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Holomorphic motions and quasiconformal extensions

ABSTRACT. In this article we consider holomorphic families of univalent functions parametrized by the unit disk in such a way that the origin corresponds to the identity. Then, thanks to the λ-lemma, each member of the family has a quasiconformal extension. This method enables us not only to give simple proofs of some known results, but also to provide new results. Actually, we derive several results about quasiconformal extendability for typical classes of univalent functions.

1. Introduction. Univalence criteria sometimes produce quasiconformal extension criteria. A number of authors obtained such a kind of results by means of Grunsky's inequality or Löwner's method (see, e.g., [6] for a comprehensive account, also cf. [2]). Grunsky's inequality does not yield an explicit bound for the maximal dilatations of the quasiconformal extensions in general, however Löwner's method is sometimes satisfactory but difficult to use. On the other hand, the so-called λ-lemma and its variants generate quantitative results on quasiconformal extendability in various situations which are often sharp. However, we should notice the reader that this method has the disadvantage that the quasiconformal extension cannot be given explicitly in many cases.

In this article we shall demonstrate the power of the λ-lemma. First, compare the following two results.

Key words and phrases. Holomorphic motion, quasiconformal extension, univalent function.

Theorem A (Krzyż [15], cf. [21], p. 294). *Let ω be an analytic function on the unit disk Δ with $|\omega'(z)| \le k$, where $0 \le k < 1$ is a constant. Then the function $f(z) = z + \omega(1/z)$ on the outside of Δ can be extended to a k-quasiconformal automorphism of the Riemann sphere by setting $f(z) = z + \omega(\overline{z})$ on Δ.*

Theorem B (Fait, Krzyż and Zygmunt [12, Theorem 2′]). *Let ω be an analytic function on the unit disk Δ with $|\omega'(z)| \le k$, where $0 \le k < 1$ is a constant. Then the function $f(z) = z + \omega(z)$ on Δ can be extended to a k-quasiconformal automorphism of the Riemann sphere by setting $f(z) = z + \omega(1/\overline{z})$ outside Δ.*

As the function $f(z) = z + k/z$ shows, Theorem A is best possible, however Theorem B can be improved by using the λ-lemma as follows: *Under the same hypothesis as in* Theorem B, *the function $f(z) = z + \omega(z)$ can be extended to a k'-quasiconformal automorphism of the Riemann sphere, where $k' = k/(2-k) < k$.* In fact, this can be easily obtained as a corollary of the following result, but a way of k'-quasiconformal extension of f is not clear from our method.

Theorem 1.1. *Let k be a constant in $[0,1)$. For an analytic function f on the unit disk Δ with $f(0) = 0$ and $f'(0) \ne 0$, let $p(z)$ represent one of the quantities $zf'(z)/f(z)$, $1 + zf''(z)/f'(z)$ or $f'(z)$. If*

$$|p(z) - (1+k^2)/(1-k^2)| \le 2k/(1-k^2)$$

for all $z \in \Delta$, the function f can be extended to a k-quasiconformal automorphism of the Riemann sphere.

In the case $p(z) = zf'(z)/f(z)$ or $f'(z)$ the above result is best possible as the function $p(z) = (1+kz^2)/(1-kz^2)$ indicates (see Section 4 for details).

The remainder of this paper is organized as follows. In Section 2, we present a precise definition of holomorphic motions and state the λ-lemma and its consequences. We also introduce (the Bers embedding of) the universal Teichmüller space and recall its fundamental properties. In Section 3, we establish a general principle of strengthening univalence criteria into those of quasiconformal extension thanks to the λ-lemma. Section 4 is devoted to applications of the general principle to several concrete cases. Consequently, we obtain a number of quasiconformal extension criteria, most of which seem not to appear in the literature. From these, Theorem 1.1 above immediately follows.

2. Holomorphic motions and the universal Teichmüller space. We recall here the definition of holomorphic motions and its properties. Let Δ denote the unit disk in the complex plane. A holomorphic motion of the subset E of the Riemann sphere $\widehat{\mathbb{C}}$ is a map $F : E \times \widehat{\mathbb{C}}$ satisfying the following three condtions.

1. For each $z \in E$, the map $F(z, \cdot) : \Delta \to \widehat{\mathbb{C}}$ is holomorphic,
2. For each $\lambda \in \Delta$, the map $F_\lambda := F(\cdot, \lambda) : E \to \widehat{\mathbb{C}}$ is injective, and
3. $F_0 = \mathrm{id}_E$.

The notion of holomorphic motions was first introduced by Mañé, Sad and Sullivan [18] in order to investigate the complex dynamics on the Riemann sphere and nowadays it proves to be very useful in various aspects. The following results will be fundamental in our argument.

Theorem C (Mañé-Sad-Sullivan [18] and Bers-Royden [7]).
Let $F : E \times \Delta \to \widehat{\mathbb{C}}$ be a holomorphic motion of E. Then the following hold.
1. *The map F is uniformly jointly continuous in two variables. Therefore, F uniquely extends to a holomorphic motion of $\overline{E}$, which will be denoted still by the same letter.*
2. *For each $\lambda \in \Delta$, the map F_λ is quasiconformal in the interior* $\mathrm{Int}(\overline{E})$ *of* $\overline{E}$.
3. *The Beltrami coefficient $\mu(\lambda) = \overline{\partial} F_\lambda / \partial F_\lambda$ is a holomorphic map from Δ to the unit ball of the complex Banach space* $L^\infty(\mathrm{Int}(\overline{E}))$.

The next striking result was first established by Slodkowski [22]. For another proof, see also Astala-Martin [4] and Douady [9].

Theorem D. *Every holomorphic motion of E can be extended to a holomorphic motion of the whole sphere $\widehat{\mathbb{C}}$.*

We should note that the above extension is not necessarily unique. These two theorems have various applications to the Teichmüller theory (see [11]).

Now we remind the reader of the definition of the universal Teichmüller space and relevant notions (see [17], as a general reference).

Let $B_j(D), (j = 1, 2)$ denote the complex Banach space consisting of all holomorphic functions φ on the hyperbolic domain D in $\widehat{\mathbb{C}}$ with hyperbolic (Poincaré) metric $\rho_D(z)|dz|$ of constant negative curvature -4 such that the norms

$$||\varphi_{j,D}|| = \sup_{z \in D} \rho_D(z)^{-j} |\varphi(z)|$$

are finite. The universal Teichmüller space $T(\Delta)$ is the set of Schwarzian derivatives $S_f = (f''/f')' - \frac{1}{2}(f''/f')^2$ of those univalent functions f on the unit disk which can be quasiconformally extended to the whole sphere. The famous Nehari-Kraus theorem claims that $||S_f||_{2,\Delta} \leq 6$ for a univalent function f on Δ, hence $T(\Delta) \subset B_2(\Delta)$. Moreover, by Ahlfors, $T(\Delta)$ is shown to be a bounded contractible domain of $B_2(\Delta)$.

We also note that $||T_f||_{1,\Delta} \leq 6$ holds for a univalent function f, where T_f denotes the pre-Schwarzian derivative f''/f' of f. Thus, the set $T_1(\Delta)$ of T_f of all univalent functions $f : \Delta \to \mathbb{C}$ which admit quasiconformal extensions to the whole sphere, is sometimes thought to be another model of the universal Teichmüller space (cf. [3] and [24]).

The Teichmüller distance between the points S_f and S_g of $T(\Delta)$ is defined by

$$d_T(S_f, S_g) = \inf_h d_\Delta(0, ||\mu_h||_\infty),$$

where h runs over all quasiconformal automorphisms of $\widehat{\mathbb{C}}$ such that $h = f \circ g^{-1}$ on $g(\Delta)$ with Beltrami coefficient $\mu_h = h_{\bar{z}}/h_z$ and d_Δ denotes the hyperbolic distance determined by the hyperbolic metric $\rho_\Delta(z)|dz| = |dz|/(1-|z|^2)$ on Δ, i.e., $d_\Delta(z,w) = \operatorname{arctanh}(|z-w|/|1-\bar{z}w|)$. We note that the infimum is always attained by some h. In particular, we note that the quantity $d_T(S_f, 0)$ measures the smallest maximal dilatation of the quasiconformal extension of f. A quasiconformal extension $\tilde{f}$ to $\widehat{\mathbb{C}}$ of f is called extremal, if the Beltrami coefficient μ of $\tilde{f}$ satisfies $d_T(0, S_f) = d_\Delta(0, ||\mu||_\infty)$. It is known that $\tilde{f}$ is extremal if and only if μ satisfies the Hamilton-Krushkal' condition (cf. Gardiner [13, Chapter 6]):

$$\sup_\varphi \left| \iint_{\Delta^*} \mu(z)\varphi(z)\, dxdy \right| = ||\mu||_\infty,$$

where the supremum is taken over all integrable holomorphic quadratic differentials $\varphi = \varphi(z)dz^2$ on the exterior Δ^* of the unit disk with $||\varphi|| = \iint_{\Delta^*} |\varphi(z)|dxdy \leq 1$. Note that the Hamilton-Krushkal' condition is conformally invariant.

The celebrated Royden-Gardiner theorem states that the Teichmüller distance d_T coincides with the Kobayashi (pseudo-)distance on $T(\Delta)$ (for a modern proof also using the optimal λ-lemma, see [11]). By the contraction property of Kobayashi pseudo-distance, we have the following (known) result as a simple corollary.

Proposition 2.1. *Suppose that $h : \Delta \to T(\Delta)$ is a holomorphic map. Then we have $d_T(h(s), h(t)) \leq d_\Delta(s,t)$ for any $s, t \in \Delta$.*

Let $\Phi : M_1 \to T(\Delta)$ denote the Bers projection, where

$$M_k = \{\mu \in L^\infty(\mathbb{C}) : ||\mu||_\infty < k, \mu|_\Delta = 0\}.$$

Precisely speaking, $\Phi(\mu)$ is the Schwarzian derivative $w^\mu|_\Delta$, where w^μ is a homeomorphic solution of the Beltrami equation $w_{\bar{z}} = \mu w_z$ on $\widehat{\mathbb{C}}$ (note that w^μ is conformal on Δ by the assumption $\mu = 0$ on Δ). It is well-known that the map $\Phi : M_1 \to T(\Delta)$ is a holomorphic submersion.

The λ-lemma has an important application to the Teichmüller theory. Now we explain it. Let $h : \Delta \to T(\Delta)$ be a holomorphic map. We choose a $\mu_0 \in M_1$ such that $\Phi(\mu_0) = h(0)$ and fix it. For each $\lambda \in \Delta$, a meromorphic function f_λ on Δ is uniquely determined by the conditions:

$$S_{f_\lambda} = h(\lambda), \tag{1}$$

$$f_\lambda(0) = f'_\lambda(0) - 1 = f''_\lambda(0) = 0. \tag{2}$$

In such a way, we obtain a holomorphic motion $F(z, \lambda) = f_\lambda \circ f_0^{-1}(z)$ of $f_0(\Delta)$. By Theorem D, F can be extended to a holomorphic motion $\widetilde{F}$ of the whole sphere. By assumption, the univalent function f_0 has a unique quasiconformal extension $\widetilde{f}_0$ with Beltrami coefficient μ_0. Let $\widetilde{h}(\lambda)$ denote the Beltrami coefficient of the quasiconformal map $\widetilde{F}_\lambda \circ \widetilde{f}_0$. By virtue of Theorem C (3), we then see that the map $\widetilde{h} : \Delta \to M_1$ is holomorphic and satisfies $\Phi \circ \widetilde{h} = h$ and $\widetilde{h}(0) = \mu_0$. Thus we have proved the following:

Proposition 2.2 ([11]). *Suppose that a holomorphic map $h : \Delta \to T(\Delta)$ and a point $\mu_0 \in M_1$ such that $\Phi(\mu_0) = h(0)$ are given. Then, there exists a holomorphic map $\widetilde{h} : \Delta \to M_1$ such that $\Phi \circ \widetilde{h} = h$ on Δ and that $\widetilde{h}(0) = \mu_0$.*

We will call $\widetilde{h}$ a lift of h with $\widetilde{h}(0) = \mu_0$. This need not be uniquely determined only by the condition $\widetilde{h}(0) = \mu_0$.

Example 2.1. As the simplest example, we consider the holomorphic motion $F(z, t) = f(tz)/t$ of the unit disk for a normalized univalent function $f : \Delta \to \mathbb{C}$ so that $f(0) = 0$ and $f'(0) = 1$. (Of course, we set $F(z, 0) = \lim_{t\to 0} f(tz)/t = z$.) By Theorem D, F extends to a holomorphic motion $\widetilde{F}$ of the whole sphere. We then have a $|\lambda|$-quasiconformal extension $\widetilde{F}_\lambda$ of F_λ. Note that $\widetilde{F}_\lambda$ can be chosen so that $\widetilde{F}_\lambda(\infty) = \infty$. But, without this restriction, it turns out that F_λ can be extended to a $|\lambda|^2$-quasiconformal map (Krushkal' [14]). In fact, the map $h : \Delta \to T(\Delta)$ defined by $h(\lambda)(z) = S_{F_\lambda}(z) = \lambda^2 S_F(\lambda z)$ satisfies that $h'(0) = 0$. Thus the claim follows from the next theorem.

Theorem E (Krushkal' [14]). *Suppose that a holomorphic map $h : \Delta \to T(\Delta)$ satisfies that $h'(0) = \cdots = h^{(m)}(0) = 0$. Then $d_T(h(0), h(t)) \le d_\Delta(0, t^{m+1})$ holds for every $t \in \Delta$.*

Problem. For the above $h : \Delta \to T(\Delta)$, can we choose a lift $\widetilde{h} : \Delta \to M_1$ of h so that $\widetilde{h}'(0) = \cdots = \widetilde{h}^{(m)}(0) = 0$?

If this is true, we have an easy proof of Theorem E by applying Schwarz's lemma to the function $t^{-m-1}(\widetilde{h}(t) - \mu_0)/(1 - \overline{\mu}_0\widetilde{h}(t))$, where $\mu_0 = \widetilde{h}(0)$.

3. Univalence criteria and quasiconformal extensions. In this section we explain how univalence criteria generate quasiconformal extension criteria by means of the λ-lemma. First, we state a general principle which leads to quasiconformal extensions of univalent functions.

Suppose that given are a hyperbolic simply connected domain U and an operation P on meromorphic functions on a plane domain D associated with a univalence criterion. Let $\mathcal{N}(D)$ be a class of meromorphic functions on D normalized at a point $a \in D$ so that the operator $P : \mathcal{N}(D) \to \mathcal{M}(D)$ is injective, where $\mathcal{M}(D)$ is another class of meromorphic functions on D satisfying the condition A at the point a, where the condition A is empty or $\varphi(a) = p_0$, where p_0 is the point appearing just below. Further assume the following conditions:

1) (univalence criterion). If $f \in \mathcal{N}(D)$ satisfies $P_f(D) \subset U$, the function f must be univalent in D,
2) $\mathrm{id}_D \in (D)$ and $P_{\mathrm{id}_D}(z) = p_0$ for all $z \in D$, where p_0 is a point in U,
3) $\mathrm{Hol}_0(D, U) \subset P(\mathcal{N}(D))$, where $\mathrm{Hol}_0(D, U)$ is the class of holomorphic functions f on D with values in U satisfying the condition A,
4) (holomorphic dependence of P) for a holomorphic family φ_λ in $\mathrm{Hol}_0(D, U)$ (i.e., the map $\lambda \mapsto \varphi_\lambda(z)$ is a $\widehat{\mathbb{C}}$-valued holomorphic function for every fixed $z \in D$), the corresponding functions f_λ in $\mathcal{N}(D)$ such that $P_{f_\lambda} = \varphi_\lambda$ form a holomorphic family, too.

Under these circumstances, we can show the following claim.

Theorem 3.1. *Let $L : \Delta \to U$ be a Riemann mapping function of U with $L(0) = p_0$. If $f \in \mathcal{N}(D)$ satisfies $P_f(D) \subset L(B_k)$ for some $k \in [0, 1)$, the function f can be extended to a k-quasiconformal automorphism $\widetilde{f}$ of $\widehat{C}$, where $B_k = \{z \in \mathbb{C} : |z| \leq k\}$. Further such an $\widetilde{f}$ can be chosen so that $\widetilde{f}(\infty) = \infty$ when $Hol_0(D, U) \subset P(\mathcal{N}(D) \cap Hol(D, \mathbb{C}))$.*

Proof. We define a holomorphic family φ_λ in $\mathrm{Hol}_0(D, U)$ parametrized through $\lambda \in \Delta$ by

$$\varphi_\lambda(z) = L\left(\frac{\lambda}{k} L^{-1} \circ P_f(z)\right).$$

Let F_λ be the holomorphic family in $\mathcal{N}(D)$ uniquely determined by $P_{F_\lambda} = \varphi_\lambda$. By the univalence criterion, we then observe that $F(z, \lambda) = F_\lambda(z)$ is a holomorphic motion of D. When $\mathrm{Hol}_0(D, U) \subset P(\mathcal{N}(D) \cap \mathrm{Hol}(D, \mathbb{C}))$, we have $D \subset \mathbb{C}$, since $P_{\mathrm{id}} = p_0 \in \mathrm{Hol}_0(D, U)$ implies $\mathrm{id}_D \in \mathrm{Hol}(D, \mathbb{C})$. Therefore we can extend F to a holomorphic motion of $D \cup \{\infty\}$ by setting $F(\infty, \lambda) = \infty$. Now the optimal λ-lemma (Theorem D) produces a holomorphic motion $\widetilde{F}$ of $\widehat{C}$ whose restriction to D (or $D \cup \{\infty\}$) coincides with F. By Theorem C, each $\widetilde{F}_\lambda$ is a $|\lambda|$-quasiconformal extension of F_λ. Since $P_{F_k} = \varphi_k = P_f$, we have $f = F_k$, thus the proof is now complete. □

Remarks.

1. In the above proof, the condition A was needed only to make sure that $L(k^{-1}\lambda L^{-1} \circ \varphi) \in P(\mathcal{N}(D))$ for all $\varphi \in \mathcal{M}(D)$ with $\varphi(\Delta) \subset B_k$ and $\lambda \in \Delta$.

2. One may think that the above theorem can be generalized as follows: *If $P_f(D) \subset g_k(\Delta)$, the function $f \in \mathcal{N}(D)$ can be extended to a k-quasiconformal automorphism of $\widehat{\mathbb{C}}$, where $g(z, \lambda) = g_\lambda(z)$ is a holomorphic function from $\Delta \times \Delta$ into U such that g_λ is univalent for each $\lambda \in \Delta \setminus \{0\}$, while g_0 is the constant function p_0.* Actually, this can be shown by taking $\varphi_\lambda = g_\lambda \circ g_k^{-1} \circ P_f$. However, it is not a proper generalization because it is always true that $g_k(\Delta) \subset L(B_k)$. The last assertion is easily obtained by applying Schwarz's lemma to the function $\lambda \mapsto L^{-1}(g(z, \lambda))$.

Corollary 3.2. *Under the same hypothesis as in* Theorem 3.1, *the norm of the Schwarzian derivative of the function f satisfies $||S_f||_{2,D} \leq 12k$. When D is a unit disk Δ, we have a better estimate: $||S_f||_{2,\Delta} \leq 6k$. Furthermore if $\mathrm{Hol}_0(\Delta, U) \subset P(\mathcal{N}(\Delta) \cap \mathrm{Hol}(\Delta, \mathbb{C}))$, we also have $||T_f||_{1,\Delta} \leq 6k$.*

Proof. By a theorem of Beardon and Gehring [5], we know that $||S_f||_{2,D} \leq 12$ for any univalent function f on the hyperbolic plane domain D. Hence our estimate follows from Lehto's majorant principle. The other cases can be treated in the same fashion. □

In the case $D = \Delta$, we can consider the map $h : \Delta \to T(\Delta)$ defined by $h(\lambda) = S_{F_\lambda}$, where F_λ is the holomorphic family constructed above. It is easy to see that $h : \Delta \to T(\Delta)$ is actually holomorphic and $h(0) = 0$. If $h'(0) = 0$, we have by Theorem E a better estimate $d_T(h(\lambda), 0) \leq d_\Delta(0, \lambda^2)$, which implies that F_λ can be extended to a $|\lambda|^2$-quasiconformal automorphism of $\widehat{\mathbb{C}}$. We will see below that $h'(0) = 0$ happens in concrete examples.

4. Applications. Now we apply the theorem shown in the preceeding section to several examples.

First, we consider the operation $P_f = f'$ for non-constant holomorphic functions on a convex domain $D \subset \mathbb{C}$. Now fix a point $a \in D$, and consider the class

$$\mathcal{N}(D) = \{f \in \mathrm{Hol}(D, \mathbb{C}) : f(a) = 0\} \quad \text{and} \quad \mathcal{M}(D) = \mathrm{Hol}(D, \mathbb{C}).$$

(Therefore, the condition A is vacuous here.) For $\beta \in (-\pi/2, \pi/2)$, we set $U_\beta = \{z \in \mathbb{C}^* : |\arg z - \beta| < \pi/2\}$. Then the Noshiro-Warschawski theorem yields that if $P_f(D) \subset U_\beta$ the function $f \in \mathcal{N}(D)$ should be univalent. Hence Theorem 3.1 is now applicable. The Riemann mapping function $L_\beta : \Delta \to U_\beta$ of U_β with $L_\beta(0) = 1$ is given by $L_\beta(z) = (1 + e^{2i\beta}z)/(1-z)$. Since the assumption $f(a) = 0$ is not essential for quasiconformal extendability, we have the next

Theorem 4.1. *Let D be a convex domain and take a point a in D. Suppose that a holomorphic function f on D satisfies $f'(D) \subset L_\beta(B_k)$. Then the function f can be extended to a k-quasiconformal automorphism of the Riemann sphere fixing ∞.*

Note that

$$L_\beta = \left\{ z \in \mathbb{C} : \left| z - \frac{1 + e^{2i\beta}k^2}{1-k^2} \right| \le \frac{2k\cos\beta}{1-k^2} \right\}.$$

Suppose that the convex domain D is also a quasidisk, in other words, the Riemann mapping function $g : \Delta \to D$ of D can be extended to a quasiconformal mapping. Under the same notation as in Section 3, we consider $h(\lambda) = S_{F_\lambda \circ g} = g^* S_{K_\lambda} + S_g$, where $g^* : B_2(D) \to B_2(\Delta)$ is the isometric isomorphism defined by the pullback $g^*\varphi(z) = \varphi(g(z))g'(z)^2$ by g. We set $\psi = k^{-1} L_\beta^{-1} \circ f'$. Then $\varphi_\lambda = L_\beta(\lambda\psi) = 1 + (1 + e^{2i\beta})\lambda\psi + O(\lambda^2)$ as $\lambda \to 0$. Hence, $h(\lambda) = g^*(\varphi_\lambda''/\varphi_\lambda - 3(\varphi_\lambda'/\varphi_\lambda)^2/2) + S_g = S_g + (1 + e^{2i\beta})\lambda g^*(\psi'') + O(\lambda^2)$ as $\lambda \to 0$. Theorefore, we can see that $h'(0) = (1 + e^{2i\beta})g^*(\psi'')$. In particular, $h'(0) = 0$ if and only if $\psi'' = 0$, i.e., $f'(z) = L_\beta(c(z+d))$ with constants c and d satisfying $\sup_{w \in \partial D} |w + d| \le k/|c|$, and in that case f can be extended to a k^2-quasiconformal automorphism of $\widehat{\mathbb{C}}$ by Theorem E.

It is not clear from the proof whether Theorem 4.1 is best possible or not. In the case $\beta = 0$ and $D = \Delta$, actually, this is best possible. In fact, take the normalized holomorphic function $f_2 : \Delta \to \mathbb{C}$ determined by the relation $f_2'(z) = L_0(kz^2) = 1 - 2kz^2 + \dots$. Explicitly,

$$f_2(z) = -z + \frac{1}{\sqrt{k}} \log \frac{1 + \sqrt{k}z}{1 - \sqrt{k}z}.$$

Then the meromorphic function $g_2(z) = 1/f_2(1/z)$ belongs to the familiar class Σ_0 of normalized univalent meromorphic function on $|z|>1$ and can be represented by the power series

$$g_2(z) = z - \frac{2k}{3}z^{-1} + \frac{2k^2}{45}z^{-3} + \frac{46k^3}{945}z^{-5} + \cdots$$

in $|z|>1$.

On the other hand, by a theorem of Schiffer (see [10, §4.7]), any $g(z) = z+b_1/z+b_2/z^2+\cdots$ in Σ_0 satisfies $|b_2| \le 2/3$. Now, from the Lehto majorant principle [17, §§3.5, 3.6], it follows that a function $g \in \Sigma_0$ admitting κ-quasiconformal mapping of $\widehat{\mathbb{C}}$ should satisfy $|b_2| \le 2\kappa/3$. Hence, f_2 cannot be extended to any κ-quasiconformal mapping for an arbitrary $\kappa < k$.

A normalized analytic function f on the unit disk is called *close-to-convex* if $\mathrm{Re}[f'(z)/g'(z)] > 0$, $\forall_z \in \Delta$, for some univalent function g on Δ whose image domain is convex. Note that the function g need not be normalized here. For subclasses of close-to-convex functions, the following statement is immediately deduced from Theorem 4.1.

Corollary 4.2. *Let g be a conformal mapping from the unit disk onto a convex domain D which admits a k_1-quasiconformal automorphism of the Riemann sphere. For constants $k_2 \in [0,1)$ and $\beta \in (-\pi/2, \pi/2)$, suppose that an analytic function f on the unit disk satisfies*

$$f'(z)/g'(z) \in L_\beta(B_{k_2}) \ \text{ for any } \ z \in D.$$

Then f can be extended to a $(k_1+k_2)/(1+k_1k_2)$-quasiconformal automorphism of the Riemann sphere.

Proof. The function $h(z) = f(g^{-1}(z))$ satisfies $h'(D) \subset L_\beta(B_{k_2})$. Now the result follows from Theorem 4.1 and from the fact that the composition of k_1 and k_2 -quasiconformal mappings becomes a $(k_1+k_2)/(1+k_1k_2)$-quasiconformal mapping.

Remark. The λ-lemma does not tell us about any explicit construction of a quasiconformal extension. At least, in this case, we have a concrete way of k'-quasiconformal extension of a function $f \in \mathcal{N}(\Delta)$ with $f'(\Delta) \subset L_0(B_k)$, where $k' = 2k/(1+k^2)$. In fact, writing as $f(z) = (1+k^2)z/(1-k^2) + p(z)$, we can extend f by setting $f(z) = (1+k^2)z/(1-k^2) + p(1/\overline{z})$ outside the unit disk Δ.

Secondly, we consider the operation $P_f(z) = zf'(z)/f(z)$ on holomorphic functions f on the unit disk Δ. Let $\mathcal{N}(\Delta)$ be the set of holomorphic functions

f on the unit disk normalized so that $f(0) = f'(0) - 1 = 0$ and $\mathcal{M}(\Delta)$ the set of holomorphic functions φ on Δ satisfying the condition A: $\varphi(0) = 1$. Fix $\beta \in (-\pi/2, \pi/2)$. Then the condition $P_f(\Delta) \subset U_\beta$ says that f is β-spirallike, in particular, univalent on Δ (see, for example, [10]). For a $\varphi \in \mathcal{M}(D)$, we have the following expression of the function $f \in (\Delta)$ with $P_f = \varphi$:

$$f(z) = z \exp\left(\int_0^z \frac{\varphi(\zeta) - 1}{\zeta}\, d\zeta\right).$$

In particular, the holomorphic dependence of the operator P is evident. So, Theorem 3.1 is now applicable.

Theorem 4.3. *Let $\beta \in (-\pi/2, \pi/2)$ and f be a holomorphic function on the unit disk with $f'(0) = 1$. Suppose that f satisfies $zf'(z)/f(z) \in L_\beta(B_k)$ on Δ for some constant $k \in [0,1)$. Then the function f can be extended to a k-quasiconformal automorphism of the Riemann sphere fixing ∞. In particular, we have $||T_f||_{1,\Delta} \le 6k$ and $||S_f||_{2,\Delta} \le 6k$.*

When $\beta = 0$ (i.e., the case of ordinary starlike functions), the above result is best possible. In fact, the function $f \in \mathcal{N}(\Delta)$ satisfying $P_f(z) = L_0(kz^2) = (1 + kz^2)/(1 - kz^2)$ has the form $z/(1 - kz^2)$, therefore has the k-quasiconformal extension by setting $f(z) = z/(1 - kz/\overline{z})$ outside Δ, which turns out to be conjugate to the affine map $z \mapsto z - k\overline{z}$ by the inversion $z \mapsto 1/z$, thus to be extremal.

On the other hand, for general $\beta \in (-\pi/2, \pi/2)$, it is unknown whether the above result is best possible or not. At least we know so far that the function $f \in \mathcal{N}(\Delta)$ determined by $f'(z) = L_\beta(kz^2)$ with $\beta \ne 0$ has a k'-quasiconformal extension with some $k' < k$, which we show now. The function f has the form $z(1 - kz^2)^{-\alpha}$, where $\alpha = (1 + e^{2i\alpha})/2$. Then the function $F(z) = 1/f(1/z) = z(1 - k/z^2)^\alpha$ on Δ^* can be extended to a quasiconformal mapping by setting $F(z) = z(1 - k\overline{z}^2|z|^{-2/\alpha})^\alpha$ on Δ. The Beltrami coefficient μ of F has the form $\mu(z) = -ke^{2i\beta}|z|^{i\tan\beta}$ on Δ, in particular, $|\mu(z)| = k$ a.e. on Δ, thus F is a k-quasiconformal automorphism of $\widehat{\mathbb{C}}$. This mapping F is not extremal. In fact, for any integrable holomorphic quadratic differential $\varphi(z)dz^2$ on the unit disk Δ, we have

$$\begin{aligned}\iint_\Delta \mu(z)\varphi(z)dxdy &= \int_0^1 \int_0^{2\pi} \mu(r)\varphi(re^{i\theta})rd\theta dr \\ &= 2\pi\varphi(0)\int_0^1 r\mu(r)dr = \frac{-2\pi ke^{2i\beta}\varphi(0)}{2 + i\tan\beta}.\end{aligned}$$

From the sharp estimate $\pi|\varphi(0)| \le ||\varphi|| = \iint_\Delta |\varphi(z)|dxdy$, it follows that

$$\sup_{||\varphi|| \le 1} \left|\iint_\Delta \mu(z)\varphi(z)dxdy\right| = \frac{2k}{|2 + i\tan\beta|} < k = ||\mu||_\infty$$

if $\beta \neq 0$. Hence F does not satisfy the Hamilton-Krushkal' condition, equivalently, F is not extremal.

Now we examine the condition for $h'(0) = 0$, where $h(\lambda) = S_\lambda$ and F_λ is as in Section 3. Letting $\psi = k^{-1}L_\beta^{-1} \circ P_f$, we have $\varphi_\lambda = L_\beta(\lambda\psi) = 1+(1+e^{2i\beta})\lambda\psi+O(\lambda^2)$ as $\lambda \to 0$. A straightforward calculation shows that

$$z^2 S_f(z) = z\left(\frac{z\varphi'(z)}{\varphi(z)}\right)' - \frac{1}{2}\left(\frac{z\varphi'(z)}{\varphi(z)}\right)^2 + \frac{1}{2}(1-\varphi(z)^2)$$

if a non-constant holomorphic function f satisfies $zf'(z)/f(z) = \varphi(z)$ (cf. [23]). Thus we have $z^2 S_{F_\lambda}(z) = (1+e^{2i\beta})\lambda(z(\psi'(z))' - \psi(z)) + O(\lambda^2)$, which implies

$$h'(0)(z) = (1 + e^{2i\beta})z^{-2}(z^2\psi''(z) + z\psi'(z) - \psi(z)).$$

In particular, $h'(0) = 0$ if and only if the function ψ is a holomorphic solution of the Fuchsian differential equation $z^2\psi''(z) + z\psi'(z) - \psi(z) = 0$. The indicial equation $\rho(\rho - 1) + \rho - 1 = \rho^2 - 1 = 0$ has the roots 1 and -1, thus a holomorphic solution near the origin is a constant multiple of the fundamental solution $\psi(z) = z$ corresponding to the root 1. Therefore $P_f(z) = L_\beta(cz)$, where c is a constant with $|c| \leq k$. By Theorem E, we conclude that the function f has a k^2-quasiconformal extension to the sphere if f satisfies $zf'(z)/f(z) = P_\beta(kz)$. Such an f is nothing else but the function $z/(1 - kz)^{1+\exp(2i\beta)}$. The function $z/(1 - z)^{1+\exp(2i\beta)}$ is sometimes called the β-spirallike Koebe function.

Remark. Here we mention a relation with strongly starlike functions. A normalized holomorphic function f on the unit disk is called *strongly starlike of order* α if $|\arg P_f(z)| = |\arg(zf'(z)/f(z))| \leq \pi\alpha/2$ on Δ, where α is a constant with $0 \leq \alpha \leq 1$. In [12], it is shown that such a function can be extended to a $\sin(\pi\alpha/2)$-quasiconformal automorphism of $\widehat{\mathbb{C}}$ fixing ∞ in an explicit way. In particular, we have $||S_f||_{2,\Delta} \leq 6\sin(\pi\alpha/2)$. There is a problem: Can one extend a strongly starlike function of order α to an α- quasiconformal automorphism of $\widehat{\mathbb{C}}$? If this is true, we should have $||S_f||_{2,\Delta} \leq 6\alpha$. So far, the author only knows that $|S_f(0)| \leq 6\alpha$ for a strongly starlike function f of order α and the equality holds if and only if f is a rotation of the function F determined by the relation $zF'(z)/F(z) = \{(1+z^2)/(1-z^2)\}^\alpha$. This fact can be shown in the same fashion as in [8]. See also [23].

On the other hand, an elementary geometry shows that a function $f \in \mathcal{N}(\Delta)$ satisfying $P_f(\Delta) \subset L_0(B_k)$ is strongly starlike of order α, where α is the number determined by $\sin(\pi\alpha/2) = 2k/(1+k^2)$, equivalently, $\tan(\pi\alpha/4) = k$. Because $\pi\alpha/4 < k < \alpha$ ($< \sin(\pi\alpha/2)$) for $0 < k < 1$, our result supports the affirmative answer to the problem stated above.

Next we consider the operation $P_f(z) = 1 + zf''(z)/f'(z)$. Let $\mathcal{N}(\Delta)$ be the set of holomorphic functions f on the unit disk normalized so that $f(0) = f'(0) - 1 = 0$ and $\mathcal{M}(\Delta)$ the set of holomorphic functions φ on Δ with the condition A: $\varphi(0) = 1$. Then the condition

$$P_f(\Delta) \subset U_0 = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$$

means that the function f is convex, in particular univalent in Δ. Letting $g(z) = zf'(z)$, we have, by the Alexander theorem, $P_f(z) = zg'(z)/g(z)$. Since f is recovered by the formula $f(z) = \int_0^z g(\zeta)/\zeta d\zeta$ from the function g, in combination with the last example we obtain the holomorphic dependence of the operation P. Therefore, by virtue of Theorem 3.1, we have the following

Theorem 4.4. *Suppose that a non-constant holomorphic function f on the unit disk Δ satisfies $1 + zf''(z)/f'(z) \in L_0(B_k)$ on Δ for some constant $k \in [0, 1)$. Then f can be extended to a k-quasiconformal automorphism of the Riemann sphere fixing ∞.*

Corollary 4.5. *Under the same situation in the above theorem, we have the norm estimate of the Schwarzian derivative: $||S_f||_{2,\Delta} \le 2k$.*

This follows from the fact $||S_f||_{2,\Delta} \le 2$ for any convex function f on Δ (see [20] or [16]) and the Lehto majorant principle. We remark that this corollary also implies k-quasiconformal extendability of such a function f as above by the Ahlfors-Weill theorem [1].

Finally, we consider the derivative of $h(\lambda) = S_{F_\lambda}$. We set $\psi = k^{-1} L_0^{-1} \circ P_f$. Then $\varphi_\lambda = L_0(\lambda\psi) = 1 + 2\lambda\psi + O(\lambda^2)$ as $\lambda \to 0$.

In this case,

$$z^2 S_{F_\lambda}(z) = z\varphi'_\lambda(z) - (\varphi_\lambda(z)^2 - 1)/2 = 2\lambda(z\psi'(z) - \psi(z)) + O(\lambda^2),$$

in particular, $h'(0)(z) = 2z^{-2}(z\psi'(z) - \psi(z))$. Hence, $h'(0) = 0$ if and only if $\psi(z) = cz$, where c is a constant with $|c| \le 1$. However, the function f such that $P_f(z) = L_0(kz)$ turns out to be the Möbius transformation $f(z) = z/(1 - kz)$.

We conclude this section by giving an example of another type. Consider here the Schwarzian derivative as the operation P, i.e., $P_f = S_f$. Let D be a hyperbolic simply connected plane domain and take a finite point $a \in D$. Moreover, let $\mathcal{N}(D)$ be the set of locally univalent meromorphic functions f on D normalized so that $f(a) = f'(a) - 1 = f''(a) = 0$ and $\mathcal{M}(D)$ the set of all holomorphic functions φ on D such that $\varphi(z) = O(z^{-4})$ as

$z \to \infty$ if $\infty \in D$. In this case, the condition A is vacuous. Now we restrict our attention to the case $D = \Delta$ and $a = 0$. Then, by Nehari's result [19], the condition $|S_f(z)| \leq \pi^2/2$ on the unit disk Δ forces $f \in \mathcal{N}(\Delta)$ to be univalent. The holomorphic dependence of the Schwarzian derivative operator is well-known in the complex analytic theory of Teichmüller spaces, so we can apply Theorem 3.1 to show the following

Theorem 4.6. *Let k be a constant in $[0,1)$. If a function $f \in \mathcal{N}(\Delta)$ satisfies $|S_f(z)| \leq \pi^2 k/2$ on Δ, then f can be extended to a k-quasiconformal automorphism of the whole sphere.*

This result is, of course, not new and we remark that the above used method is also applicable for any quasidisk D.

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