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THOMAS H. MacGREGOR and M. P. STERNER

Boundedness of an Analytic Function and the Range of its Fractional Derivative

Dedicated to Professor Eligiusz Złotkiewicz on the occasion of his sixtieth birthday

ABSTRACT. This paper considers the question of what conditions on the α-fractional derivative of a function analytic in the open unit disk imply the function is bounded. Therein result gives such a condition expressed in terms of a geometric quantity about the range of the fractional derivative.

In this paper conditions are obtained about the fractional derivative of an analytic function implying that the function is bounded in the open unit disk. The range of the fractional derivative is assumed to be contained in a simply connected domain, and the conditions are given in terms of that domain. The arguments depend on properties of conformal mappings and the Ahlfors' distortion theorem provides the crucial step in the analysis. This approach also was used by F. Rønning and the first author in [3], where the boundedness of the analytic function was derived from properties of its logarithmic derivative.

Let $\Delta = \{z \in \mathbf{C} : |z| < 1\}$. Suppose that the function f is analytic in Δ and let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1}$$

for $|z| < 1$. Suppose that $\alpha > 0$ and let k denote the greatest integer in α. The operator D^α is defined by

$$D^\alpha f(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha+1)}{n!} a_{n+k} z^n \tag{2}$$

for $|z| < 1$, where Γ denotes the gamma function. If α is a positive integer then $k = \alpha$ and hence

$$D^\alpha f(z) = \sum_{n=0}^{\infty} \frac{(n+k)!}{n!} a_{n+k} z^n = \sum_{m=k}^{\infty} m(m-1)\dots(m-(k-1)) a_m z^{m-k}$$
$$= f^{(k)}(z), \quad \text{the } k^{\text{th}} \text{ derivative of } f.$$

$D^\alpha f$ may be viewed as the α-fractional derivative of f. There are a number of definitions of fractional derivative and an historical survey of this concept is given in [5]. One definition in terms of power series is due to Hadamard and is given by

$$f^{(\alpha)}(z) = z^{-\alpha} \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n-\alpha+1)} a_n z^n . \tag{3}$$

We want $D^\alpha f$ to be analytic in Δ. Besides doing that (2) is a convenient definition for obtaining the results in this paper. Other definitions in terms of the Taylor coefficients which yield a sequence having essentially the same asymptotic expansion as that given by (2) would yield similar results.

We begin by inverting the relation $f \mapsto D^\alpha f$ given by (2). To do this we need the following known formula from the theory of hypergeometric series.

$$\int_0^1 (1-t)^{\alpha-1} t^n \, dt = \frac{\Gamma(\alpha) n!}{\Gamma(n+\alpha+1)} \tag{4}$$

for $\alpha > 0$ and $n = 1, 2, 3, \dots$. From (2) we obtain

$$\int_0^1 (1-t)^{\alpha-1} D^\alpha f(tz) dt = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha+1)}{n!} a_{n+k} \int_0^1 (1-t)^{\alpha-1} t^n dt z^n$$
$$= \Gamma(\alpha) \sum_{n=0}^{\infty} a_{n+k} z^n .$$

Therefore

$$f(z) = \sum_{n=0}^{k-1} a_n z^n + \frac{z^k}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} D^\alpha f(tz) dt . \tag{5}$$

For $0 < r < 1$, define $D(r) = D(r; f, \alpha)$ by

$$D(r) = \max_{|z| \le r} |D^\alpha f(z)| . \tag{6}$$

If $|z| < 1$ then

$$\begin{aligned} \left| \int_0^1 (1-t)^{\alpha-1} D^\alpha f(tz) dt \right| &\le \int_0^1 (1-t)^{\alpha-1} |D^\alpha f(tz)| dt \\ &\le \int_0^1 (1-t)^{\alpha-1} D(t|z|) dt \\ &\le \int_0^1 (1-t)^{\alpha-1} D(t) dt . \end{aligned}$$

This inequality and (5) yield the following result.

Lemma 1. *If $\alpha > 0$ and*

$$\int_0^1 (1-t)^{\alpha-1} D(t) dt < \infty , \tag{7}$$

then f is bounded.

Henceforth it is assumed that the range of $D^\alpha f$ is contained in a simply connected domain Ω and $\Omega \neq \mathbf{C}$. This is the same as the assumption that the range of $D^\alpha f$ avoids some unbounded continuum.

Let G denote the unique conformal mapping of Δ onto Ω such that $G(0) = \Gamma(\alpha+1) a_k$ and $G'(0) > 0$. Since $G(0) = D^\alpha f(0)$, these conditions imply that $D^\alpha f$ is subordinate to G in Δ. For $0 < r < 1$, let

$$M(r) = \max_{|z| \le r} |G(z)| . \tag{8}$$

The subordination implies that

$$D(r) \le M(r) \tag{9}$$

for $0 < r < 1$ [1, p.191]. Lemma 1 and (9) yield the following result.

Lemma 2. *If $\alpha > 0$ and*

$$\int_0^1 (1-t)^{\alpha-1} M(t) dt < \infty , \tag{10}$$

then f is bounded.

Theorem 1. *If $\alpha > 2$ and $D^\alpha f(\Delta) \subset \Omega$ where Ω is a simply connected domain and $\Omega \neq \mathbf{C}$, then f is bounded.*

Proof. The assumptions imply that $D^\alpha f$ is subordinate to the function G described above. Because G is analytic and univalent in Δ

$$|G(z)| \leq |G(0)| + \frac{|z|}{(1-|z|)^2}|G'(0)| \tag{11}$$

for $|z| < 1$ [1, p.33]. Hence $|G(z)| \leq \frac{A}{(1-|z|)^2}$ for some constant $A > 0$. This implies

$$\int_0^1 (1-t)^{\alpha-1} M(t)dt \leq A \int_0^1 (1-t)^{\alpha-3}dt.$$

The last integral is finite because $\alpha > 2$. Thus (10) holds and consequently f is bounded.

□

In what follows the question of the boundedness of f is considered in the case $0 < \alpha \leq 2$. Conditions on Ω will be obtained which imply (10).

Assume that Ω is an unbounded Jordan domain and that the origin is on $\partial\Omega$. By the Carathéodory extension theorem, G extends continuously to $\bar{\Delta}$ and there are unique points z_0 and z_∞ on $\partial\Delta$ such that $G(z_0) = 0$ and $G(z_\infty) = \infty$. We choose the normalization $z_\infty = 1$.

The points 0 and ∞ break up $\partial\Omega$ into two Jordan curves which are denoted Γ^- and Γ^+. Let Γ be a Jordan curve with endpoints 0 and ∞ all of who other points belong to Ω. For each $R > 0$ there is an arc denoted γ_R which is contained in $\Omega \cap \{w : |w| = R\}$, meets Γ and has one endpoint on Γ^- and the other oint on Γ^+. It is assumed that except possibly for isolated values of R there is a tangent to $\partial\Omega$ at each endpoint of γ_R and these tangents are not tangent to the circle $\{w : |w| = R\}$. Let $\ell(R)$ denote the length of γ_R and let $\varphi(R) = \ell(R)/R$. Then $\varphi(R)$ is the angular variation of γ_R.

Theorem 2. *Suppose that $0 < \alpha \leq 2$, $D^\alpha f(\Delta) \subset \Omega$ and Ω is a simply connected domain having the properties described above. If*

$$\int_1^\infty \frac{1}{\varphi(y)\exp\{\alpha\pi \int_1^y \frac{1}{x\varphi(x)}dx\}}dy < \infty \tag{12}$$

then f is bounded.

Proof. There are complex numbers c and d such that $|c| = 1$ and $|d| < 1$ and the Möbius transformation

$$\tau(z) = c\frac{z+d}{1+\bar{d}z} \tag{13}$$

satisfies $\tau(1) = 1$ and $\tau(z_0) = -1$. From

$$1 - |\tau(z)|^2 = \frac{(1-|d|^2)(1-|z|^2)}{|1+\bar{d}z|^2}$$

it follows that

$$\frac{1-|\tau(z)|}{1-|z|} \geq \frac{1-|d|}{1+|\tau(z)|} \geq \frac{1-|d|}{2}.$$

Hence

$$\frac{1}{1-|\tau(z)|} \leq \frac{A}{1-|z|} \tag{14}$$

for $|z| < 1$, where A is a constant.

For $|z| < 1$, let $w = G(z)$ and $\zeta = \log w$. The composition $z \mapsto \zeta$ gives a conformal mapping of Δ onto a domain Φ. The boundary of Φ consists of two curves denoted Λ^- and Λ^+ which correspond to Γ^- and Γ^+, respectively, and every vertical line intersects Φ. For each R $(0 < R < \infty)$ the image of γ_R under the map $w \mapsto \zeta$ is a line segment denoted θ_s with $s = \log R$ and $-\infty < s < \infty$. If $\theta(s)$ denotes the length of θ_s then $\theta(s) = \varphi(R)$. Except possibly for isolated values of s, at each endpoint of θ_s there are tangents to $\partial\Lambda^-$ and $\partial\Lambda^+$ and these tangents are not vertical. It follows that θ is continuous except possibly for isolated values of s [4, p.93].

For $|z| < 1$, let $\sigma(z) = \log[(1+\tau(z))/(1-\tau(z))]$. Then $z \mapsto \sigma$ gives a conformal mapping of Δ onto the strip $S = \{\sigma : |\operatorname{Im} \sigma| < \pi/2\}$. The composition of the maps $\zeta \mapsto w$, $w \mapsto z$, $z \mapsto \tau$ and $\tau \mapsto \sigma$ yields a conformal mapping of Φ onto S, which is denoted g. Let $a_\circ$ be a fixed real number and choose $\zeta_\circ \in \theta_{a_\circ}$. For $|z| < 1$ and z sufficiently close to 1 the corresponding ζ belongs to θ_a with $a > a_\circ$. Let $R_\circ = e^{a_\circ}$ and $R = e^a$. Then

$$\int_{a_\circ}^{a} \frac{1}{\theta(s)} ds = \int_{R_\circ}^{R} \frac{1}{x\varphi(x)} dx \geq \int_{R_\circ}^{R} \frac{1}{2\pi x} dx = \frac{1}{2\pi} \log \frac{R}{R_\circ}.$$

Since the last quantity exceeds 2 for $|z| < 1$ and z sufficiently close to 1, Ahlfors' distortion theorem is applicable [4, p.97]. Therefore

$$\operatorname{Re}[g(\zeta) - g(\zeta_\circ)] > \pi \int_{a_\circ}^{a} \frac{1}{\theta(s)} ds - 4\pi \; . \tag{15}$$

Since $\operatorname{Re} g(\zeta) = \log|(1+\tau(z))/(1-\tau(z))| \leq \log[2/|1-\tau(z)|]$, (14) implies that $\operatorname{Re} g(\zeta) \leq \log[1/(1-|z|)] + B$ for some constant B. Hence (15)

$$\int_{a_\circ}^{a} \frac{1}{\theta(s)} ds \leq \frac{1}{\pi} \log \frac{1}{1-|z|} + C \tag{16}$$

for some constant C. Inequality (16) is equivalent to

$$\int_{R_\circ}^{R} \frac{1}{x\varphi(x)} dx \le \frac{1}{\pi} \log \frac{1}{1-r} + C \, , \tag{17}$$

and thus (17) holds for $|z| < 1$ and z sufficiently near 1, where $r = |z|$ and $R = |G(z)|$.

For each r $(0 < r < 1)$ there is a unique number $y = y(r)$ such that $y \ge R_\circ$ and

$$\int_{R_\circ}^{y} \frac{1}{x\varphi(x)} dx = \frac{1}{\pi} \log \frac{1}{1-r} + C \, . \tag{18}$$

If (18) is solved for $1-r$ and dr/dy is computed, we find that

$$(1-r)^{\alpha-1} y \frac{dr}{dy} = \frac{\pi e^{\alpha\pi C}}{\varphi(y)} \exp\left[-\alpha\pi \int_{R_\circ}^{y} \frac{1}{x\varphi(x)} dx\right]$$

except possibly for a discrete set of values of r. This equality and the assumption (12) imply that

$$\int_0^1 (1-r)^{\alpha-1} y(r) dr < \infty \, . \tag{19}$$

We have $R = |G(z)| \le y(r)$ for $z \in \Delta$ sufficiently near 1. Also G is bounded in $\Delta \backslash \{z : |z-1| < \varepsilon\}$ for each ε $(0 < \varepsilon < 1)$. Thus (19) implies $\int_0^1 (1-r)^{\alpha-1} M(r) dr < \infty$. Hence Lemma 2 yields the conclusion that f is bounded.

□

We note that in the cases $\alpha = 1$ and $\alpha = 2$ Theorem 2 gives a condition about the range of f' and f'' which implies f is bounded.

When the angular variation φ is nondecreasing, Theorem 2 takes on a more simplified form. Specifically, suppose that φ is nondecreasing on $[b, \infty)$ for some $b > 0$. Then for $y > b$ we have

$$\int_b^y \frac{1}{x\varphi(x)} dx \ge \frac{1}{\varphi(y)} \log \frac{y}{b} .$$

Hence

$$\exp\left\{-\alpha\pi \int_b^y \frac{1}{x\varphi(x)} dx\right\} \le [b]^{\alpha\pi/\varphi(y)} [y]^{-\alpha\pi/\varphi(y)} .$$

Since φ is nondecreasing and bounded above (by 2π), $\lim_{y\to\infty}\varphi(y)$ exists. Hence from the inequality above we see that

$$\int_1^\infty [y]^{-\alpha\pi/\varphi(y)}\,dy < \infty \tag{20}$$

implies (12). This shows that when φ is nondecreasing (20) implies that f is bounded.

Theorem 3 below is a converse of Theorem 2 valid for domains Ω which are symmetric and sufficiently smooth. Specifically, assume that Ω is symmetric with respect to the real axis and that for each R $(0 < R < \infty)$ the set $\Omega \cap \{w : |w| = R\}$ is a single arc, which is still denoted γ_R. Also suppose that the angular variation φ is differentiable and there is a constant N such that

$$|\varphi'(R)| \le \frac{N}{R} \tag{21}$$

for $0 < R < \infty$.

Theorem 3. *Suppose that $0 < \alpha \le 2$ and Ω is a domain having the properties stated above. If*

$$\int_1^\infty \frac{1}{\varphi(y)\exp\{\alpha\pi\int_1^y \frac{1}{x\varphi(x)}dx\}}\,dy = \infty \tag{22}$$

then there is an analytic function f such that $D^\alpha f(\Delta) \subset \Omega$ and f is unbounded.

Proof. Let G denote the conformal mapping of Δ onto Ω such that $G(0) = 1$ and $G'(0) > 0$. Let $G(z) = \sum_{n=0}^\infty b_n z^n$ for $|z| < 1$ and define f by $\sum_{n=k}^\infty a_n z^n$, where k is the greatest integer in α and $a_n = [(n-k)!/\Gamma(n-k+\alpha+1)]b_{n-k}$ for $n = k, k+1, \ldots$. Then $D^\alpha f = G$ and from (5) we obtain

$$f(z) = \frac{z^k}{\Gamma(\alpha)}\int_0^1 (1-t)^{\alpha-1}G(tz)dt\,. \tag{23}$$

We consider the various mappings defined in the proof of Theorem 2 and use the same notation. The image of Ω under the map $w \mapsto \zeta$ is a domain Φ which is symmetric with respect to the real axis, and every vertical line $\{\zeta : \operatorname{Re}\zeta = s\}$ meets Φ in exactly one line segment θ_s. The length of θ_s

is $\theta(s) = \varphi(R)$. Since $s = \log R$, we have $\frac{d\theta}{ds} = R\frac{d\varphi}{dR}$ and hence (21) is the same as

$$|\theta'(s)| \leq N \tag{24}$$

for $-\infty < s < \infty$. The curves Γ^- and Γ^+ which form $\partial\Phi$ now are given by $v = -\theta(s)/2$ and $v = \theta(s)/2$ where $v = \operatorname{Im}\zeta$.

Since Ω is symmetric with respect to the real axis, $G(0)$ is real and $G'(0)$ is real a simple argument about $\overline{G(\bar{z})}$ shows that $G(z)$ is real when z is real. Let $h : \Delta \to \Phi$ denote the map $z \mapsto \zeta$. Then $h(z)$ is real when z is real, $h(\Delta)$ is convex in the direction of the imaginary axis, $\lim_{\substack{z\to -1\\ z\in\Delta}} \operatorname{Re} h(z) = -\infty$ and $\lim_{\substack{z\to 1\\ z\in\Delta}} \operatorname{Re} h(z) = \infty$. Therefore h belongs to the class of functions studied by W. Hengartner and G. Schober in [2]. A consequence of this membership, shown in [2], is that for each r $(0 < r < 1)$ the domain $h(\{z : |z| < r\})$ is convex in the direction of the imaginary axis. This property of $h(\{z : |z| < r\})$ and the fact this domain is symmetric with respect to the real axis implies that

$$\max_{|z|\leq r} \operatorname{Re} h(z) = \operatorname{Re} h(r) \tag{25}$$

for every r $(0 < r < 1)$. Since $w = G(z)$ and $\zeta = h(z) = \log w$, we have $|G(z)| = \exp[\operatorname{Re} h(z)]$. Hence (25) shows that

$$\max_{|z|\leq r} |G(z)| = G(r) \tag{26}$$

for every r $(0 < r < 1)$.

As in the proof of Theorem 2, let g be the conformal mapping of Ω onto the strip S. A theorem of S. Warschawski [6] yields the following inequality

$$\operatorname{Re}[g(\zeta) - g(\zeta_\circ)] \tag{27}$$

$$\leq \pi \int_{a_\circ}^{a} \frac{1}{\theta(x)} dx + \frac{\pi}{12} \int_{a_\circ}^{a} \frac{[\theta'(x)]^2}{\theta(x)} dx + 12\pi(1 + N^2)$$

where ζ, $\zeta_\circ$, a, and $a_\circ$ have the meaning as before. This uses (24). The condition (24) also implies

$$\int_{a_\circ}^{a} \frac{[\theta'(x)]^2}{\theta(x)} dx \leq N^2 \int_{a_\circ}^{a} \frac{1}{\theta(x)} dx \leq N^2 \log \frac{2\pi}{\theta(a_\circ)}.$$

Hence (27) shows that

$$\operatorname{Re} g(\zeta) \leq \pi \int_{a_0}^{a} \frac{1}{\theta(x)} dx + A \tag{28}$$

for some constant A.

Let τ be defined by (13). Since $\tau(1) = 1$ and τ is differentiable at 1 we have $|1 + \tau(r)| \geq 1/2$ and $|1 - \tau(r)| \leq B(1 - r)$ for r sufficiently close to 1 $(0 < r < 1)$, where B is a positive constant. Hence for ζ corresponding to such r,

$$\operatorname{Re} g(\zeta) = \log \left| \frac{1 + \tau(r)}{1 - \tau(r)} \right| \geq \log \frac{1}{2B} + \log \frac{1}{1 - r} .$$

This inequality and (28) imply that

$$\int_{a_0}^{a} \frac{1}{\theta(x)} dx \geq \frac{1}{\pi} \log \frac{1}{1 - r} + C \tag{29}$$

for r sufficiently close to 1, where C is some constant. This inequality is equivalent to

$$\int_{R_0}^{R} \frac{1}{x\varphi(x)} dx \geq \frac{1}{\pi} \log \frac{1}{1 - r} + C . \tag{30}$$

For each r define $y = y(r)$ by

$$\int_{R_0}^{y} \frac{1}{x\varphi(x)} dx = \frac{1}{\pi} \log \frac{1}{1 - r} + C . \tag{31}$$

Then $y(r) \leq R = |G(r)| = G(r)$. Using the argument given directly after (18) we see that (31) and (22) yield

$$\int_0^1 (1 - r)^{\alpha - 1} y(r) dr = \infty . \tag{32}$$

Since $G(r) \geq y(r)$, (26) and (32) imply that

$$\int_0^1 (1 - r)^{\alpha - 1} M(r) dr = \infty . \tag{33}$$

An easy consequence of (33) is that

$$\lim_{r \to 1^-} \int_0^1 (1 - t)^{\alpha - 1} M(tr) dt = \infty . \tag{34}$$

From (23), (34) and $M(r) = G(r)$ we obtain $\lim_{r \to 1^-} f(r) = \infty$. Therefore f is not bounded.

References

[1] Duren, P. L., *Univalent Functions*, Springer-Verlag, New York 1983.

[2] Hengartner, W. and G. Schober, *On schlicht mappings to domains convex in one direction*, Comment. Math. Helv. **45** (1970), 303-314.

[3] MacGregor, T. H. and F. Rønning, *Conditions on the logarithmic derivative of a function implying boundedness*, Trans. Amer. Math. Soc. **347** (1995), 2245-2254.

[4] Nevanlinna, R., *Analytic Functions*, Springer-Verlag, New York 1970.

[5] Ross, B., *A brief history and exposition of the fundamental theory of fractional calculus*, Lecture Notes in Mathematics **457**, Springer-Verlag 1975, 1-36.

[6] Warschawski, S. E., *On conformal mapping of infinite strips*, Trans. Amer. Math. Soc. **51** (1942), 280-335.

Department of Mathematics
and Statistics
SUNY at Albany
Albany, New York 12222, USA

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Department of Mathematics
University of Montevallo
Montevallo, Alabama 35115, USA