)} \int_{z_0}^{1} k \binom{n}{k} t^{k-1}(1-t)^{n-k} dt = \frac{1}{A(k,n)} [1 - I_{z_0}(k, n-k+1)].$$

This completes the proof of Proposition 1. □

Remark 1. The bounds given in Proposition 1 (i)–(iv) may be meaningful even when $\mathrm{E}\, X^{\alpha}$ does not exist (see Example 4 below). Therefore from now on we assume that the corresponding lower and upper bounds on $\mathrm{E}\, X^{\alpha}_{k,n}$ are finite, without making stronger assumptions.

Remark 2. Note that if supp $F \subseteq [0; \infty)$ then (iii) and (iv) follow from (ii) and (i) respectively. A similar remark concerns the case supp $F \subseteq (-\infty; 0]$.

Proposition 2. *Let $k, n \in \mathrm{N}$ be such that $k \leq n$.*

(i) *If the quantile function F^{-1} is convex on $[0; 1/A(k,n)]$, where $A(k,n)$ is defined by (5), then*

$$\mathrm{E}\, X_{k,n} \geq \begin{cases} F^{-1}(1/2A(k,n)) & \text{for } k < k_0 \\ F^{-1}(1/2A(k_0,n)) & \text{for } k \geq k_0, \end{cases}$$

where $k_0 = [(n+1)/2]$.

(ii) *If F^{-1} is concave on $[1 - 1/A(k,n); 1]$, then*

$$\mathrm{E}\, X_{k,n} \leq \begin{cases} F^{-1}(1 - 1/2A(k,n)) & \text{for } k \geq k_0 \\ F^{-1}(1 - 1/2A(k_0,n)) & \text{for } k < k_0. \end{cases}$$

Proof. Applying the Jensen inequality to the bound (i) of Proposition 1, taken with $\alpha = 1$, we get

$$\mathrm{E}\, X_{k,n} \geq F^{-1}\left(A(k,n) \int_0^{1/A(k,n)} t dt \right) = F^{-1}\left(1/2A(k,n)\right),$$

for $k < k_0$. Other cases can be proven in a similar way. □

Remark 3. An important fact to note here is that in comparison with the bounds (1) and (2) our inequalities have got several advantages. First of all, they work under weaker assumptions than (1) (we even do not need $\mathrm{E}\,X$ be finite). Secondly they always give upper and lower bounds. Finally, for some distributions our upper bounds are more precise than the bound (1) and for some k (closed to n) our bound (ii) is better than (2) for all distribution functions. Moreover, for $\alpha = 2$ (or, more generally, α even) our bounds (iii) and (iv) are meaningful while (1) and (2) are useless if the support of X contains both positive and negative reals. The advantages are illustrated in the examples below.

Example 1. For

$$F(x) = \begin{cases} 1 - \frac{1}{x^2} & \text{for } x \geq 1 \\ 0 & \text{else} \end{cases}$$

it is easy to check that: $\mathrm{E}\,X = 2, \mathrm{E}\,X_{2,3} = 1.6$ ($\mathrm{E}\,X > \mathrm{E}\,X_{2,3}$) and the second moment does not exist. From Proposition 1 (i) and (ii) we have $\mathrm{E}\,X_{2,3} \geq 3 - \sqrt{3} \approx 1.26$ and $\mathrm{E}\,X_{2,3} \leq \sqrt{6} \approx 2.50$, respectively. Observe that the same upper bound follows from (2).

Example 2. Let

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ 0.1(x-0.1)^3 + (0.1)^4 & \text{for } x \in [0; 0.1) \\ x^4 & \text{for } x \in [0.1; 1] \\ 1 & \text{for } x > 1. \end{cases}$$

Computations show (see Table 1) that bounds on e.g. $\mathrm{E}\,X_{k,10}$ obtained from Proposition 1 (ii) are better than the ones derived from (1) for $k = 1, \ldots, 10$. We also compare the former bounds with the quantile bounds provided in Proposition 2.

TABLE 1

k	Lower bound Proposition 1(i)	Upper bound Proposition 1(ii)	Upper bound Proposition 2	Upper bound given in (1)
1	0.4498	0.9464	0.9481	1.1373
2	0.5692	0.9464	0.9481	1.0183
3	0.6048	0.9464	0.9481	0.9811
4	0.6229	0.9464	0.9481	0.9634
5	0.6295	0.9464	0.9481	0.9658
6	0.6295	0.9464	0.9481	0.9658
7	0.6295	0.9492	0.9558	0.9634
8	0.6295	0.9553	0.9634	0.9811
9	0.6295	0.9657	0.9663	1.0183
10	0.6295	0.9872	0.9873	1.1373

Example 3. Let

$$F^{-1}(x) = \begin{cases} x - \frac{1}{2} & \text{for } x \in [0; \frac{1}{2}] \\ x^2 - x + \frac{1}{4} & \text{for } x \in (\frac{1}{2}; 1]. \end{cases}$$

In Table 2 below we compare the exact value of $\mathrm{E}\, X_{k,10}^2$ with the lower and upper bounds obtained from Proposition 1. Observe that (1) and (2) are useless in this example.

TABLE 2

k	Lower bound Proposition 1(iv)	Upper bound Proposition 1(iii)	The exact value of $\mathrm{E}\, X_{k,10}^2$
1	0.00332	0.20324	0.17424
2	0.02124	0.14368	0.11358
3	0.03005	0.12360	0.06779
4	0.02538	0.11522	0.03609
5	0.01194	0.10639	0.01678
6	0.00305	0.08521	0.00726
7	0.00163	0.05470	0.00484
8	0.00172	0.03216	0.00794
9	0.00082	0.02633	0.01685
10	0.00002	0.04224	0.03378

Now, we shall use the Steffensen inequalities to obtain some moment bounds for order statistics based on samples from distributions with bounded support.

Proposition 3. *Let* supp $F = [0; A]$, $A > 0$. *Then for every real* $\alpha > 0$, $k, n \in \mathbb{N}$ *such that* $k \leq n$

$$\mathrm{E}\, X_{k,n}^{\alpha} \geq \binom{n}{k-1} A^{\alpha} \left(\frac{\mathrm{E}\, X_{k,k}^{\alpha}}{A^{\alpha} k} \right)^{n-k+1}. \tag{7}$$

The equality holds iff either F *has one atom at 0, or* $k = 1$ *and* F *has one atom in* 1, *or* $k = n$. *Given* n, *the right hand side of* (7) *may decrease as a function of* k. *Therefore* (7) *implies immediately a better inequality*

$$\mathrm{E}\, X_{k,n}^{\alpha} \geq \max_{1 \leq i \leq k} \left\{ \binom{n}{i-1} A^{\alpha} \left(\frac{\mathrm{E}\, X_{i,i}^{\alpha}}{A^{\alpha} i} \right)^{n-i+1} \right\}. \tag{7'}$$

Proof. Let us denote $f(t) = (1-t)^{n-k}$, $g(t) = \left[F^{-1}(t)\right]^{\alpha} t^{k-1}$ for $t \in [0; 1]$ and observe that f is nonincreasing and $0 \leq g(t) \leq A^{\alpha}$ on $[0; 1]$. Applying

Corollary 1 we get by (6)

$$\mathrm{E}\, X_{k,n}^{\alpha} \geq A^{\alpha} k \binom{n}{k} \int_{1-\lambda}^{1} f(t)dt = A^{\alpha} \binom{n}{k-1} \lambda^{n-k+1},$$

where

$$\lambda = \frac{1}{A^{\alpha}} \int_0^1 g(t)dt = \frac{1}{A^{\alpha} k} \mathrm{E}\, X_{k,k}^{\alpha}. \qquad \square$$

Remark 4. Using Lemma 1 with $f(t) = t^{k-1}$ and $g(t) = \left[F^{-1}(t)\right]^{\alpha} \times (1-t)^{n-k}$, one gets the following inequality

$$\mathrm{E}\, X_{k,n}^{\alpha} \leq \binom{n}{k} A^{\alpha} \left[1 - \left(1 - \frac{\mathrm{E}\, X_{1,n-k+1}^{\alpha}}{A^{\alpha}(n-k+1)} \right)^{k} \right]. \tag{8}$$

Inequality (8), though obtained like (7) by the use of Steffensen's inequality, works much worse than (7). One explanation is that (7) and (8) were derived under the same assumption $0 \leq g(t) \leq A^{\alpha}$. In the case $g(t) = \left[F^{-1}(t)\right]^{\alpha} t^{k-1}$, A^{α} is the best possible upper bound on g for every F; in the case $g(t) = \left[F^{-1}(t)\right]^{\alpha} (1-t)^{n-k}$ the upper bound A^{α} on g is attainable iff $n = k$ or F has one atom in A. Therefore the right hand side of (8) is evidently too large (may be even larger than A^{α}).

Remark 5. It is possible to check that for i.i.d. r.v.'s the following inequalities hold:

$$\mathrm{E}\, X_{k,n} \leq \mathrm{E}\, X_{k,k} \quad \text{and} \quad \mathrm{E}\, X_{k,n} \geq \mathrm{E}\, X_{1,n-k+1}, \tag{9}$$

for every $1 \leq k \leq n$. Let us observe that even in the case $\alpha = 1$, (7) does not follow from (9). In fact (7) is somehow reverse to (9).

Remark 6. Inequality (7) has the following very interesting features:

1) The lower bound on $\mathrm{E}\, X_{k,n}^{\alpha}$ involves quantities which, by definition, are greater than $\mathrm{E}\, X_{k,n}^{\alpha}$, i.e. A^{α} and $\mathrm{E}\, X_{k,k}^{\alpha}$.

2) The assumption that supp F be bounded is necessary to get a lower bound on $\mathrm{E}\, X_{k,n}^{\alpha}$ in the form $(\mathrm{E}\, X_{k,k}^{\alpha})^{n-k+1}$ times a positive constant (compare Remark 8 and Example 4 below).

Remark 7. When supp $F \subseteq [0;\infty)$, the inequalities proven above can be extended easily to all $\alpha \in \mathbb{R}_{+}$. On the other hand, the corresponding inequalities for every $\alpha < 0$ can be obtained immediately by introducing a new random variable $Z = 1/X$ and observing that since $Z^{\alpha}_{k,n} = X^{-\alpha}_{n-k+1,n}$, the bounds for $Z^{\alpha}_{k,n}$ follow from the ones for $X^{-\alpha}_{n-k+1,n}$ ($-\alpha$ being positive now).

Remark 8. Putting $\alpha = k = 1$ we get from (7) the following simple bound

$$\mathrm{E}\,X_{1,n} \geq A\left(\frac{\mathrm{E}\,X}{A}\right)^{n} \quad \text{for } n = 1, 2, \ldots . \tag{10}$$

Example 4. The example presented below shows necessity of our assumption that the support be bounded. Let

$$F(t) = \begin{cases} 1 - (1+t)^{-1} & \text{for } t \geq 0 \\ 0 & \text{elsewhere .} \end{cases}$$

It is easy to check that $\mathrm{E}\,X_{1,2} = 1$ but the first moment of X is $+\infty$. So it is impossible to get a lower bound on $\mathrm{E}\,X_{1,n}$ in the form $(\mathrm{E}\,X)^n$ times a positive constant. The same example shows that the lower bound in Proposition 1 (i) is meaningful (and equals 0.39) while $\mathrm{E}\,X$ is not finite.

Now we shall extand (10) to cover the case $1 < k \leq n$.

Proposition 4. *Let* supp $F = [0; A]$, $A > 0$. *Then for every* $k, n \in \mathrm{N}$ *such that* $k \leq n$

(i)
$$\mathrm{E}\,X_{k,n} \geq A\left(\frac{\mathrm{E}\,X}{A}\right)^{n-k+1},$$

(ii)
$$\mathrm{E}\,X_{k,n} \leq A\left[1 - \left(1 - \frac{\mathrm{E}\,X}{A}\right)^{k}\right].$$

Proof. (i) From (9) and (7) we get

$$\mathrm{E}\,X_{k,n} \geq \mathrm{E}\,X_{1,n-k+1} \geq A\left(\frac{\mathrm{E}\,X}{A}\right)^{n-k+1}.$$

(ii) From (9) and (8) we obtain

$$\mathrm{E}\,X_{k,n} \leq \mathrm{E}\,X_{k,k} \leq A\left[1 - \left(1 - \frac{\mathrm{E}\,X}{A}\right)^{k}\right]. \qquad \square$$

Remark 9. Though obtained via (8), the bound given in (ii) is not greater than A, so in many cases it is better than (8) (the trick relied on using first (9) in order to work with $X_{k,k}$ in which case (8) may be attainable).

Example 5. Let us consider the random variable X with Lebesgue density

$$f(x) = \begin{cases} c(1.1-x)^{-5} & \text{when } x \in [0;\, 1] \\ 0 & \text{otherwise,} \end{cases}$$

where c is a norming constant. In Table 3 below we present lower bounds, upper bounds and exact values of $\mathrm{E}\, X_{k,n}$, $n = 10$, $k = 1, \ldots, 10$.

TABLE 3

k	Lower bound (7′)	Lower bound Proposition 4(i)	The exact value of $\mathrm{E}\, X_{k,10}$	Upper bound Proposition 4(ii)
1	0.713143	0.713143	0.880904	0.966758
2	0.713143	0.737665	0.935072	0.998895
3	0.713143	0.763029	0.955670	0.999963
4	0.713143	0.789266	0.967693	0.999999
5	0.713143	0.816406	0.975960	1.000000
6	0.713143	0.844478	0.982161	1.000000
7	0.713143	0.873516	0.987071	1.000000
8	0.713143	0.903552	0.991103	1.000000
9	0.713143	0.934621	0.994506	1.000000
10	0.997436	0.966758	0.997436	1.000000

Remark 10. It is interesting to compare Proposition 4 (ii) with the following inequality

$$\mathrm{E}\, X_{k,n} \le \frac{n}{n+1-k}\, \mathrm{E}\, X \tag{11}$$

which follows from (2). Since (11) does not take into account that the support of F is bounded, for $\mathrm{E}\, X$ close to A and k large the right hand side of (11) may be of order nA, while the bound given in (ii) is never greater than A.

Remark 11. In Propositions 1–4 above several lower and upper bounds on $\mathrm{E}\, X_{k,n}$ were proven. Combining them with known inequalities enables one to get further improvements.

Remark 12. Grudzień and Szynal (1980, 1982) considered distributions and moments of order statistics from sample with random size. The question arises whether bounds similar to the ones given in the present paper can be derived in their case too.

3. Inequalities for record statistics. An analogous result to Proposition 1 can be obtained for k-th record statistics.

Proposition 5. *Let $n = 0, 1, 2, \ldots, k = 2, 3, \ldots$. If $\alpha = 2i - 1$, $i \in \mathrm{N}$, then*

(i)

$$\mathrm{E}\left[Y_n^{(k)}\right]^\alpha \geq \begin{cases} C(k,n) \int_0^{1/C(k,n)} \left[F^{-1}(t)\right]^\alpha dt & \text{for } n \leq [k/2] \\ C\left(k,[k/2]\right) \int_0^{1/C(k,[k/2])} \left[F^{-1}(t)\right]^\alpha dt & \text{for } n > [k/2], \end{cases}$$

(ii)

$$\mathrm{E}\left[Y_n^{(k)}\right]^\alpha \leq \begin{cases} C(k,n) \int_{1-1/C(k,n)}^1 \left[F^{-1}(t)\right]^\alpha dt & \text{for } n > [k/2] \\ C\left(k,[k/2]\right) \int_{1-1/C(k,[k/2])}^1 \left[F^{-1}(t)\right]^\alpha dt & \text{for } n \leq [k/2]. \end{cases}$$

If $\alpha = 2i$, $i \in \mathrm{N}$, then

(iii)

$$\mathrm{E}\left[Y_n^{(k)}\right]^\alpha \leq C(k,n) \left\{ \int_0^{\lambda_3} \left[F^{-1}(t)\right]^\alpha dt + \int_{1-\lambda_4}^1 \left[F^{-1}(t)\right]^\alpha dt \right\},$$

(iv)

$$\mathrm{E}\left[Y_n^{(k)}\right]^\alpha \geq C(k,n) \int_{z_0-\lambda_3}^{z_0+\lambda_4} \left[F^{-1}(t)\right]^\alpha dt,$$

where $z_0 = F(0)$,

$$C(k,n) = \frac{k}{n!}\left[\frac{kn}{e(k-1)}\right]^n, \tag{12}$$

$$\lambda_3 = \begin{cases} \gamma(n+1, -k\log(1-z_0))/C(k,n) & \text{for } z_0 \in [0;1) \\ 1/C(k,n) & \text{for } z_0 = 1, \end{cases}$$

$$\lambda_4 = \frac{1}{C(k,n)} - \lambda_3$$

and

$$\gamma(m,x) = \frac{\int_0^x x^{m-1} e^{-x} dx}{\int_0^\infty x^{m-1} e^{-x} dx}, \quad m > 0,$$

denotes the normalized incomplete Gamma-function. The equalities in (i)–(iv) *are attained iff F has one atom only.*

Proof. We shall apply the following formula for α-th moment of the k-th record statistics which was proved by Grudzień and Szynal (1983),

$$\mathrm{E}\left[Y_n^{(k)}\right]^\alpha = \int_0^1 \left[F^{-1}(t)\right]^\alpha \frac{k}{n!}[-k\log(1-t)]^n(1-t)^{k-1}dt, \tag{13}$$

$$\alpha \in \mathrm{N},\ k = 2,3,4,\dots,\ n = 0,1,2,\dots.$$

Let us define

$$g_2(t,k,n) = \begin{cases} k(n!)^{-1}[-k\log(1-t)]^n(1-t)^{k-1} & \text{for } t \in [0;1) \\ 0 & \text{for } t = 1, \end{cases}$$

and

$$C(k,n) = \max\{g_2(t,k,n);\ t \in [0;1]\} \quad k = 2,3,\dots,\ n = 0,1,2,\dots.$$

It is easy to check that (here the assumption $k \geq 2$ is essential to have g_2 bounded)

$$C(k,n) = \frac{k}{n!}\left[\frac{nk}{e(k-1)}\right]^n.$$

Using Lemma 1 to functions

$$f(t) = \left[F^{-1}(t)\right]^\alpha \quad \text{and} \quad g(t) = g_2(t,k,n)/C(k,n),\ t \in [0;1],$$

we get

$$\mathrm{E}\left[Y_n^{(k)}\right]^\alpha \geq C(k,n)\int_0^\lambda \left[F^{-1}(t)\right]^\alpha dt$$

and

$$\mathrm{E}\left[Y_n^{(k)}\right]^\alpha \leq C(k,n)\int_{1-\lambda}^1 \left[F^{-1}(t)\right]^\alpha dt,$$

where

$$\lambda = \frac{1}{C(k,n)}\int_0^1 g(t)dt = \frac{1}{C(k,n)}.$$

Using Lemma 3 below gives (i) and (ii). In order to prove (iii) and (iv) we can rewrite (13) as

$$\mathrm{E}\left[Y_n^{(k)}\right]^\alpha = \int_0^{z_0} \left[F^{-1}(t)\right]^\alpha \frac{k}{n!}[-k\log(1-t)]^n(1-t)^{k-1}dt +$$

$$+ \int_{z_0}^1 \left[F^{-1}(t)\right]^\alpha \frac{k}{n!}[-k\log(1-t)]^n(1-t)^{k-1}dt,$$

where $z_0 = F(0)$. Applying now Corollary 1 and Lemma 1 in the same way as in the first part of the proof we have

$$\mathrm{E}\left[Y_n^{(k)}\right]^\alpha \le C(k,n)\left\{\int_0^{\lambda_3}\left[F^{-1}(t)\right]^\alpha dt + \int_{1-\lambda_4}^{1}\left[F^{-1}(t)\right]^\alpha dt\right\}$$

and

$$\mathrm{E}\left[Y_n^{(k)}\right]^\alpha \ge C(k,n)\int_{z_0-\lambda_3}^{z_0+\lambda_4}\left[F^{-1}(t)\right]^\alpha dt,$$

where

$$\begin{aligned}\lambda_3 &= \int_0^{z_0} g(t)dt = \frac{1}{C(k,n)}\int_0^{-k\log(1-z_0)}\frac{1}{n!}u^n e^{-u}du \\ &= \frac{1}{C(k,n)}\gamma(n+1,-k\log(1-z_0)) \\ \lambda_4 &= \int_{z_0}^1 g(t)dt = \frac{1}{C(k,n)}(1-\lambda_3).\end{aligned}$$

When $z_0 = 0$, λ_3 is understood to equal $1/C(k,n)$. This completes the proof. □

Lemma 2. *For every real $x \ge 4$, it holds*

$$\left(1+\frac{1}{(x/2)-1}\right)^{(x/2)-1} < e\left(1-\frac{1}{x}\right) < \left(1+\frac{1}{(x-1)/2}\right)^{(x-1)/2}$$

Proof. Let us consider an auxiliary function $f_1 : [4;\infty) \to \mathbb{R}$ defined by

$$f_1(x) = \left(1-\frac{1}{(x-1)/2}\right)^{(x-1)/2} - e\left(1-\frac{1}{x}\right).$$

The first derivative of f_1 has the following property

$$\begin{aligned} f_1'(x) &= \left(1+\frac{1}{(x-1)/2}\right)^{(x-1)/2}\left\{\frac{1}{2}\ln\left(1+\frac{2}{x-1}\right)-\frac{1}{x+1}\right\}-\frac{e}{x^2} \\ &< \left(1+\frac{1}{(x-1)/2}\right)^{(x-1)/2}\left\{\frac{1}{2}\ln\left(1+\frac{2}{x-1}\right)-\frac{1}{x+1}-\frac{1}{x^2}\right\} \\ &\equiv \left(1+\frac{1}{(x-1)/2}\right)^{(x-1)/2} g_1(x).\end{aligned}$$

Simple calculation shows that $g_1(x) < 0$ for $x \geq 4$ and in consequence $f_1(x) > 0$ for $x \geq 4$. Defining the function $f_2 : [4; \infty) \to \mathbb{R}$ as

$$f_2(x) = \left(1 + \frac{1}{(x/2) - 1}\right)^{(x/2)-1} - e\left(1 - \frac{1}{x}\right)$$

and deriving the first derivative we have

$$\begin{aligned} f_2'(x) &= \left(1 + \frac{2}{x-2}\right)^{(x-2)/2} \left\{\frac{1}{2}\ln\frac{x}{x-2} - \frac{1}{x}\right\} - \frac{e}{x^2} \\ &> \left(1 + \frac{2}{x-2}\right)^{(x-2)/2} \left\{\frac{1}{2}\ln\frac{x}{x-2} - \frac{1}{x} - \frac{1}{x^2}\left(1 + \frac{2}{x-2}\right)^{1/2}\right\} \\ &\equiv \left(1 + \frac{2}{x-2}\right)^{(x-2)/2} g_2(x) > 0 \end{aligned}$$

because $\lim_{x\to\infty} g_2(x) = 0$ and

$$\begin{aligned} g_2'(x) &= -\frac{2}{x^2(x-2)} + \frac{2}{x^3}\sqrt{1 + \frac{2}{x-2}} + \frac{1}{x^2(x-2)^2}\sqrt{\frac{x-2}{x}} \\ &< -\frac{2}{x^2(x-2)} + \frac{2}{x^3}\left(1 + \frac{1}{x-2}\right) + \frac{1}{x^2(x-2)^2} \\ &= \frac{-x+4}{x^3(x-2)^2} \leq 0 \end{aligned}$$

for $x \geq 4$. Since $\lim_{x\to\infty} f_2(x) = 0$, the assertion holds. □

Lemma 3. *Let $C(k,n)$ be defined by (12), where $k = 2, 3, \ldots$, $n = 0, 1, 2, \ldots$. For fixed k the following inequalities hold:*

$$C(k, n+1) < C(k,n) \quad \text{for } n < [k/2]$$

and

$$C(k, n+1) > C(k,n) \quad \text{for } n \geq [k/2].$$

Proof. Let us consider the quotient $Q(k,n) = C(k,n+1)/C(k,n)$ and rewrite it as $Q(k,n) = \left(1 + \frac{1}{n}\right)^n \left[e\left(1 - \frac{1}{k}\right)\right]^{-1}$ for $n > 0$ and $Q(k,0) = \left[e\left(1 - \frac{1}{k}\right)\right]^{-1}$ for $n = 0$. Denoting $n_0 = \min\{n \in \mathbb{N};\ \left(1 + \frac{1}{n}\right)^n > e\left(1 - \frac{1}{k}\right)\}$ we can easy observe that $Q(k,n) < 1$ for $n < n_0$ and $Q(k,n) > 1$ for $n \geq n_0$. The proof will be completed if we show that $n_0 = [k/2]$ for $k = 2, 3, \ldots$.

In case $k = 2$ and 3 we can do it by direct computations. So, it suffices to check that

$$\left(1+\frac{1}{[k/2]}\right)^{[k/2]} > e\left(1-\frac{1}{k}\right)$$

and

$$\left(1+\frac{1}{[k/2]-1}\right)^{[k/2]-1} < e\left(1-\frac{1}{k}\right)$$

for each $k = 4, 5, \ldots$ or equivalently that the following inequalities are satisfied for all $x \geq 4$

(i) $$\left(1+\frac{1}{(x/2)}\right)^{x/2} > e\left(1-\frac{1}{x}\right),$$

(ii) $$\left(1+\frac{1}{(x-1)/2}\right)^{(x-1)/2} > e\left(1-\frac{1}{x}\right),$$

(iii) $$\left(1+\frac{1}{(x/2)-1}\right)^{(x/2)-1} < e\left(1-\frac{1}{x}\right),$$

(iv) $$\left(1+\frac{1}{(x-1)/2-1}\right)^{(x-1)/2-1} < e\left(1-\frac{1}{x}\right).$$

Let us observe that (i) and (iv) follow from (ii) and (iii) which hold by Lemma 2, respectively. Since

$$[k/2] = \begin{cases} k/2, & \text{when } k \text{ is even} \\ (k-1)/2, & \text{when } k \text{ is odd} \end{cases}$$

the proof is completed. □

Remark 13. Analogous observations to Remarks 1 and 2 are valid.

Proposition 6. *Let $k = 2, 3, \ldots, n = 0, 1, \ldots$.*

(i) *If the quantile function F^{-1} is convex on $[0; 1/C(k,n)]$, where $C(k,n)$ is defined by (12), then*

$$\mathrm{E}\,Y_n^{(k)} \geq \begin{cases} F^{-1}(1/2C(k,n)) & \text{for } n \leq [k/2] \\ F^{-1}(1/2C(k,[k/2])) & \text{for } n > [k/2]. \end{cases}$$

(ii) *If F^{-1} is concave on $[1-1/C(k,n);1]$, then*

$$\mathrm{E}\,Y_n^{(k)} \le \begin{cases} F^{-1}(1-1/2C(k,n)) & \text{for } n > [k/2] \\ F^{-1}(1-1/2C(k,[k/2])) & \text{for } n \le [k/2]. \end{cases}$$

Proof. The same reasoning as in the proof of Proposition 2 applied for Proposition 5 (i) and (ii) gives the result. □

Remark 14. Note that (i)–(iv) in Proposition 5 always give upper and lower bounds and in comparison with (3) (for $\alpha = 1$) they work under weaker assumptions (we even do not need $\mathrm{E}\,X$ be finite). Moreover, for $\alpha = 2i$, $i \in \mathrm{N}$, our bounds (iii) and (iv) are meaningful while (3) is useless if the support of X contains both positive and negative reals. The advantages are illustrated in the example below.

Example 6. Let

$$F^{-1}(t) = \begin{cases} t^2 - 0.0016 & \text{for } t \in [0; 0.04] \\ (t-0.04)^{1/4} & \text{for } t \in (0.04; 1]. \end{cases}$$

Computations show (see Table 4) that bounds on e.g. $\mathrm{E}Y_n^{(7)}$ obtained from Propositions 5 and 6 are better then the ones derived from (3) for n=1,...,10. In Table 5 we also compare the exact value of $\mathrm{E}\left[Y_n^{(7)}\right]^2$ with the lower and upper bounds from Proposition 5.

TABLE 4

n	Lower bound Proposition 5(i)	Upper bound Proposition 5(ii)	Upper bound Proposition 6(ii)	Upper bound given in (3)
1	0.5176	0.9314	0.9335	1.0018
2	0.5509	0.9314	0.9335	0.9708
3	0.5585	0.9314	0.9335	0.9632
4	0.5585	0.9325	0.9345	0.9665
5	0.5585	0.9355	0.9373	0.9760
6	0.5585	0.9394	0.9409	0.9896
7	0.5585	0.9436	0.9447	1.0061
8	0.5585	0.9479	0.9489	1.0249
9	0.5585	0.9521	0.9529	1.0455
10	0.5585	0.9561	0.9567	1.0679

TABLE 5

n	Lower bound Proposition 5(iv)	Upper bound Proposition 5(iii)	The exact value of $\mathrm{E}\left[Y_n^{(7)}\right]^2$
1	0.3653	0.8623	0.4070
2	0.4132	0.8705	0.5183
3	0.4223	0.8686	0.5976
4	0.4188	0.8708	0.6586
5	0.4092	0.8762	0.7075
6	0.3960	0.8834	0.7474
7	0.3806	0.8912	0.7807
8	0.3640	0.8992	0.8085
9	0.3469	0.9071	0.8321
10	0.3296	0.9145	0.8521

Remark 15. For extensions to real α, see Remark 7.

References

[1] Arnold, B.C. and N. Balakrishnan, *Relations, Bounds and Approximations for Order Statistics*, Springer, Berlin, 1989.

[2] Caraux, G. and O. Gascuel, *Bounds on Expectations of Order Statistics via Extremal Dependences*, Statist. Probab. Lett. **15** (1992), 143–148.

[3] David, H.A., *Order Statistics*, 2nd ed. Wiley, New York, 1981.

[4] Dziubdziela, W. and B. Kopociński, *Limiting Properties of the k-th Record Value*, Appl. Math. **15** (1976), 187–190.

[5] Gajek, L. and U. Gather, *Moment Inequalities for Order Statistics with Applications to Characterization of Distributions*, Metrika **38** (1991), 357–367.

[6] Gajek, L. and E. Lenic, *Moment Inequalities for Order and Record Statistics under Restrictions on Their Distributions*, Ann. Univ. Mariae Curie–Skodowska Sect. A **33** (1993), 27–35.

[7] Gajek, L. and T. Rychlik, *Projection Method for Moment Bounds on Order Statistics from Restricted Families*. I. Dependent Case, J. Multivariate Anal. **57** (1996 a), 156–174.

[8] ———, *Projection Method for Moment Bounds on Order Statistics from Restricted Families*. II. Independent Case (1996 b) (submitted for publication).

[9] Grudzień, Z. and D. Szynal, *On Distributions and Moments of Order Statistics for Random Sample Size*, Ann. Univ. Mariae Curie–Skodowska Sect. A **34** (1980), 51–63.

[10] ———, *On Order Statistics for Random Sample Size Having Compound Binomial and Poisson Distributions*, Zastos. Mat. **17(2)** (1982), 259–270.

[11] ______, *On the Expected Values of the k-th Record Values and Associated Characterizations of Distributions*, Probab. Statist. Decision Theory, Vol. A, Proc. 4-th Pannonian Symp. Math. Statist., Badtatzmannsdorf (1983).
[12] Ludwig, O., *Über Erwartungsworte und Varianzen von Ranggrössen in kleinen Stichproben*, Metrika **38** (1960), 218–233.
[13] Mitrinović, D.S., *Analytic Inequalities*, Springer, Berlin, 1972.
[14] Moriguti, S., *Extremal Properties of Extreme Value Distributions*, Ann. Math. Statist. **22** (1951), 523–536.
[15] Rychlik, T., Stochastically Extremal Distributions of Order Statistics for Dependent Samples, Statist. Probab. Lett. **13** (1992), 337–341.

Institute of Mathematics
Technical University of Łódź
ul. Żwirki 36
90-924 Łódź, Poland

received February 24, 1997