ANNALES
UNIVERSITATIS MARIAE CURIE-SKLODOWSKA
LUBLIN-POLONIA

VOL. LII.1, 5 SECTIO A 1998

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On Some Formal and Regular Identities

To E. J. Złotkiewicz on his 60-th birthday

ABSTRACT. We discuss some identities related to the Bieberbach-Milin problem and de Branges' functions.

1. Introduction. In his solution to the Bieberbach-Milin problem [1], L. de Branges introduced certain functions (de Branges' functions) associated with Loewner's differential equation [8] and Milin's functionals [9, Chapter 3]. He gave a representation of their derivatives implying that these functions are nondecreasing in Loewner's parameter.

C. FitzGerald and Ch. Pommerenke [3] used the St. Petersburg modification of de Branges' proof (see I.M. Milin's comments [10]) to show that it was sufficient to consider de Branges' functions of a less complicated form. Later L. Weinstein [12] found an integral transformation leading to an alternative representation. In turn, D. Zeilberger [14] generalized Weinstein's transformation. He collaborated with a computer specialist (Shalosh B. Ekhad) to produce an identity in terms of formal functions. Here we give a short proof of this general result and other identities related to de Branges' functions. A detailed solution to the Bieberbach-Milin problem, involving a relatively simple coefficient form of de Branges' functions, is given in [5].

1991 *Mathematics Subject Classification.* 30C50, 30C75, 33C45.
Key words and phrases. The Bieberbach-Milin problem, Loewner's chain, Milin's functionals, de Branges' functions.

2. The Pick function and auxiliary polynomials. Let $E = \{z : |z| < 1\}$ and let $K(z)$ denote the Koebe function $z/(1-z)^2$, $z \in E$. For each $t \geq 0$, the Pick function $w = w(z,t)$ is defined implicitly by the equation

$$e^t K(w(z,t)) = K(z), \quad z \in E. \tag{1}$$

It follows that

$$w_t/w = (w-1)/(w+1). \tag{2}$$

All known representations of the derivatives of de Branges' functions involve certain polynomials that are nonnegative on the interval [0, 1]. L. de Branges defined these polynomials by means of a system of linear differential equations and recognized their nonpositivity as a particular case of the Askey-Gasper inequalities for hypergeometric series (1976), [1]. Weinstein's approach is based on a Pick function representation of the same polynomials (see Remark below) and the addition theorem for Legendre polynomials (1785) [12]. Zeilberger used a formal expansion and a computer for this purpose [14]. An elementary and self-contained proof of the polynomial inequalities in question was found by the author and M.E.H. Ismail [6]. A simplified version of this proof is given in [5].

The needed polynomials $P_{m,n}(x)$ can be defined by the formal expansion (cf. [12], [14], [5])

$$\begin{aligned}&\left[1-\left(2(1-x)+x\left(\zeta+\zeta^{-1}\right)\right)z+z^2\right]^{-1}\\ &\qquad = \sum_{n=0}^{\infty}\left[\sum_{m=0}^{n} P_{m,n}(x)(\zeta^m+\zeta^{-m})\right]z^n.\end{aligned} \tag{3}$$

The nonnegativity property above is as follows :

$$P_{m,n}(x) \geq 0, \quad x \in [0,1]. \tag{4}$$

An identity discovered by Weinstein implies that (cf. [12], [5])

$$P_{m,n}\left(e^{-t}\right) = \left\{K(z)\frac{1-w(z,t)}{1+w(z,t)}w^m(z,t)\right\}_{n+1}, \ n \geq 1, \tag{5}$$

where $w(z,t)$ is defined by (1), $t \geq 0$, and $n \geq m \geq 1$. Equation (5) links de Branges' functions and (4).

Here and below the notation $\{f\}_n$ stands for the coefficient of z^n in the Taylor (or Laurent) expansion about $z = 0$ of a function $f(z)$.

Remark. Two different proofs that de Branges [1] and Weinstein [12] used the same polynomials were published by P.Todorov [11] and H.Wilf [13]. Another way (in two versions) of showing this fact can be found in reviewer's remarks [4]. Finally, one can consider the left-hand side of (3) as the sum of a geometric series to verify that the polynomials from [1] and [12] are identical.

3. Four lemmas on identities.

Lemma 1. *Let*

$$f(z,t) = z\exp\left(t + \sum_{m=1}^{\infty} c_m z^m\right),$$

$$\Phi_m(z,t) = 2\left(1 + \sum_{\nu=1}^{m} \nu c_\nu z^\nu\right) - m c_m z^m, \tag{6}$$

and

$$V_m(t) = \{\Phi_m(z,t)\overline{\Phi_m(\overline{z^{-1}},t)} f_t(z,t)/[z f_z(z,t)]\}_0, \tag{7}$$

where $c_m = c_m(t), m \geq 1$, are formal functions of t. Then the following formal identities hold:

$$V_m(t) = 4\left(1 + \sum_{\nu=1}^{m} \nu c'_\nu \overline{c_\nu}\right) - 2m c'_m \overline{c_m} - |m c_m|^2, \quad m \geq 1. \tag{8}$$

Proof. Using the formal equations

$$\left\{\frac{f_t(z,t)}{z f_z(z,t)}\right\}_k = \left\{\frac{1 + \sum_{\nu=1}^{m} c'_\nu z^\nu}{1 + \sum_{\nu=1}^{m} \nu c_\nu z^\nu}\right\}_k \quad (k = 0, 1, ..., m)$$

for a given $m \geq 1$, we have

$$V_m(t) = \left\{\left[2\left(1 + \sum_{\nu=1}^{m-1} c'_\nu z^\nu\right) + (2c'_m - m c_m) z^m\right] \overline{\Phi_m(\overline{z^{-1}},t)}\right\}_0.$$

The last equation implies (8). □

Lemma 2. *Let*

$$(9)\qquad h(z,t)=\sum_{m=1}^{\infty}\left(m\,|c_m|^2-4/m\right)w^m,\quad t\geq 0,\quad z\in E,$$

where $c_m = c_m(t)$, $m \geq 1$, *are formal functions of* t *and* $w = w(z,t)$ *is defined by (1). Then the following formal equation holds:*

$$(10)\quad \frac{\partial h}{\partial t}=\frac{1-w}{1+w}\sum_{m=1}^{\infty}\operatorname{Re}\left[4\left(1+\sum_{\nu=1}^{m}\nu c'_\nu\overline{c_\nu}\right)-2mc'_m\overline{c_m}-|mc_m|^2\right]w^m.$$

Proof. For each $m \geq 1$,

$$\frac{\partial}{\partial t}\left[\left(m\,|c_m|^2-4/m\right)w^m\right]=[2\operatorname{Re}(mc'_m\overline{c_m})+(|mc_m|^2-4)w_t/w]w^m.$$

Now use (2) and the expansion $(1+w)/(1-w) = 1+2w+2w^2+\cdots$ to get (10).

□

Lemmas 1 and 2 imply Lemma 3 which gives Zeilberger's identity [14].

Lemma 3. *Under the conditions of* Lemmas 1 and 2, *the following formal identity holds:*

$$(11)\qquad \frac{\partial h}{\partial t}=\frac{1-w}{1+w}\sum_{m=1}^{\infty}\operatorname{Re}(V_m)w^m,\quad t\geq 0,\quad z\in E.$$

Lemma 4 [5]. *Let* $a_m,\ m=1,2,\ldots,$ *be given and define*

$$b_m=2\left(1+\sum_{\nu=1}^{m}a_\nu\right)-a_m,\quad m=1,2,\ldots.$$

Then

$$4\operatorname{Re}\left(1+\sum_{\nu=1}^{m}\overline{a}_\nu b_\nu\right)=|a_m+b_m|^2,\quad m\geq 1.$$

4. The representations of the derivatives of de Branges' functions

Let $\{f(z,t) : t \geq 0\}$ be a Loewner chain generated by a continuously increasing family of simply connected domains (for each $t \geq 0$, $f(z,t)$ is 1-1 and analytic in z, $z \in E$). We assume that $f0,t) = 0$ and $f_z(0,t) = e^t, t \geq 0$, and that $f(z,t)$ satisfies the partial differential equation

$$f_t = z f_z \mathcal{P}(z,t), \quad z \in E, t \geq 0, \tag{12}$$

where for each $t \geq 0$, $\mathcal{P}$ is analytic in z, $z \in E$, with $\operatorname{Re}\mathcal{P} > 0$ and $\mathcal{P}(0,t) = 1$; and the coefficients of $\mathcal{P}$ are measurable functions of t ([8], [7]; see also Loewner's equation and the Loewner-Kufarev equation in [2, Chapter 3]).

For each $t \geq 0$, we use the Herglotz representation formula

$$\mathcal{P}(z,t) = \int_0^{2\pi} \frac{e^{i\Theta} + z}{e^{i\Theta} - z} d\mu_t(\Theta), \tag{13}$$

where μ_t is a nonnegative unit measure on the Borel subsets of $[0, 2\pi]$. In the simplest case, when μ_t is a point mass for each t,

$$\mathcal{P}(z,t) = \frac{\gamma + z}{\gamma - z}, \tag{14}$$

where $\gamma = \gamma(t)$ is a continuous complex-valued function on $[0, \infty)$ with $|\gamma| = 1$.

It is convenient for us to use de Branges' functions $\varphi_n(t)$, $n \geq 1$, in the following coefficient form (cf. [5])

$$\varphi_n(t) = \{K(z)h(z,t)\}_{n+1}, \quad t \geq 0, \tag{15}$$

where h is defined by (9) and c_m (in (9)) are the logarithmic coefficients of the function $f(z,t)$ described above

$$c_m(t) = \{\log[f(z,t)/z]\}_m, \quad m \geq 1. \tag{16}$$

Below we give three representations of the derivatives $\varphi_n'(t)$ $(n \geq 1, t \geq 0)$ in terms of the polynomials $P_{m,n}$, the coefficients c_m, and the functions Φ_m defined by (3), (16), and (6) respectively.

a) The simplest representation of φ_n' corresponds to $\mathcal{P}$ defined by (14). In this case, (12) and (16) imply

$$c_m' \gamma^m = 2\left(1 + \sum_{\nu=1}^{m} \nu c_\nu \gamma^\nu\right) - m c_m \gamma^m, \quad m \geq 1. \tag{17}$$

Using (15), Lemma 2, (5), (17), and Lemma 4 (with $a_m = mc_m\gamma^m$, $m = 1, \ldots, n$), we get [5] (compare with [3])

$$\varphi_n'(t) = \sum_{m=1}^{n} P_{m,n}\left(e^{-t}\right)\left|c_m'(t)\right|^2. \tag{18}$$

b) Let $\mathcal{P}$ be defined by (13). Then (12) and (16) give

(19)

$$c_m'(t) = \int_0^{2\pi} e^{-i\Theta m}\left[2\left(1+\sum_{\nu=1}^{m}\nu c_\nu e^{i\Theta\nu}\right) - mc_m e^{i\Theta m}\right] d\mu_t(\Theta),\ m \geq 1.$$

Now we use (15), Lemma 2, (5), (19), Lemma 4 (for each $\Theta \in [0, 2\pi]$), and (6). It follows (compare with [1]) that

$$\varphi_n'(t) = \sum_{m=1}^{n} P_{m,n}\left(e^{-t}\right)\int_0^{2\pi}\left|\Phi_m(e^{i\Theta}, t)\right|^2 d\mu_t(\Theta). \tag{20}$$

Additionally, equations (19), (6), and (20) imply

$$\varphi_n'(t) \geq \sum_{m=1}^{n} P_{m,n}\left(e^{-t}\right)\left|c_m'(t)\right|^2.$$

c) Equation (15), Lemma 3, (6), (7), and (5) imply Weinstein's integral representation (compare [12] and equation (21) below). We have

$$\varphi_n'(t) = \sum_{m=1}^{n} P_{m,n}(e^{-t})\ \mathrm{Re}(V_m),$$

and hence

$$\varphi_n'(t) = \frac{1}{2\pi}\sum_{m=1}^{n} P_{m,n}\left(e^{-t}\right)\int_0^{2\pi}\mathrm{Re}\left[\frac{f_t(z,t)}{zf_z(z,t)}\right]|\Phi_m(z,t)|^2 d\alpha,\ z = e^{i\alpha}, \tag{21}$$

provided that the integral exists for every $m = 1, \ldots, n$. Using the limiting values of the integrals along the circle $|z| = r < 1$ as $r \to 1^-$, one can combine (21) with (12) (cf. Weinstein's approach in [12]). If $\mathcal{P}$ is defined by (14), this approach leads to an integral representation of φ_n' which is equivalent to (18). In the general case ($\mathcal{P}$ is defined by (13)), the limiting representation is equivalent to (20).

We remind the reader that each of the representations (18), (20), and (21) (together with inequality (4)) implies that Milin's functionals ($\varphi_n(0)$, $n \geq 1$) are nonpositive (cf. [9, Chapter 3], [1], ..., and [5]).

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received December 29, 1997