ANNALES
UNIVERSITATIS MARIAE CURIE-SKLODOWSKA
LUBLIN-POLONIA

VOL. LII.1, 1 SECTIO A 1998

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Cesàro Sum Approximation of Outer Functions

Dedicated to Professor Eligiusz Zlotkiewicz
on the occasion of his 60th birthday

Abstract. It is well known that outer functions are zero-free on the unit disk. If an outer function, f, is given as an infinite series and a finite (polynomial) approximation is chosen, then it is desirable that the approximants retain the zero-free property of f. We observe for outer functions that the standard Taylor approximants do not, in general, retain the zero-free property – even when fairly restrictive conditions are placed on the permissible outer functions. We show, using methods of geometric function theory, that Cesàro sum approximants for outer functions which arise as the derivatives of bounded convex functions do inherit the desired zero-free property. We, also, find that a "cone-like" condition holds for the boundaries of the ranges of these approximants.

Introduction. Let $\mathbb{D}$ denote the open unit disk in $\mathbb{C}$. It is well known that outer functions are zero-free on the unit disk. Outer functions, which play an important role in H^p theory, arise in the characteristic equation which determines the stability of certain nonlinear systems of differential equations. The solutions of such characteristic equations frequently involve ratios of the form h/f where f is an outer function (see [Cu], p. 288).

If f is given as an infinite series and a finite (polynomial) approximation is chosen, then it is desirable, in order to justify the choice of the approximant, that the zeros of the approximant retain the zero-free property of f.

In this note we consider questions about when approximating sequences of polynomials for outer functions inherit the zero-free property on $\mathbb{D}$ possessed by outer functions. This leads to an investigation of the location of the zeros and behavior of such approximating sequences of polynomials.

We show by examples that for outer functions the standard Taylor approximants do not, in general, retain the zero-free property. This is especially the case for low order Taylor approximants, even when fairly restrictive conditions are placed on the permissible outer functions, such as requiring these to be generated by Smirnov domains or requiring these to satisfy certain geometric conditions. The case for high order Taylor approximants to functions analytic and non-vanishing on the closed unit disk is covered by Hurwitz's theorem which assures that for n, the degree of the approximant, sufficiently large, that the Taylor approximant will be zero-free on $\mathbb{D}$ since the target, the outer function, is zero-free on $\mathbb{D}$. For practical purposes, however, the n required above may be prohibitively large. Thus, we seek conditions on outer functions and classes of approximants for which the zero-free property on $\mathbb{D}$ will be inherited by all of the approximants.

Jentzsch's classical results ([Di], p. 352) show that the circle of convergence for a Taylor series is a subset of the set of limit points of the zero sets of the sequence of Taylor approximants. This suggests that, in general, a careful analysis will be required in order to affirm the desired zero-free inheritance.

We, thus, consider other approximants such as Cesàro sums. We find that when considering geometric restrictions, such as convexity on Smirnov domains, the methods of geometric function theory can be applied to verify the desired zero-free inheritance for appropriate approximants. We, also, find that a "cone-like" condition holds for the boundaries of the ranges of these approximants.

Outer Functions. Recall that an outer function is a function f in H^p of the form

$$f(z) = e^{i\gamma} \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1+e^{it}z}{1-e^{it}z} \log \psi(t) dt \right\}$$

where $\psi(t) \geq 0$, $\log \psi(t)$ is in L^1 and $\psi(t)$ is in L^p. See [Du1] for the definitions and classical properties of outer functions. Since any function f in H^1 which has $1/f$ in H^1 is an outer function, then typical examples of outer functions can be generated by functions of the form $\prod_{k=1}^{n}(1-e^{i\theta_k} z)^{\alpha_k}$ for $-1 < \alpha_k < 1$.

Taylor Approximants. We will look at some examples which show that the standard Taylor approximants are not necessarily zero-free on $\mathbb{D}$. Consider $f(z) = (1 + e^{i/10}z)^{1/2}/(1-z)^{71/100}$. If we let $p_n(z) = \sum_{k=0}^{n} a_k z^k$ be the Taylor approximant for $f(z) = \sum_{k=0}^{\infty} a_k z^k$, then computer computations show that p_3 has 1 zero inside $\mathbb{D}$, p_4 has no zeros inside $\mathbb{D}$ and p_5 again has a zero inside $\mathbb{D}$. Thus, not only do some of the p_n's have zeros inside $\mathbb{D}$, there is also no guarantee that once they are zero-free on $\mathbb{D}$, that they will remain so for higher orders.

Next, we show that even imposing restrictive geometric constraints on the outer functions may not suffice to assure that their Taylor approximants will inherit their zero-free property on $\mathbb{D}$. A common and useful generation for outer functions is found by considering Smirnov domains. See [Po] for their definition and properties. We recall that if a simply connected domain G with a rectifiable boundary is a Smirnov domain, then the conformal map f, mapping $\mathbb{D}$ onto G, has the property that its derivative f' is an outer function. Also, boundedness and convexity is a sufficient condition on a domain to guarantee it to be a Smirnov domain. Thus, the derivatives of bounded convex functions are always outer functions.

For r, $0 < r < 1$, let $f_r(z) = z/(1 - rz)$. Then, f_r maps $\mathbb{D}$ onto a bounded convex domain so that f_r is an outer function. Since $f_r(z) = 1 + 2rz + \ldots$ it is easy to see that many of its early Taylor approximants are not zero-free on $\mathbb{D}$ for r near 1.

Cesàro Approximants. The previous observations suggest using different approximants. We consider the Cesàro means for a function $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Let $s_n(z) = \sum_{k=0}^{n} a_k z^k$. The Cesàro means σ_n are defined by $\sigma_n(z) = \frac{1}{n+1}\sum_{k=0}^{n} s_k(z)$. We note that s_n can be written as the linear combination

$$(*) \qquad s_n = (n+1)\sigma_n - n\sigma_{n-1}.$$

The proof of Jentzsch's theorem ([Di], p. 352) can be modified to show that if an assumption is made that $\limsup \sqrt[n]{|\sigma_n(z)|} \leq 1$ for some z outside the circle of convergence, then a contradiction ensues. Specifically, for such a z and for any $\rho > 0$ there exists an $n(\rho)$ such that for $n > n(\rho)$ we have that $\sqrt[j]{|\sigma_j(z)|} \leq (1+\rho)$ for $j = n-1$ and n. It follows from (2) that

$$\begin{aligned} |s_n(z)| &\leq (n+1)|\sigma_n(z)| + n|\sigma_{n-1}(z)| \\ &\leq (n+1)(1+\rho)^n + n(1+\rho)^{n-1} \leq 2(n+1)(1+\rho)^n. \end{aligned}$$

Thus, $\limsup|s_n(z)| \leq \limsup \sqrt{[n]2(n+1)(1+\rho)} = 1+\rho$. The arbitrariness of ρ implies that $\limsup|s_n(z)| \leq 1$, which contradicts a conclusion in the proof of Jentzsch's Theorem. Continuing as in the proof of Jentzsch's Theorem, it follows that the circle of convergence is also a subset of the limit set of the zeros of $\{\sigma_n\}$.

We now consider the Cesàro approximants of outer function which are the derivatives of a convex function. We introduce the following notation (see [Du2]). Let

$$
\begin{aligned}
A &= \{f : f(z) = a_0 + a_1 z + a_2 z^2 + ..., \ f \text{ is analytic on } \mathbb{D}\},\\
S &= \{f \in A : f(0) = 0, f(0) = 1, \ f \text{ is univalent on } \mathbb{D}\},\\
K &= \{f \in S : f \text{ is close-to-convex}\},\\
S^* &= \{f \in S : f \text{ is starlike w.r.t. } 0\},\\
C &= \{f \in S : f \text{ is convex}\}.
\end{aligned}
$$

We note Kaplan's relationship that $f \in K$ if and only if there exists a $g \in S^*$ such that $zf(z) = g(z)p(z)$ for some $p \in A$ such that $p(0) = 1$ and $p(\mathbb{D})$ lies in a half plane H with $0 \in \partial H$. Also, we note that close-to-convexity and convexity are geometric conditions on a domain which are independent of the normalization of an associated function mapping $\mathbb{D}$ onto the domain.

For $f, g \in A$ with $f(z) = a_0 + a_1 z + a_2 z^2 + ...$ and $g(z) = b_0 + b_1 z + b_2 z^2 + ...$, define $f * g$ by $f * g(z) = a_0 b_0 + a_1 b_1 z + a_2 b_2 z^2 + \ldots$, i.e., $f * g$ is the (Hadamard) convolution of f and g. We note that we will employ, in the arguments we give, a common abuse of notation which interchanges the function $f * g$ with the function values $f(z) * g(z)$. The two major Sheil-Small-Ruscheweyh results (see [Du2], Section 8.3) state that

(A) if $f \in C$ and $g \in S^*$, then $f * g \in S^*$ and
(B) if $f \in C$ and $g \in S^*, p \in A$ with $p(0) = 1$, then $f * gp = (f * g)p_1$ where $p_1(\mathbb{D}) \subset$ closed convex hull of $p(\mathbb{D})$.

Let $h \in A$ with $h(z) = a_0 + a_1 z + a_2 z^2 + \ldots$. From the partial sums $s_n(z) = \sum_{k=0}^{n} a_k z^k$ we can construct the Cesàro means σ_n of h by

$$\sigma_n(z) = \frac{1}{n+1}\sum_{k=0}^{n} s_k(z) = h(z) * \sum_{k=0}^{n} \frac{n-k+1}{n+1} z^k = h * g_n(z)$$

where $g_n(z) = \sum_{k=0}^{n} \frac{n-k+1}{n+1} z^k$. Note that g_n is the Cesàro mean of the identity function w.r.t. convolution, i.e., $z/(1-z)$.

We have now

Proposition 1. *Let $f \in A$ be such that $f(\mathbb{D})$ is convex. Then, the Cesàro means σ_n of f' are zero-free on $\mathbb{D}$ for all n. In particular, if $f(\mathbb{D})$ is a bounded convex domain, then for the outer function f' we have that the Cesàro means σ_n are zero-free on $\mathbb{D}$ for all n.*

Proof. Let $h = f'$. Let k be defined by $k(z) = z/(1-z)^2$ and note that $zf'(z) = k * f(z)$. Then,

$$\sigma_n(z) = h * g_n(z) = f' * g_n(z) = \frac{zf'(z) * zg_n(z)}{z}$$
$$= \frac{f(z) * k(z) * zg_n(z)}{z} = \frac{f(z) * z(zg_n(z))'}{z}.$$

If it can be shown that $zg_n \in K$, then applying (A) and using Kaplan's relationship would yield a $g \in S^*$ and a p [with $p(0) = 1$ and $p(\mathbb{D})$ lying in a half plane H with $0 \in \partial H$] such that

$$\frac{f(z) * z(zg_n(z))'}{z} = \frac{f * gp(z)}{z} = \frac{(f * g(z))p_1(z)}{z} \neq 0$$

since $p_1(z) \neq 0$ [since $p_1(\mathbb{D}) \subset H$] and $f * g \in S^*$ is 0 only for $z = 0$.

J. Lewis's ([Le], Lemma 3, p. 1118) result on Jacobi polynomials implies that $zg_n \in K$. An alternate approach to this can be made by noting that Egervary [Eg] showed that $G_n(z) = zg_n(z)$ is starlike w.r.t. $G_n(1)$. Since another characterization of K is that $f \in K$ if and only if the complement of $f(\mathbb{D}), \mathbb{C}\backslash f(\mathbb{D})$, can be written as the union of non-intersecting half-rays, then the fact that $G_n(\mathbb{D})$ is starlike with respect to $G_n(1)$ implies that $\mathbb{C}\backslash G_n(\mathbb{D})$ is the union of half-rays emanating from $G_n(1)$.

Cone Condition. We now show that the ranges of the Cesàro approximants satisfy a cone-like condition on the boundary. We will recall the following additional notation from geometric function theory. Let

$$P = \{p \in A : p(0) = 1,\ \operatorname{Re} p(z) > 0 \ \text{ for } z \in \mathbb{D}\},$$
$$P(1/2) = \{p \in P : \operatorname{Re} p(z) > 1/2 \ \text{ for } z \in \mathbb{D}\},$$
$$S^*(\alpha) = \{f \in S^* :\ \operatorname{Re} zf'(z)/f(z) > \alpha \text{ for } z \in \mathbb{D}\}.$$

Recall (see [Du2]) that $h \in C$ implies that $h(z)/z \in P(1/2)$ and $zh'(z)/h(z) \in P(1/2)$. Ruscheweyh ([Ru], p. 55) has generalized the principal results (A) and (B) on convolution to show

(C) if $f, g \in S^*(1/2), p \in A$ with $p(0) = 1$, then $f * gp = (f * g)p_1$ where $p_1(\mathbb{D}) \subset$ closed convex hull of $p(\mathbb{D})$.

Note (C) also implies that $f * g \in S^*(1/2)$. We have then

Proposition 2. *Let $f \in A$ such that $f(\mathbb{D})$ is convex and $f(z) = a_1 z + a_2 z^2 + ..., a_1 \neq 0$. Then, the Cesàro means $\sigma_n^{(2)}$ of f' of order 2 have their ranges contained in a cone (from 0) with opening $2\beta_n \pi$, where $\beta_n < 1$.*

Proof. We may assume that $a_1 = 1$, otherwise apply the argument to $f(z)/a_1$.

First, note that for the function f the Cesàro means of f' of order γ can be given by (see [Ru], p. 142]) $f * g_n^{(\gamma)}$ where

$$g_n^{(\gamma)}(z) = \sum_{k=0}^{n} \frac{\binom{n-k+\gamma}{n-k}}{\binom{n+\gamma}{n}} z^k.$$

In Egervary's notation $s_{n-1}^{(\gamma)} = z g_n^{(\gamma)}$. Egervary proved for each $n \geq 1$

(a) $s_n^{(1)}(\mathbb{D})$ is starlike w.r.t. $s_n^{(1)}(1)$,
(b) $s_n^{(2)} \in S^*(1/2)$,
(c) $s_n^{(3)} \in C$.

We will use (b) as follows to yield the cone condition on the ranges $f * g_n^{(\gamma)}(\mathbb{D})$. Suppose $f \in C \subset S^*(1/2)$. Then, using (b) and (C) we have

$$f' * g_n^{(2)} = \frac{zf'(z) * zg_n^{(2)}(z)}{z} = \frac{f(z) * z(zg_n^{(2)}(z))'}{z}$$
$$= \frac{f(z) * \frac{zg_n^{(2)}\, z(zg_n^{(2)}(z))'}{zg_n^{(2)}}}{z} = \frac{f(z) * zg_n^{(2)}(z)}{z} p_1(z),$$

where $p_1 \in P(1/2)$ and $f(z) * zg_n^{(2)}(z) \in S^*(1/2)$. By a result of Brickman, et al., [BHMW] we have $\frac{f(z) * zg_n^{(2)}(z)}{z} \in P(1/2)$. Since $(f(z) * zg_n^{(2)})/z$ is a polynomial it is bounded, hence there exists $\beta_1 < 1$ such that

$$\left| \arg \frac{f(z) * zg_n^{(2)}(z)}{z} \right| < \frac{\beta_1 \pi}{2}.$$

Hence, we have

$$\left| \arg f'(z) * g_n^{(2)}(z) \right| = \left| \arg \frac{f(z) * zg_n^{(2)}(z)}{z} . p_1(z) \right|$$
$$\leq \left| \arg \frac{f(z) * zg_n^{(2)}(z)}{z} \right| + |\arg p_1(z)|$$
$$\leq \beta_1 \frac{\pi}{2} + \frac{\pi}{2} = \beta \pi.$$

Since a subordination argument can be made to show that

$$\frac{f(z) * zg_n^{(2)}(z)}{z} \prec g_n^{(3)}(z),$$

then β_1 can be chosen independently of f.

We note that the above conclusion is valid for $f \in S^*(1/2)$ and that it is not generally extendable to $f \in S^*$, since $k'(z) * g_n^{(2)}$ does not satisfy a cone condition and $k \in S^*$.

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