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On Regularity Theorems for Linearly Invariant Families of Analytic Functions in the Unit Polydisk, II

Dedicated to Professor Eligiusz Złotkiewicz on the occasion of his 60th birthday

ABSTRACT. This paper is a continuation of our research ([GS2], [GS3]) concerning the regularity theorems for linearly invariant families of functions defined on the unit polydisk. In particular we show, that the higher dimensional cases differ significantly from one dimensional. Moreover, we pay special attention to the relationship between various linearly invariant families.

In [P] Ch. Pommerenke introduced and studied the notion of a linearly invariant family of functions holomorphic in the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Linearly invariant families play an important role in the theory of conformal mappings. Furthermore, an interest in these families grows because of their relationship with the Bloch class ([GS1]). In [C] and [S] regularity theorems were obtained for such families.

In [GS1] we defined linearly invariant families of functions analytic in the unit polydisk $\Delta^m \subset \mathbb{C}^m$, $m \geq 1$.

Key words and phrases. Linearly invariant family, regularity theorem.

In this paper we continue the study of the regularity theorem for linearly invariant families of functions defined on the unit polydisk Δ^m. As we will see (Theorem 2), the effect of higher dimensions makes problems different from those for $m = 1$, and consequently for $m > 1$ we obtain more complete results. Moreover, we show connections between subfamilies $\mathcal{U}'_\alpha(\delta)$ of $\mathcal{U}^l_\alpha$ and connections between families $\mathcal{U}^l_\alpha$ for various m.

Let $\mathbf{T} = \{z \in \mathbb{C} : \ |z| = 1\}$ and $\mathbf{T}^m$ be the unit torus. We will consider the class $\mathcal{H}(\Delta^m)$ of all functions $f : \Delta^m \longrightarrow \mathbb{C}$ analytic in Δ^m. For $z = (z_1, \dots, z_m) \in \mathbb{C}^m$ we define the norm $\|z\| = \max_{1 \le j \le m} |z_j|$. Let $\mathbb{O} = (0, \cdots, 0) \in \mathbb{C}^m$. Recall that to every $a \in \Delta$ there corresponds an automorphism ϕ_a of Δ: $\phi_a(z) = (a + z)/(1 + \bar{a}z)$, $z \in \Delta$. The same can be done in the polydisk Δ^m. For $a = (a_1, \cdots, a_m) \in \Delta^m$ the Möbius map ϕ_a of Δ^m onto Δ^m is defined by the formula $\phi_a(z) = (\phi_1(z_1), \cdots, \phi_m(z_m))$, where $\phi_j(z_j) = \frac{z_j + a_j}{1 + \overline{a_j} z_j}$, $j = 1, \cdots, m$. Now, we are ready to give the following

Definition 1. Let $l = 1, \dots, m$ be fixed. The l-linearly invariant family $\mathfrak{M}_l$ is the class of all functions f, $f \in \mathcal{H}(\Delta^m)$, such that

1) $f(\mathbb{O}) = 0$, $\frac{\partial f}{\partial z_l}(\mathbb{O}) = 1$, $\frac{\partial f}{\partial z_l}(z) \neq 0$, for $z \in \Delta^m$,
2) for all $f \in \mathcal{M}_l$ and $\theta = (\theta_1, \dots, \theta_m) \in \mathbb{R}^m$, $f(ze^{i\theta})e^{-i\theta_l} \in \mathfrak{M}_l$, where $ze^{i\theta} = (z_1 e^{i\theta_1}, \dots, z_m e^{i\theta_m})$,
3) for all $f \in \mathfrak{M}_l$ and $a = (a_1, \dots, a_m) \in \Delta^m$

$$f(a, z) := \frac{f(\phi_a(z)) - f(\phi_a(\mathbb{O}))}{\frac{\partial f}{\partial z_l}(a)(1 - |a_l|^2)} \quad \text{in } \mathfrak{M}_l.$$

The following definition extends the Pommerenke's notion of the order of a function, ([P]). Let $\frac{\partial f}{\partial z_l}(z) = 1 + c_1(f)z_1 + \dots + c_m(f)z_m + o(\|z\|)$ as $z \to \mathbb{O}$.

Definition 2. Let f satisfy condition 1) of Definition 1. The order of the function f is defined as

$$\operatorname{ord} f = \sup_{a \in \Delta^m} \frac{1}{2} \left\| \nabla \frac{\partial f(a, \mathbb{O})}{\partial z_l} \right\| = \frac{1}{2} \sup_{a \in \Delta^m} \|(c_1(f(a, \cdot)), \dots, c_m(f(a, \cdot)))\|.$$

The order of a linearly invariant family $\mathfrak{M}_l$ is given by

$$\operatorname{ord} \mathfrak{M}_l = \sup_{f \in \mathfrak{M}_l} \operatorname{ord} f.$$

Definition 3. The universal l-linearly invariant family $\mathcal{U}^l_\alpha$ of order α is defined as

$$\mathcal{U}^l_\alpha = \bigcup \{\mathfrak{M}_l : \operatorname{ord} \mathfrak{M}_l \le \alpha\}.$$

For $m = 1$ the above definitions coincide with the classical definitions from the paper [P] and in this case the linearly invariant family is denoted by $\mathcal{U}_\alpha$.

It is a very important and interesting problem to study the behaviour of functions "near" the torus $\mathbf{T}^m$. For classes of functions analytic in the unit disk Δ there are known theorems of regularity and growth of the modulus of functions, as z tends to $\mathbf{T}$ along a radius of Δ. A result of this type is known for the class $\mathcal{U}_\alpha$, too ([C], [S]). In [GS2] we showed that an analogue is true for the class $\mathcal{U}_\alpha^l$.

Write $\mathbf{r} = (r_1, ..., r_m)$, $\theta = (\theta_1, ..., \theta_m) \in \mathbb{R}^m$, $\mathbf{r}e^{i\theta} = (r_1 e^{i\theta_1}, ..., r_m e^{i\theta_m})$, $\mathbb{I}^- = (1^-, \ldots, 1^-)$; moreover, let $M(r,p) = \max_{\|z\| \le r} |p(z)|$. In [GS2] and [GS3] we showed (regularity theorem) that for every $f \in \mathcal{U}_\alpha^l$ there exists $\theta \in \mathbb{R}^m$ such that

$$\lim_{\mathbf{r}\to\mathbb{I}^-} \left|\frac{\partial f}{\partial z_l}(\mathbf{r}e^{i\theta})\right| \prod_{k=1}^{m} \left(\frac{1-r_k}{1+r_k}\right)^\alpha (1-r_l^2) =$$

$$(*) \qquad \lim_{\mathbf{r}\to\mathbb{I}^-} M(r,F) 2\alpha \prod_{k=1}^{m} \left(\frac{1-r_k}{1+r_k}\right)^\alpha = \delta \in [0,1],$$

where $F(z) = \int_0^{z_l} \frac{\partial f}{\partial z_l}(z_1, \ldots, z_{l-1}, s, z_{l+1}, \ldots, z_m)\, ds$. Denote by $\mathcal{U}_\alpha^l(\delta)$ the family of all functions from $\mathcal{U}_\alpha^l$ for which the last limit is equal δ.

Let us denote by $\mathcal{L}$ the set of analytic one-to-one maps $\phi(z) = (\phi_1(z_1), ..., \phi_m(z_m))$ from Δ^m into Δ^m, such that for every $k = 1, \ldots, m$ the function $\phi_k(z_k)$ is analytic and univalent in Δ and $|\phi_k(z_k)| < 1$ in Δ. By the definition of $\mathcal{U}_\alpha^l$ its invariance with respect to maps $\phi(z)$ follows in the case $\phi_k(z_k)$ are conformal authomorphisms of Δ. The problem of invariance of $\mathcal{U}_\alpha^l$ with respect to $\phi \in \mathcal{L}$ is interesting for us; that is if

$$\mathfrak{U}_\alpha^l = \left\{ \Lambda_\phi[f] = \frac{f(\phi(z)) - f(\phi(\mathbb{O}))}{\frac{\partial f}{\partial z_l}(\phi(\mathbb{O}))\phi_l'(\mathbb{O})} : \; f \in \mathcal{U}_\alpha^l, \; \phi \in \mathcal{L} \right\},$$

one can ask about a relationship between $\mathcal{U}_\alpha^l$ and $\mathfrak{U}_\alpha^l$ (it is clear that $\mathcal{U}_\alpha^l \subset \mathfrak{U}_\alpha^l$) and about the linear invariance of $\mathfrak{U}_\alpha^l$. Pommerenke in [P, Theorem 1.2] studied the above problem in the case $m = 1$.

Theorem 1. *$\mathfrak{U}_\alpha^l$ is an l-linearly invariant family of order $\beta = \max(\alpha, 2)$. Thus $\mathfrak{U}_\alpha^l \subset \mathcal{U}_\beta^l$; $\mathfrak{U}_\alpha^l = \mathcal{U}_\alpha^l$ for $\alpha \ge 2$.*

Proof. If $\phi \in \mathcal{L}$ and $\tilde{\phi}_k(z_k) = \phi_k\left(\frac{z_k + a}{1 + \bar{a} z_k}\right)$, $a \in \Delta$, $k = 1, \ldots, m$ then $(\tilde{\phi}_1, \ldots, \tilde{\phi}_m) \in \mathcal{L}$, since $\tilde{\phi}_k(z_k)$ are analytic and univalent in Δ and $|\tilde{\phi}_k(z_k)| < 1$. Therefore $\mathfrak{U}_\alpha^l$ is l-linearly invariant.

Now, let $f \in \mathcal{U}_\alpha^l$, $\phi \in \mathcal{L}$. Write $g = \Lambda_\phi[f]$. We will estimate ord g. Let $a_k = \phi_k(0)$, $\omega_k(w) = \frac{w+a_k}{1+\bar{a}_k w}$ $(w \in \Delta)$ and $\chi_k(z_k) = \frac{\phi_k(z_k)-a_k}{1-\bar{a}_k\phi_k(z_k)}$, $k = 1, \dots, m$. The functions $\chi_k(z_k)$ are analytic and univalent in Δ, $\chi_k(0) = 0$, $|\chi_k(z_k)| < 1$ in Δ. Set $\omega = (\omega_1(z_1), \dots, \omega_m(z_m))$, $\chi = (\chi_1(z_1), \dots, \chi_m(z_m))$, $h = \Lambda_\omega[f]$. Then $h \in \mathcal{U}_\alpha^l$, $\phi_k(z_k) = \omega_k[\chi_k(z_k)]$ and $g = \Lambda_\chi[\Lambda_\omega[f]] = \Lambda_\chi[h] = h(\chi_1(z_1), \dots, \chi_m(z_m))/\chi_l'(0)$. Therefore

$$\frac{\partial g}{\partial z_l}(z) = \frac{\partial h}{\partial z_l}(\chi_1(z_1), \dots, \chi_m(z_m))\frac{\chi_l'(z_l)}{\chi_l'(0)},$$

$$\frac{\partial^2 g}{\partial z_l^2}(\mathbb{O}) = \frac{\partial^2 h}{\partial z_l^2}(\mathbb{O})\chi_l'(0) + \frac{\partial h}{\partial z_l}(\mathbb{O})\frac{\chi_l''(0)}{\chi_l'(0)},$$

$$\frac{\partial^2 g}{\partial z_l \partial z_k}(\mathbb{O}) = \frac{\partial^2 h}{\partial z_l \partial z_k}(\mathbb{O})\chi_k'(0), \qquad \text{for} \qquad k \neq l.$$

The following inequality holds ([Pi]) $|\frac{\chi_l''(0)}{\chi_l'(0)}| \leq 4(1 - |\chi_l'(0)|)$. By the definition of ord h and the inequality ord $h \leq \alpha$ we obtain $|\frac{\partial^2 h}{\partial z_l \partial z_k}(\mathbb{O})| \leq 2\alpha$, for all $k = 1, \dots, m$.

Since $|\chi_l'(0)| \leq 1$, by the Schwarz Lemma we obtain

$$\frac{1}{2}\left|\frac{\partial^2 g}{\partial z_l^2}(\mathbb{O})\right| \leq \alpha|\chi_l'(0)| + 2(1 - |\chi_l'(0)|) \leq \max(\alpha, 2),$$

$$\frac{1}{2}\left|\frac{\partial^2 g}{\partial z_l \partial z_k}(\mathbb{O})\right| \leq \alpha, \qquad k \neq l.$$

Since the above is true for every function $g \in \mathfrak{U}_\alpha^l$ and $\mathfrak{U}_\alpha^l$ is l-linearly invariant, we have ord $g \leq \max(\alpha, 2)$; therefore $\mathfrak{U}_\alpha^l \subset \mathcal{U}_\beta^l$.

If $\alpha \geq 2$ then ord $g = \alpha$. Thus $g \in \mathcal{U}_\alpha$ and $\mathfrak{U}_\alpha^l \subset \mathcal{U}_\alpha^l \subset \mathfrak{U}_\alpha^l$, which implies $\mathfrak{U}_\alpha^l = \mathcal{U}_\alpha^l$.

□

Remark 1. If we consider in Theorem 1 a family $\tilde{\mathcal{L}} \subset \mathcal{L}$ (instead of $\mathcal{L}$) of maps with $\phi_l(z_l) = e^{i\theta}\frac{z_l + a}{1+\bar{a}z_l}$, $a \in \Delta$, $\theta \in \mathbb{R}$ then, as follows from the proof of Theorem 1, $\chi_l(z_l) = z_l$. Thus $\frac{1}{2}|\frac{\partial^2 g}{\partial z_l^2}(\mathbb{O})| \leq \alpha$ and ord $g \leq \alpha$. Consequently, if we consider $\tilde{\mathfrak{U}}_\alpha^l \subset \mathfrak{U}_\alpha^l$ (connected with $\tilde{\mathcal{L}}$ instead of $\mathcal{L}$) then ord $\tilde{\mathfrak{U}}_\alpha^l = \alpha$ and $\tilde{\mathfrak{U}}_\alpha^l = \mathcal{U}_\alpha^l$ for all $\alpha \geq 1$.

Corollary 1. *For every $\alpha \geq 1$ the family $\mathcal{U}_\alpha^l$ is invariant with respect to the class $\tilde{\mathcal{L}}$.*

Remark 2. If $\alpha < 2$ then (by Theorem 1) $\mathfrak{U}_\alpha^l$ is a proper subset of $\mathcal{U}_2^l$.

Indeed, in the case $m = 1$ the function $\xi/(1-\xi)^2 \in \mathcal{U}_2$, but for any function $f \in \mathcal{U}_\alpha$, $\alpha < 2$, and any function ψ regular in Δ such that $\psi(0) = 0$, $|\psi(\xi)| < 1$ in Δ, the function $\frac{f[\psi(\xi)]}{f'[\psi(0)]\psi'(0)}$ is different from $\frac{\xi}{(1-\xi)^2}$, since (see [P], p. 115)

$$|f[\psi(\xi)]| \leq \left(\frac{1+|\psi(\xi)|}{1-|\psi(\xi)|}\right)^\alpha - 1 \leq \left(\frac{1+|\xi|}{1-|\xi|}\right)^\alpha - 1 < \frac{|\xi|}{(1-|\xi|)^2}.$$

Similar considerations in the case $m \geq 2$ imply Remark 2.

In [S] the second author showed that in the case $\alpha \geq 2$, for a positive function $\varepsilon(r)$, arbitrarily small for $r \to 1^-$ and every $\delta \in [0,1)$ there exists $f \in \mathcal{U}_\alpha(\delta)$ such that $\lim_{r\to 1^-} \frac{2\alpha M(r,f)(\frac{1-r}{1+r})^\alpha - \delta}{\varepsilon(r)} = \infty$, that is, the expression $2\alpha M(r,f)(\frac{1-r}{1+r})^\alpha$ may tend to δ arbitrary slowly, as $r \to 1^-$. The proof of this result was based on an analogous result of N.A. Shirokov ([Sh]) for the class S, $S \subset \mathcal{U}_2$. An analogous result is also true in the case $m \geq 2$ for the class $\mathcal{U}_\alpha^l$. But in this case the result is stronger. Using Corollary 1 we are able to reject the restriction $\alpha \geq 2$. It is the effect of the higher dimension.

Theorem 2. *For every positive arbitrarily small (as $r \to 1^-$) function $\varepsilon(r)$, $r \in [0,1)$, $\delta_0 \in [0,1)$ and $\alpha \geq 1$ there exists a function $\Phi(z) = \Phi(z_1, \ldots, z_m) \in \mathcal{U}_\alpha^l(\delta_0), m \geq 2$, such that $\lim_{r\to 1^-} \frac{2\alpha M(r,\Phi)(\frac{1-r}{1+r})^{\alpha m} - \delta_0}{\varepsilon(r)} = \infty$.*

Proof. 1^0 Let $\delta_0 \in (0,1)$ and let the function f_θ be given by the formula

$$f_\theta(z) = \frac{e^{i\theta_l}}{2\alpha}\left[\prod_{k=1}^m \left(\frac{1+z_k e^{-i\theta_k}}{1-z_k e^{-i\theta_k}}\right)^\alpha - 1\right] \in \mathcal{U}_\alpha^l(1).$$

As noted in [GS3], the function f_0 (with $\theta = \mathbb{O}$) belongs to $\mathcal{U}_\alpha^l$ and thus the function

$$\begin{aligned} F(z) &= \int_0^{z_l} \frac{\partial f_0}{\partial z_l}(z_1, \ldots, z_{l-1}, s, z_{l+1}, \ldots, z_m)\, ds \\ &= \frac{1}{2\alpha}\left[\prod_{k=1}^m \left(\frac{1+z_k}{1-z_k}\right)^\alpha - \prod_{k\neq l}\left(\frac{1+z_k}{1-z_k}\right)^\alpha\right] \end{aligned}$$

also belongs to $\mathcal{U}_\alpha^l$. In [Sh] a family consisting of convex functions ϕ in Δ was constructed such that $\phi(0) = \phi'(0) - 1 = 0$, $|\phi(\xi)| \leq \phi(|\xi|)$ and $1 < \lim_{r\to 1^-} \phi(r) = \phi(1) = a < b = \lim_{r\to 1^-} (\phi(1) - \phi(r))/(1-r)$.

Write

$$\omega(\xi) = \frac{\xi(2\rho-1)/\rho}{1-\xi(1-\rho)/\rho}, \quad \rho \in (\tfrac{1}{2}, 1]; \qquad \psi(\xi) = \frac{\phi(\omega(\xi))}{\phi(1)},$$

$$H(\xi) = \frac{\rho a}{4(2\rho-1)}[(\frac{1+\psi(\xi)}{1-\psi(\xi)})^2 - 1].$$

The function ψ is univalent in Δ, $\psi(0) = 0$, $|\psi(\xi)| < 1$ in Δ; $H \in S \subset \mathcal{U}_2$. For any arbitrarily small $\varepsilon(r) \to 0$, as $r \to 1^-$, one can choose ([Sh]) a function ϕ (defined above) such that for $1 < a < b$, b being arbitrarily close to 1 and for every $\rho \in (\frac{1}{2}, 1]$ the following condition holds: $\lim_{r\to 1^-} \frac{M(r,H)(1-r)^2 - \delta'}{\varepsilon(r)} = \infty$, where $\delta' = \lim_{r\to 1^-} M(r,H)(1-r)^2 = \frac{\rho a}{2\rho-1} \lim_{r\to 1^-} (\frac{1-r}{1-\psi(r)})^2 = \frac{2\rho-1}{\rho}\frac{a^3}{b^2}$. One can assume that $\lim_{r\to 1^-} \frac{(1-r)^{1/2}}{\varepsilon(r)} = 0$. Thus

$$\begin{aligned}
\infty &= \lim_{r\to 1^-} \frac{M(r,H)(1-r)^2 - \delta'}{\varepsilon(r)} \\
&= \lim_{r\to 1^-} \frac{4M(r,H)(\frac{1-r}{1+r})^2 - \delta' + M(r,H)(1-r)^2(1-\frac{4}{(1+r)^2})}{\varepsilon(r)} \\
&= \lim_{r\to 1^-} \frac{4M(r,H)(\frac{1-r}{1+r})^2 - \delta'}{\varepsilon(r)} + \lim_{r\to 1^-} M(r,H)(1-r)^2 \frac{(r-1)(3+r)}{\varepsilon(r)(1+r)^2}
\end{aligned}$$

(1)

$$\begin{aligned}
&= \lim_{r\to 1^-} \frac{4M(r,H)(\frac{1-r}{1+r})^2 - \delta'}{\varepsilon(r)} \\
&= \lim_{r\to 1^-} \frac{\frac{\rho a}{2\rho-1}[(\frac{1+\psi(r)}{1-\psi(r)}\frac{1-r}{1+r})^2 - (\frac{1-r}{1+r})^2] - \frac{2\rho-1}{\rho}\frac{a^3}{b^2}}{\varepsilon(r)} \\
&= \frac{\rho a}{2\rho-1} \lim_{r\to 1^-} \frac{\chi^2(r) - C^2}{\varepsilon(r)},
\end{aligned}$$

where $\chi(r) = \frac{1+\psi(r)}{1-\psi(r)}\frac{1-r}{1+r}$, $C = \frac{a}{b}(2-\frac{1}{\rho})$. By Corollary 1 the function $\Phi(z) = F(\psi(z_1), \dots, \psi(z_{l-1}), z_l, \psi(z_{l+1}), \dots, \psi(z_m))$ belongs to $\mathcal{U}_\alpha^l(\delta)$, for some $\delta \in [0,1]$. Now from the construction of F and from the equality $(*)$ (see [GS3]) it follows that

$$\delta = \lim_{r\to 1^-} M(r,\Phi) 2\alpha \left(\frac{1-r}{1+r}\right)^{\alpha m}$$

(2) $$= \lim_{r\to 1^-} \left(\frac{1+\psi(r)}{1-\psi(r)}\right)^{\alpha(m-1)} \left(\left(\frac{1+r}{1-r}\right)^{\alpha} - 1\right)\left(\frac{1-r}{1+r}\right)^{\alpha m}$$

$$= \lim_{r\to 1^-} \left(\frac{1-r}{1-\psi(r)}\right)^{\alpha(m-1)} = \left(\frac{a}{b}\frac{2\rho-1}{\rho}\right)^{\alpha(m-1)}.$$

Choose a function ϕ and a number $\rho \in (\frac{1}{2}, 1]$ such that $\delta_0^{\frac{1}{\alpha(m-1)}} < \frac{a}{b} < 1$, and $(\frac{a}{b}\frac{2\rho-1}{\rho})^{\alpha(m-1)} = \delta_0$. Then $\Phi \in \mathcal{U}_\alpha^l(\delta_0)$. Observe that with the above notation

$$\lim_{\chi\to C}\frac{\chi^{\alpha(m-1)} - C^{\alpha(m-1)}}{\chi^2 - C^2} > 0.$$

By (1) and the fact that $\lim_{r\to 1^-}\frac{1-r}{\varepsilon(r)} = 0$ we get

$$\begin{aligned}(3)\qquad &\lim_{r\to 1^-}\frac{M(r,\Phi)2\alpha(\frac{1-r}{1+r})^{\alpha m} - \delta_0}{\varepsilon(r)}\\ &= \lim_{r\to 1^-}\frac{\chi^{\alpha(m-1)}(r) - C^{\alpha(m-1)} - \chi^{\alpha(m-1)}(r)(\frac{1-r}{1+r})^{\alpha}}{\varepsilon(r)}\\ &= \lim_{r\to 1^-}\frac{\chi^{\alpha(m-1)}(r) - C^{\alpha(m-1)}}{\varepsilon(r)}\\ &= \lim_{\chi\to C}\frac{\chi^{\alpha(m-1)} - C^{\alpha(m-1)}}{\chi^2 - C^2}\lim_{r\to 1^-}\frac{\chi^2(r) - C^2}{\varepsilon(r)} = \infty.\end{aligned}$$

This gives the result in the case $\delta_0 \in (0,1)$.

2^0 Let $\delta_0 = 0$. In [Sh] it was shown that one can choose a convex function ϕ in Δ such that

$$(4)\qquad \begin{aligned}&\phi(0) = 0,\ \ \phi'(0) = 1,\ \ |\phi(\xi)| \le \phi(|\xi|),\\ &\phi(1) = a < \infty,\ \ b(r) = \frac{\phi(1) - \phi(r)}{1-r} \to \infty\end{aligned}$$

as $r \to 1^-$, and for $\rho = 1$ (that is $\omega(\xi) = \xi$) holds

$$\lim_{r\to 1^-}\frac{M(r,H)(1-r)^2}{(\varepsilon(r))^{2/(\alpha(m-1))}} = \infty.$$

Here we have taken $(\varepsilon(r))^{\frac{2}{\alpha(m-1)}}$ as an arbitrarily small term. From (3) it follows that $\lim_{r\to 1^-} M(r,H)(1-r)^2 = 0$. For the function ϕ the above defined function Φ belongs to $\mathcal{U}_\alpha^l$ by Corollary 1. From (2) and (4) it follows that $\Phi \in \mathcal{U}_\alpha^l(0)$ and (see (3))

$$\begin{aligned}\lim_{r\to 1^-}\frac{M(r,\Phi)(\frac{1-r}{1+r})^{\alpha m}}{\varepsilon(r)} &= \lim_{r\to 1^-}\frac{\chi^{\alpha(m-1)}(r)}{\varepsilon(r)}\\ &= \lim_{r\to 1^-}\left[\frac{\chi^2(r)}{(\varepsilon(r))^{2/\alpha(m-1)}}\right]^{\frac{\alpha(m-1)}{2}} = \infty,\end{aligned}$$

since
$$\infty = \lim_{r\to 1^-} \frac{M(r,H)(1-r)^2}{(\varepsilon(r))^{2/(\alpha(m-1))}} = a \lim_{r\to 1^-} \frac{\chi^2(r)}{(\varepsilon(r))^{2/(\alpha(m-1))}}.$$

□

The next theorem allows us to connect families $\mathcal{U}^l_\alpha(\delta)$ of functions analytic in Δ^m with families $\mathcal{U}^l_\alpha(\delta)$ of functions analytic in Δ^n, $n < m$.

Theorem 3. *Let $f(z_1,\dots,z_m) \in \mathcal{U}^l_\alpha(\delta_0)$ and $m > l$. Let us fix a variable $z_m = a_m \in \Delta$ for the function f. Then the function*

$$\Phi(z_1,\dots,z_{m-1}) = \frac{f(z_1,\dots,z_{m-1},a_m) - f(0,\dots,0,a_m)}{\frac{\partial f}{\partial z_l}(0,\dots,0,a_m)}$$

belongs to the family $\mathcal{U}^l_\alpha(\delta)$ of functions analytic in Δ^{m-1}. Moreover, if $\delta_0 > 0$ then ord $\Phi = \alpha$ and $\delta > 0$, and if $a_m = 0$ then $\delta \geq \delta_0$. Furthermore, the set $\{\Phi(z_1,\dots,z_{m-1}) : f \in \mathcal{U}^l_\alpha\}$ coincides with the family $\mathcal{U}^l_\alpha$ of analytic functions in Δ^{m-1}.

Proof. Denote $z_* = (z_1,\dots,z_{m-1})$. Since $\Phi(\mathbb{O}) = 0$, $\frac{\partial \Phi}{\partial z_l}(\mathbb{O}) = 1$, $\frac{\partial \Phi}{\partial z_l}(z_*) \neq 0$ in Δ^{m-1}, $\Phi(z_*)$ belongs to the family $\mathcal{U}^l_\alpha(\delta)$ of functions analytic in Δ^{m-1}, if ord$\Phi \leq \alpha$. By Theorem 1.1 of [GS1] we have

$$\text{ord } f = \max_{1\leq k\leq m} \sup_{z\in\Delta^m} \left| \frac{\frac{\partial^2 f}{\partial z_l \partial z_k}(z)}{\frac{\partial f}{\partial z_l}(z)} \frac{1-|z_k|^2}{2} - \overline{z_k}\delta^l_k \right| \leq \alpha$$

and then

$$\sup_{z_*\in\Delta^{m-1}} \left| \frac{\frac{\partial^2 \Phi}{\partial z_l^2}(z_*)}{\frac{\partial \Phi}{\partial z_l}(z_*)} \frac{1-|z_l|^2}{2} - \overline{z_l} \right| \leq \sup_{z\in\Delta^m} \left| \frac{\frac{\partial^2 f}{\partial z_l^2}(z)}{\frac{\partial f}{\partial z_l}(z)} \frac{1-|z_l|^2}{2} - \overline{z_l} \right| \leq \alpha,$$

$$\sup_{z_*\in\Delta^{m-1}} \left| \frac{\frac{\partial^2 \Phi}{\partial z_l \partial z_k}(z_*)}{\frac{\partial \Phi}{\partial z_l}(z_*)} \frac{1-|z_k|^2}{2} \right| \leq \sup_{z\in\Delta^m} \left| \frac{\frac{\partial^2 f}{\partial z_l \partial z_k}(z)}{\frac{\partial f}{\partial z_l}(z)} \frac{1-|z_l|^2}{2} \right| \leq \alpha, \quad k \neq l.$$

Thus ord $\Phi \leq \alpha$ and $\Phi(z_*)$ belongs to the family $\mathcal{U}^l_\alpha(\delta)$ of functions analytic in Δ^{m-1} with some $\delta \in [0,1]$. On the other hand , if $\Phi(z_*)$ belongs to the family $\mathcal{U}^l_\alpha$ of functions analytic in Δ^{m-1} then it belongs to the family $\mathcal{U}^l_\alpha$ of functions analytic in Δ^m. In this way we get the last statement of our Theorem.

If $\delta_0 > 0$ then by the regularity theorem (see [GS2]) there exists a direction of the maximal growth $\theta = (\theta_1, \dots, \theta_m)$ of f such that

$$\lim_{\mathbf{r}\to\mathbb{I}^-}\left[\left|\frac{\partial f}{\partial z_l}(\mathbf{r}e^{i\theta})\right|\prod_{k=1}^{m}\left(\frac{1-r_k}{1+r_k}\right)^{\alpha}(1-r_l^2)\right] = \delta_0,$$

where the expression in the brackets decreases with respect to every variable $r_k \in [0,1)$. Consequently

$$\lim_{\mathbf{r}\to\mathbb{I}^-}\left[\left|\frac{\partial f}{\partial z_l}(r_1e^{i\theta_1}, \dots, r_{m-1}e^{i\theta_{m-1}}, 0)\right|\prod_{k=1}^{m-1}\left(\frac{1-r_k}{1+r_k}\right)^{\alpha}(1-r_l^2)\right] \geq \delta_0.$$

Thus (if $a_m = 0$)

$$\lim_{\mathbf{r}\to\mathbb{I}^-}\left[\left|\frac{\partial \Phi}{\partial z_l}(r_1e^{i\theta_1}, \dots, r_{m-1}e^{i\theta_{m-1}})\right|\prod_{k=1}^{m-1}\left(\frac{1-r_k}{1+r_k}\right)^{\alpha}(1-r_l^2)\right] \geq \delta_0$$

and $\Phi \in \mathcal{U}_\alpha^l(\delta)$ (in Δ^{m-1}), $\delta \geq \delta_0$. If $a_m \neq 0$, consider the function

$$\tilde{f}(z) = \frac{f(z_1, \dots, z_{m-1}, \frac{z_m + a_m}{1+\bar{a}_m z_m}) - f(0, \dots, 0, a_m)}{\frac{\partial f}{\partial z_l}(0, \dots, 0, a_m)}.$$

By Lemma 2 of [GS2] $\tilde{f} \in \mathcal{U}_\alpha^l(\delta^0)$, $\delta^0 > 0$. Then

$$\Phi(z_*) = \tilde{f}(z_1, \dots, z_{m-1}, 0) = \frac{f(z_1, \dots, z_{m-1}, a_m) - f(0, \dots, 0, a_m)}{\frac{\partial f}{\partial z_l}(0, \dots, 0, a_m)}.$$

and by the above reasoning $\Phi(z_*)$ belongs to the family $\mathcal{U}_\alpha^l(\delta)$ of functions analytic in Δ^{m-1}, with $\delta \geq \delta^0 > 0$. $\square$

Remark 3. In Theorem 3 one can fix any other variable z_k, $k \neq l$. In the case $k < l$ the function $\Phi(z_*)$ belongs to $\mathcal{U}_\alpha^{l-1}$ (in Δ^{m-1}), since in z_* the variable z_l stays on $l-1$ position.

Corollary 2. *Let $f \in \mathcal{U}_\alpha^l(\delta_0)$ (in Δ^m). For f let variables $z_{k_1}, \dots, z_{k_n}$, $1 \leq n \leq m-1$ be free and let the rest variables be fixed. Moreover let z_l be one of the free variables, $l = k_j$. Then the normalized function $\Phi(z_*) = \Phi(z_{k_1}, \dots, z_{k_n})$ belongs to the family $\mathcal{U}_\alpha^j(\delta)$ of analytic function in Δ^n, where for $\delta_0 > 0$, $\operatorname{ord}\Phi = \alpha$ and $\delta > 0$, and if the fixed variables are zeros then $\delta \geq \delta_0$. Moreover $\{\Phi : f \in \mathcal{U}_\alpha^l$ (in Δ^m)$\}$ is identical with $\mathcal{U}_\alpha^j$ (in Δ^n).*

For $n = 1$ $(k_1 = l)$, $z_* = z_l$, by Corollary 2 we obtain that with fixed all variables except for z_l the family of corresponding normalized functions $\Phi(z_l)$ coincides with $\mathcal{U}_\alpha$ of functions analytic in Δ. It seems to us that if we fix a variable z_k, $k \neq l$ then the problems are not interesting. For example it follows from Theorem 1.1 of [GS1] that

$$\Phi(\xi) = \frac{\int_0^\xi \frac{\partial f}{\partial z_l}(s, a_2, \ldots, a_m)\, ds}{\frac{\partial f}{\partial z_l}(0, a_2, \ldots, a_m)} \in \mathcal{U}_{\alpha+1},$$

for $f \in \mathcal{U}_\alpha^l$, $l \neq 1$ and fixed $z_2, \ldots, z_m$. The example of the function $f_0 \in \mathcal{U}_\alpha^l$ for $a_2 = \ldots = a_m = 0$ shows that $\operatorname{ord}\Phi = \alpha + 1$. Thus after the above operation the order of a function can be greater than before.

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