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LISA LORENTZEN

Convergence of compositions of self – mappings

ABSTRACT. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of self-mappings of a set V, i.e., $f_n(V) \subseteq V$. Under what conditions will the sequence $\{F_n\}_{n=1}^{\infty}$ given by the compositions $F_n := f_1 \circ f_2 \circ \cdots \circ f_n$ converge to a constant function in V? Answers to this question have applications in dynamical systems, Schur analysis, continued fractions and other similar structures like towers of exponentials and infinite radicals. The purpose of this paper is to collect some known answers from different areas of mathematics and give them a unified presentation.

1. Introduction

The problem.
Let U be the unit disk $|z| < 1$ in the complex plane $\mathbb{C}$, and let $\mathcal{F}$ be a family of functions analytic in U, mapping U into itself. Let further $\{f_n\}$ be a sequence from $\mathcal{F}$, and let $\{F_n\}$ be derived from $\{f_n\}$ by compositions

$$(1.1) \qquad F_n := f_1 \circ f_2 \circ \cdots \circ f_n \quad \text{for } n = 1, 2, 3, \ldots .$$

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The question we ask is the following: When will $\{F_n\}$ converge in U to a constant function? It is evident that some conditions are needed. Otherwise, $f_n(z) \equiv z$ for all n would constitute a counterexample.

If $\{F_n\}$ converges to a constant in U, it is clear that this convergence is uniform on compact subsets of U, i.e., locally uniform in U. It is however of great interest to study the additional question: When is this convergence uniform in U? That is, when will $F_n(U)$ shrink to a point as $n \to \infty$? If we can get an estimate for the diameter diam $F_n(U)$, then we also have an estimate for the speed of convergence.

There is a second kind of uniformity connected with this problem which is equally important: When is the convergence of $\{F_n\}$ uniform with respect to $\{f_n\}$ from $\mathcal{F}$ for a given z? Or: When is the convergence of diam $F_n(U)$ uniform with respect to $\{f_n\}$ from $\mathcal{F}$? We shall also address these questions in this paper.

Our problem is more general than it may seem at first. If V is a simply connected domain in $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, omitting more than one point, then there exists a univalent analytic function φ in V such that $\varphi(V) = U$ (the Riemann mapping theorem). If $f(V) \subseteq V$, then $g := \varphi \circ f \circ \varphi^{-1}$ maps U into U. Since

$$g_1 \circ g_2 \circ \cdots \circ g_n = (\varphi \circ f_1 \circ \varphi^{-1}) \circ (\varphi \circ f_2 \circ \varphi^{-1}) \circ \cdots \circ (\varphi \circ f_n \circ \varphi^{-1}) = \varphi \circ F_n \circ \varphi^{-1},$$

our results extend to more general subsets of $\widehat{\mathbb{C}}$. Moreover, if V is a multiply connected domain, and $\mathbb{C} \setminus \overline{V}$ has one and only one unbounded component D, and if every $f \in \mathcal{F}$ can be extended to a function $\widehat{f}$ analytic in the simply connected domain $\mathbb{C} \setminus \overline{D}$, then $\widehat{f}(\mathbb{C} \setminus \overline{D}) \subseteq \mathbb{C} \setminus \overline{D}$.

Our problem is in many ways a geometric problem, and it may be stated very generally indeed. U may be replaced for instance by any subset V of a Banach space X, and $\mathcal{F}$ can be any family of self-mappings of V. The problem can also be made very special by restricting the class $\mathcal{F}$ of functions.

Another area of interest is the following sequence of compositions

$$G_n := f_n \circ f_{n-1} \circ \cdots \circ f_1 \quad \text{for } n = 1, 2, 3, \ldots, \tag{1.2}$$

which Gill [13] calls *the outer composition* of $\{f_n\}$ as opposed to *the inner composition* (1.1). For these compositions G_n, it is natural to ask other types of questions. It is evident that $\{G_n\}$ converges only under restrictive conditions, as compared to $\{F_n\}$. For instance, if $f_n(z) = a_n/(1+z) \not\equiv 0$ for all n, then $\{G_n\}$ converges to some G only if $\lim_{n\to\infty} a_n = G(1+G)$. That is, $\{f_n\}$ converges to a limit function given by $f(z) = G(1+G)/(1+z)$ and $G(z) \equiv c$ being the fixed point of f.

Applications.
Compositions of the type (1.1) come up in several situations. For instance:

1. Continued fractions

$$(1.3)\quad \mathrm{K}(a_n/b_n) = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}}; \quad a_n \neq 0,$$

with approximants

$$(1.4)\qquad \frac{A_n}{B_n} := \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \quad \text{for } n = 1, 2, 3, \ldots$$

can be regarded as compositions

$$(1.5)\quad \frac{A_n}{B_n} = S_n(0), \quad \text{where } S_n := s_1 \circ s_2 \circ \cdots \circ s_n \text{ and } s_k(z) = \frac{a_k}{b_k + z}.$$

Here the elements a_k and b_k may be for instance complex numbers or entire functions, and we allow $A_n/B_n = \infty$. More generally, a_k and b_k may be vectors or matrices of complex numbers, p–adic numbers, Clifford algebras, quasiconformal functions or entire functions, etc., with properly defined division operators. In the following we assume that a_k and b_k are complex numbers. Extensions to more general cases are then often possible.

For applications of continued fractions it is often imperative that $K(a_n/b_n)$ converges; i.e., the limit $c := \lim_{n\to\infty} A_n/B_n$ exists in the extended complex plane $\widehat{\mathbb{C}}$. It is evident that $A_n/B_n = S_n(0) = S_{n+1}(\infty)$. Hence, if $S_n(0) \to c$, then also $S_n(\infty) \to c$, which rules out the possibility that $\{S_n\}$ converges to a linear fractional transformation. Therefore it makes sense to look for conditions which imply convergence of $\{S_n\}$ to a constant function on some set V. This is in fact the way convergence of continued fractions is proved in many cases.

2. Other limiting structures. Schur decomposition of a function $f(z)$ gives a sequence of linear fractional transformations or constant functions which map the unit disk into its closure. The composition sequence (1.1) converges then to $f(z)$. This is the basis for the Pick–Nevanlinna interpolation.

Infinite radicals and towers of exponentials, as found for instance in the work of Ramanujan [7, p.108] and Pólya and Szegö [35, p.37, p.214], are also structures of the same nature:

$$(1.6)\qquad \sqrt{a_1 + b_1\sqrt{a_2 + b_2\sqrt{a_3 + b_3\sqrt{\ldots}}}}; \qquad f_n(z) = \sqrt{a_n + b_n z},$$

$$(1.7) \qquad a_1^{a_2^{a_3^{\cdot^{\cdot^{\cdot}}}}} ; \qquad f_n(z) = a_n^z = e^{b_n z} \quad \text{where } b_n = \ln a_n,$$

where we choose appropriate analytic branches of the multivalued functions f_n. To assign values to these structures it is natural to require that $F_n(0) := f_1 \circ f_2 \circ \cdots \circ f_n(0)$ converges as $n \to \infty$, and to use its limit as the value. Alternatively one could use $\lim F_n(1)$. For (1.7) we have $F_n(0) = F_{n-1}(1)$, so that it makes no difference. Here we may again choose a_k and b_k to be complex numbers to get results which may possibly be extended to more general cases.

Infinite sums and infinite products can also be put into this framework, but for these structures there is not much to gain. For infinite sums the functions $f_n(z) := a_n + z$ do not have any contraction properties, and for infinite products the functions $f_n(z) := a_n z$ contract too strongly, either towards 0 or towards ∞, for our analysis to be applicable.

3. Dynamical systems. Iterations can be seen as compositions (1.1) or (1.2), where $\mathcal{F}$ only contains one function f. However, in some cases it is more relevant to consider functions f with arbitrary noise. That is, $\mathcal{F}$ consists of a function f and small pertubations of f. Another important situation occurs when $\mathcal{F}$ is a finite class of functions, and f_n is picked from $\mathcal{F}$ according to certain probability distributions. Karlsson and Wallin [22] refer to $\{\mathcal{F}, U\}$ or $\{\mathcal{F}, V\}$ as a generalized dynamical system, discrete if $\mathcal{F}$ is finite. In this setting we are for instance interested in properties of orbits $\{g_n\}$, where $g_{n-1} = f_n(g_n)$ for all n, and thus $g_0 = F_n(g_n)$, and orbits $\{\tilde{g}_n\}$, where $\tilde{g}_{n+1} = f_n(\tilde{g}_n)$ for all n so that $\tilde{g}_{n+1} = G_n(\tilde{g}_1)$.

In the terminology of Karlsson and Wallin [22], the orbits $\{g_n\}$ and $\{\tilde{g}_n\}$ are ρ_n-stable, or rather, the dynamical system $\{\mathcal{F}, V\}$ is ρ_n-stable, iff $\operatorname{diam} F_n(V) \leq \rho_n$ for all $\{f_k\} \subseteq \mathcal{F}$, where $\{\rho_n\}$ is a given positive null-sequence. As we shall see, this means that $\{F_n\}$ converges uniformly in V to a constant function, uniformly with respect to $\{f_n\}$ from $\mathcal{F}$. This extends the idea of Fatou sets, and thus also of Julia sets to sequences $\{f_n\}$ of functions.

In this setting compositions (1.2) are also of interest. It is natural to define ρ_n-stability as $\operatorname{diam} G_n(V) \leq \rho_n \to 0$ for all such compositions of functions from $\mathcal{F}$. Clearly this is equivalent to the ρ_n-stability defined above. We do not expect that $\{G_n\}$ converges, though. Fractal images are just the limiting set of compositions $\{G_n\}$, i.e., the assymptotic orbit $\{\tilde{g}_n\}$ of an asymptotically stable generalized dynamical system.

Some historical remarks.

The simple case of iterations of one single function f has been extensively studied for a long time. The following classical result due to Denjoy and Wolff may serve as a very adequate example:

Theorem 1.1. (The Denjoy – Wolff Theorem [9], [47].) *Let f be analytic in U with $f(U) \subseteq U$. Then f is either a linear fractional transformation mapping U onto U, or f has a fixed point α in the closure $\overline{U}$ of U, and $\{F_n\}$ converges locally uniformly to α in U, where F_n is the n-th iterate of f.*

Of course, by using the Riemann mapping theorem as described above, this can be generalized to more general simply connected domains.

The Denjoy–Wolff Theorem has been generalized in many directions. Our generalization of Theorem 1.1 is natural as seen from the modern analytic theory of continued fractions. The idea of proving convergence of continued fractions $K(a_n/b_n)$ by means of *value sets* $V \subseteq \hat{\mathbb{C}}$; i.e. $s_n(V) := a_n/(b_n + V) \subseteq V$ for all n, evolved in 1942. In that year, Leighton with his student Thron [26] and Wall with his student Paydon [33] came up with the same method: Inspired by the work of Scott and Wall [37], [38] they started with a value set $V \subseteq \mathbb{C}$, which actually was a half-plane, and considered continued fractions $K(a_n/1)$ such that $s_n(V) := a_n/(1+V) \subseteq V$ for all n. In this way they could make use of the important feature of nestedness connected with compositions (1.1)

$$F_n(V) = F_{n-1}(f_n(V)) \subseteq F_{n-1}(V) \subseteq \cdots \subseteq V. \tag{1.8}$$

Incidentally this proves that if $\operatorname{diam} F_n(V) \to 0$, then $\{F_n\}$ converges uniformly in V to a constant function c and $|F_n(z) - c| \leq \operatorname{diam} F_n(V)$. The question about convergence of (1.1) more generally, was asked by Thron [44] in 1961. His interest was mainly with continued fractions [41], infinite exponentials [42], infinite radicals [36] and linear fractional transformations in general [45]. In later years it was taken up on a more general basis by Gill [12], Baker and Rippon [2], Karlsson and Wallin [22] and the author [27–29].

Techniques to prove convergence.
Let $\{f_n\}$ be a sequence of functions mapping a subset V of $\hat{\mathbb{C}}$ into itself. Then there are several established techniques to prove that $\{F_n\}$ given by (1.1) converges in V to a constant function. We shall here describe three different ones.

1. The limit point technique. The idea here is to derive bounds for $\operatorname{diam} F_n(V)$ and to prove that these bounds converge to zero as $n \to \infty$. Or, alternatively, to derive bounds for a subsequence of $\operatorname{diam} F_n(V)$ and to prove that

$$\operatorname{diam} F_{n_k}(V) \leq M_k \to 0 \quad \text{as } k \to \infty. \tag{1.9}$$

The nestedness (1.8) then shows that $F_n(V)$ shrinks to a point. This is called the *limit point case*, and it implies that $\{F_n\}$ converges uniformly in V to a constant. This technique requires that V and $\{f_n\}$ are relatively "nice". On the other hand, it is very useful when it works, since it also gives an estimate for the speed of convergence.

2. The combination technique. To prove that $\{F_n\}$ converges in V it suffices to consider the case where we do not have the limit point case. The nestedness (1.8) shows that the limit set $\Delta := \lim_{n\to\infty} F_n(V)$ exists, but now it contains more than one point. This is a very special property of $\{F_n\}$ to possess. For some families of functions F_n it is so special that it leads to convergence of $\{F_n\}$ in V.

3. Normal families. The idea is to look for functions $\psi_n : U \times V \to \mathbb{C}$ such that

(i) The compositions Ψ_n given by

$$\Psi_1(w,z) := \psi_1(w,z), \quad \Psi_n(w,z) := \Psi_{n-1}(w, \psi_n(w,z))$$

are analytic in U for every fixed $z \in V$.

(ii) There is a $w_0 \in U$ such that $\psi_n(w_0, z) = f_n(z)$ for all $z \in V$ and all n. This makes $\Psi_n(w_0, z) = F_n(z)$.

(iii) $\Psi_n(w,z)$ converges to a constant function of z for every $w \in D$, where $D \subseteq U$ is an infinite set with at least one point of accumulation in U.

The convergence can then be extended to all $w \in U$. In particular $F_n(z) = \Psi_n(w_0, z)$ converges to a constant function.

Structure of this paper.

This paper is a survey paper. The results are taken from different areas of mathematics, in particular from the theory of continued fractions where the emphasis has been on $\{F_n(0)\}$ rather than $\{F_n(z)\}$. We have taken the liberty of adjusting the notation of such results to match the one in this paper, and in some cases to extend the results to our setting if the original proofs allow this extension or if it comes about very naturally.

The choice of results included is a personal one. To be able to bring out the ideas, we have focused on simple results, void of technicalities. More results can be found in the references, and in their references again.

We shall consider different families $\mathcal{F}$ of self-mappings. In Section 2, $\mathcal{F}$ contains linear fractional transformations mapping a set $V \subseteq \widehat{\mathbb{C}}$ into itself. Working with these nice functions gives us an idea of what to expect in more general cases. It is also a very interesting case for its own sake.

In Section 3, $\mathcal{F}$ contains analytic self–mappings of U more generally. Finally, in Section 4 we mention possibilities for more general cases.

We shall mostly concentrate on the compositions (1.1), but there are strong connections between the two types of compositions (1.1) and (1.2). For instance, if $\operatorname{diam} F_n(U) \to 0$ uniformly with respect to $\{f_n\}$ from $\mathcal{F}$, then also $\operatorname{diam} G_n(U) \to 0$. That is, every convergent subsequence of $\{G_n\}$ converges uniformly in U to a constant function in this case. In the special case where $\mathcal{F}$ consists of linear fractional transformations, there are particularly strong connections between the two. (See Section 2.)

Notation.

We shall use the notation as already introduced. In particular f and f_n shall mean functions from the family $\mathcal{F}$, and F_n shall denote their compositions (1.1). Let $\overline{V}$, V° and ∂V denote the closure, the interior and the boundary of a set V in $\widehat{\mathbb{C}}$, resp., whereas $B(c,r)$ denotes the open disk with center at $c \in \mathbb{C}$ and radius $r > 0$. The euclidean distance between two sets $A, B \subseteq \widehat{\mathbb{C}}$ is denoted by $\operatorname{dist}(A,B)$. The derivative of an analytic function f is denoted by f'. $\operatorname{Re} z$, $\operatorname{Im} z$ and $|z|$ denote the real part, the imaginary part and the modulus of a complex number z, resp..

2. Linear fractional transformations

Let V be a subset of the extended complex plane $\widehat{\mathbb{C}}$, and let $\mathcal{F}$ be a family of linear fractional transformations mapping V into V. This is an important special case. It is also a very nice special case. Linear fractional transformations (we require that they are non–singular by definition) are practical functions to work with, since they are analytic, univalent mappings of $\widehat{\mathbb{C}}$ onto $\widehat{\mathbb{C}}$ whose compositions and inverse functions are also linear fractional transformations; i.e., they form a group. In fact, if $F_n := f_1 \circ f_2 \circ \cdots \circ f_n$ are compositions of linear fractional transformations of type (1.1), then $F_n^{-1} = f_n^{-1} \circ f_{n-1}^{-1} \circ \cdots \circ f_1^{-1}$ are compositions of linear fractional transformations of type (1.2), where f_k^{-1} maps $\widehat{\mathbb{C}} \setminus V$ into $\widehat{\mathbb{C}} \setminus V$.

Another advantage of this class of functions is that if $\{t_n\}$ is a sequence of linear fractional transformations converging in a domain D, then its limit function is either a linear fractional transformation or a constant function in D (possibly minus a point) [34, Thm 1]. It is also clear that $\mathcal{F}$ is a closed set in the natural metric in the field of linear fractional transformations.

Finally, linear fractional transformations have very nice mapping properties: They map generalized circles onto generalized circles. Actually, almost all the early results in this area are based on V being a circular domain, i.e., either a halfplane, a circular disk or the complement of a circular disk. This is of course a severe restriction when $\mathcal{F}$ contains only linear fractional transformations. On the other hand, it leads to very useful convergence results for continued fractions, and we learn a lot about what to expect in more general cases. So let us first look a little closer at this situation.

V is a circular domain and $f(\infty) = 0$ for $f \in \mathcal{F}$.
When $f(\infty) = 0$, we are restricted to the continued fraction generating linear fractional transformations $f(z) = a/(b+z)$ (we can always normalize to have the coefficient 1 for the z in the denominator), and the results in this section are taken from the theory of continued fractions. Typical in this respect is the parabola theorem which was the first convergence theorem for continued fractions to be proved by this technique. In our setting it may be formulated as follows.

Theorem 2.1. (The Parabola Theorem) *Let V_α be the half-plane $\mathrm{Re}(ze^{-i\alpha}) > -\frac{1}{2}\cos\alpha$ for a fixed angle $\alpha \in \mathbb{R}$, $|\alpha| < \pi/2$, and let $\mathcal{F}_\alpha$ be the family of linear fractional transformations $f(z) = a/(1+z)$, $a \in \mathbb{C} \setminus \{0\}$, mapping V_α into V_α. Let $\{f_n\}$ be a sequence from $\mathcal{F}_\alpha$, and let $\{F_n\}$ be given by* (1.1). *Then the following hold.*

A. *$f \in \mathcal{F}_\alpha$ if and only if $a \in P_\alpha \setminus \{0\}$, where*

$$(2.1)\quad P_\alpha := \{z \in \mathbb{C} : \ |z| - \mathrm{Re}(ze^{-2i\alpha}) \le \tfrac{1}{2}\cos^2\alpha\}; \quad [38], [33], [26].$$

B. *$\{F_{2n}\}_{n=1}^{\infty}$ and $\{F_{2n+1}\}_{n=1}^{\infty}$ converge locally uniformly in V to constant functions.*

C. *$\{F_n\}_{n=1}^{\infty}$ converges locally uniformly in V to a constant function if and only if*

$$(2.2)\qquad \sum_{n=1}^{\infty}\prod_{k=1}^{n}|a_k|^{(-1)^{n+k+1}} = \infty.$$

The convergence of $\{F_n\}$ is uniform in $\overline{V}$ if $\alpha = 0$, [33].

D. $\mathrm{diam}\, F_n(V) \le d_n := \dfrac{2|a_1|/\cos\alpha}{\prod_{k=2}^{n}\left(1 + \frac{\cos^2\alpha}{4(k-1)|a_k|}\right)}$, [43, Formula (3.3)].

(It is still an open question whether the convergence in part C is uniform in $\overline{V}$ in general if $\alpha \neq 0$.) The set P_α is a closed domain in $\mathbb{C}$ containing 0, whose boundary is a parabola through $-\frac{1}{4}$ with focus at the origin and axis along the ray $\arg z = 2\alpha$. P_α is called the element set corresponding to V_α, and it characterizes the functions $f \in \mathcal{F}_\alpha$. The emphasis has been until recently on proving that $K(a_n/1)$ converges, i.e., $F_n(0) \to c$, or on proving that the even and odd parts of $K(a_n/1)$ converge, i.e., $F_{2n}(0) \to c$ and

$F_{2n+1}(0) \to c^*$. It does not take much to extend such results to convergence of $\{F_n(z)\}$, or $\{F_{2n}(z)\}$ and $\{F_{2n+1}(z)\}$, for $z \in V$, though. We shall do so here, and thus prove the parts B and C of Theorem 2.1 by using two simple, but useful results from [31]. The first one is:

Theorem 2.2. ([31, Thms 4.5, 5.1].) *Let $V \neq \emptyset$ be an open, bounded subset of $\mathbb{C}$, and let $\varepsilon > 0$ be given. Let $\mathcal{F} = \mathcal{M}_\varepsilon(V)$ be the family of linear fractional transformations f mapping V into V such that*

$$f(V) \subseteq V \setminus B(z_f, \varepsilon) \qquad \text{for some } z_f \in \overline{V}. \tag{2.3}$$

Let $\{f_n\} \subseteq \mathcal{F}$ and F_n be given by (1.1). Then $F_n'(z) \to 0$ for $z \in V$, and one of the following three situations occur:

A. $\{F_n(z)\}$ converges uniformly in $\overline{V}$ to a constant function.

B. $\{F_n(z)\}$ converges locally uniformly in $\widehat{\mathbb{C}} \setminus \partial V$ to a constant function.

C. $\{F_n(z)\}$ diverges at every $z \in \widehat{\mathbb{C}} \setminus \partial V$.

The letter $\mathcal{M}$ is chosen for these families of functions, since all linear fractional transformations form the well–known Möbius group. It is clear that $\mathcal{M}_\varepsilon(V) = \emptyset$ if $\widehat{\mathbb{C}} \setminus \overline{V} = \emptyset$. Theorem 2.2 gives a picture of what goes on when $\mathcal{F} \subseteq \mathcal{M}_\varepsilon(V) \neq \emptyset$:

1. If $F_n(0) \to c$ and $0 \in V$, then $\{F_n(z)\}$ converges locally uniformly to c in V.
2. Every convergent subsequence of $\{F_n\}$ converges to a constant function in V.
3. If V is unbounded and $\widehat{\mathbb{C}} \setminus \overline{V} \neq \emptyset$, then a linear fractional transformation φ can always be applied to make $V^* := \varphi(V)$ bounded. If f maps V into V, then $f^* := \varphi \circ f \circ \varphi^{-1}$ maps V^* into V^*. Hence, if we are dealing with linear fractional transformations, disregarding possible normalizations, then we may always assume that V is bounded.
4. If $\{F_n^{-1}(\infty)\}$ has a limit point $\alpha \notin \partial V$, then there is a subsequence $F_{n_k}^{-1}(\infty) \to \alpha$, and so $F_{n_k}'(z) \to 0$ uniformly in $\overline{V}$ if V is bounded. ($\alpha \notin V^\circ$ if V is bounded.) Hence diam $F_{n_k}(\overline{V}) \to 0$. The nestedness (1.8) then proves that we have the *limit point case*, i.e., case A in Theorem 2.2 occurs.
5. Assume that V is a circular disk and that the limit point case does not occur. This is called the *limit circle case*, since then $\gamma := \lim \partial F_n(V)$ is a circle with positive radius. Then $\{F_n^{-1}(\infty)\}$ has all its limit points in ∂V, and every limit function of $\{F_n\}$ is constant for all $z \in \widehat{\mathbb{C}} \setminus \{\text{the limit points of } \{F_n^{-1}(\infty)\}\}$. By the reflection principle it thus follows that every limit function F of $\{F_n\}$ has a value $F(z) \equiv c \in \gamma$.

In view of this, it is interesting to determine whether our family $\mathcal{F}$ of functions does belong to some $\mathcal{M}_\varepsilon(V)$:

Theorem 2.3. ([31, Thms 1.3 and 4.3].) *Let V be an open subset of $\widehat{\mathbb{C}}$, and let $\mathcal{F}$ be the family of linear fractional transformations $f(z) = a/(1+z)$ mapping V into V. If $\mathcal{F}$ contains more than one function, then there exists an $\varepsilon > 0$ such that (2.3) holds for every $f \in \mathcal{F}$.*

Our set V is not bounded, but that is easily remedied:

Proof of Theorem 2.1 B, C:
B. Scott and Wall [37] proved that $\{F_{2n}(0)\}$ and $\{F_{2n+1}(0)\}$ converge to finite values if $\alpha = 0$. That it holds for general α can for instance be seen in [32, Thm III.20, p.130]. V is unbounded, but as in Remark 3 above, $V^* := \varphi(V)$ is bounded for appropriately chosen φ. Since $\partial f_n(V)$ is a circle passing through $0 \in V^\circ$ for all n, there exists an $\varepsilon > 0$ such that $f_n^* := \varphi \circ f_n \circ \varphi^{-1} \in \mathcal{M}_\varepsilon(V^*)$ for all n. The result follows therefore from Theorem 2.2.

C. Thron [40] proved that $\{F_n(0)\}$ converges if and only if (2.2) holds. Hence the result follows as above. □

The parabola theorem showed to be very useful indeed in the theory of continued fractions. If in particular all $|a_n| \leq M$ for some $M > 0$, then (2.2) holds trivially, and Theorem 2.1D shows that

$$\text{(2.4)} \qquad \operatorname{diam} F_n(\overline{V}) \leq d_n := \frac{2M/\cos\alpha}{\prod_{k=1}^{n-1}(1 + \frac{\cos^2\alpha}{4Mk})},$$

where $\{d_n\}$ is independent of $\{a_n\}$ and converges to 0. Hence we are in the limit point case, and $\operatorname{diam} F_n(V_\alpha) \to 0$ uniformly with respect to $\{f_n\}$ from $\mathcal{F}_{\alpha,M} := \{f \in \mathcal{F}_\alpha : |f(0)| \leq M\}$. That is, the continued fractions $K(a_n/1)$ with all $a_n \in P_\alpha$, $|a_n| \leq M$, converge, uniformly with respect to $\{a_n\}$. Or, in the terminology of Karlsson and Wallin, $(\mathcal{F}_{\alpha,M}, V_\alpha)$ is a d_n-stable generalized dynamical system. Of course, the same holds true if all $|a_n| \leq M_n$ for some positive sequence $\{M_n\}$ such that $\prod_{k=1}^\infty(1+1/M_k) = \infty$; i.e., $\sum 1/M_k = \infty$, and (2.4) is changed in the obvious manner.

The picture gets simpler if we choose V to be a circular disk, but keep the form of the linear fractional transformations in $\mathcal{F}$. In particular we get uniform convergence in $\overline{V}$:

Theorem 2.4. (The cartesian oval theorem) *Let V be a circular disk with $-1 \notin \overline{V}$ and center Γ, $\operatorname{Re}\Gamma > -\frac{1}{2}$. Let $\mathcal{F}$ be the family of functions $f(z) = a/(1+z)$ mapping V into V. Let $\{f_n\} \subseteq \mathcal{F}$ and F_n be given by* (1.1). *Then the following hold with $\alpha := \arg(\Gamma + \frac{1}{2})$:*

A. *The functions in $\mathcal{F}$ are characterized by*

$$0 \neq a \in E(\Gamma, R) := \left\{z : |z-u| + \frac{R}{|1+\Gamma|}|z| \leq \frac{R}{|1+\Gamma|}(|1+\Gamma|^2 - R^2)\right\},$$

[19, formula (3.2)], *where R is the radius of V and*

$$u := \Gamma(1+\Gamma)(1 - R/|1+\Gamma|^2).$$

B. *$E(\Gamma, R) \subseteq P_\alpha$ given by* (2.1), [24].

C. *$\{F_n\}$ converges uniformly in $\overline{V}$ to a constant function,* [31, Thm 2.3].

D. $\operatorname{diam} F_n(\overline{V}) \leq \dfrac{2M}{\cos\alpha}\left(\dfrac{1+c}{1+nc}\right)^c$ *for $n \geq 0$,* [24, Thm 2.1], *where $M := (|\Gamma| + R)(|1+\Gamma| - R)$ and $c := (\cos^2\alpha)/(4M)$.*

E. *$\{F_n\}$ converges locally uniformly in $\overline{V}_\alpha \cup [\widehat{\mathbb{C}} \setminus \overline{B}(-1-\Gamma, R)]$, where V_α is the halfplane in* Theorem 2.1.

The set $E(\Gamma, R)$ is a closed region bounded by a cartesian oval. It remains to prove part E. But this follows from the following simple observation:

Lemma 2.5. ([31, Thm 3.1].) *Let $V \subseteq \widehat{\mathbb{C}}$ be such that $V^\circ \neq \emptyset$ and $W := \widehat{\mathbb{C}} \setminus (-1 - \overline{V}) \neq \emptyset$. Then $f(z) := a/(1+z)$ maps V into V if and only if f maps W into W.*

This means that in general we may extend the convergence of $\{F_n(z)\}$ for $z \in V^\circ$ to locally uniform convergence of $\{F_n(z)\}$ for $z \in V^\circ \cup W$ by use of Theorem 2.2, possibly after moving V to a bounded $V^* := \varphi(V)$). In particular, if V is bounded, then $\infty \in W$, and thus, if $\{F_n(z)\}$ converges locally uniformly in V° to a constant value, then $F_n(0) = F_{n+1}(\infty)$ converges to the same value, regardless of whether $0 \in V$ or not.

There exist many results where V is a circular domain of some sort and the functions in $\mathcal{F}$ have the form $f(z) = a/(1+z)$, or more generally, $f(z) = a/(b+z)$. For historical reasons I want to mention three very classical results in continued fraction theory. Adjusted to our set-up, they can be formulated as follows:

Theorem 2.6. (Worpitzky 1865 [48]) *Let $V := B(0, \frac{1}{2})$, and let $\mathcal{F}$ be the family of linear fractional transformations $f(z) = a/(1+z)$ mapping V into V. Let $\{f_n\} \subseteq \mathcal{F}$ and F_n be given by (1.1). Then $f \in \mathcal{F}$ if and only if $a \in \overline{B}(0, \frac{1}{4}) \setminus \{0\}$, and $\{F_n\}$ converges uniformly in $\overline{V}$ to a constant function. The convergence is uniform with respect to $\{f_n\}$ from $\mathcal{F}$.*

This work by the high-school teacher Worpitzky is the earliest work we know about, proving convergence of continued fractions with complex elements. Today this is just a consequence of the parabola theorem (or the cartesian oval theorem) since $\overline{B}(0, \frac{1}{4}) \subseteq P_0$ given by (2.1).

Theorem 2.7. (Śleszyński 1889 [39]) *Let $\mathcal{F}$ be the family of linear fractional transformations $f(z) = a/(b+z)$ mapping U into U. Let $\{f_n\} \subseteq \mathcal{F}$ and F_n be given by (1.1). Then $\{F_n(0)\}$ converges for every $\{f_n\} \subseteq \mathcal{F}$.*

This theorem by the Polish mathematician Śleszyński has been known in the West as Pringsheim's theorem. It was rather recently discovered by W.J.Thron and A.Magnus that Śleszyński had already published his result when Pringsheim found his. The reason for this confusion is probably that Śleszyński wrote his paper in Russian, a language which was not so widely spoken in the West. By Theorem 2.2 it follows that $\{F_n\}$ converges locally uniformly in U to a constant function under Śleszyński's conditions. (Śleszyński's family $\mathcal{F} \subseteq \mathcal{M}_1(U)$ since $0 \notin f(U)$ for all $f \in \mathcal{F}$.) The question of whether $\{F_n\}$ converges uniformly in U is still unsettled.

In the third result, V is the intersection of two halfplanes:

Theorem 2.8. (Van Vleck 1901 [46]) *Let*

$$V := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \frac{\pi}{2} - \varepsilon\}$$

for some given $\varepsilon > 0$, and let $\mathcal{F}$ be the family of linear fractional transformations $f(z) = 1/(b+z)$ with $b = 1/f(0) \in \overline{V} \setminus \{\infty\}$. Let $\{f_n\} \subseteq \mathcal{F}$ and F_n be given by (1.1). Then the following hold.

A. *$f(V) \subseteq V$ for all $f \in \mathcal{F}$.*

B. *$\{F_{2n}(0)\}$ and $\{F_{2n+1}(0)\}$ converge to finite values.*

C. *$\{F_n(0)\}$ converges if and only if $\sum |b_n| = \infty$.*

In this case Theorem 2.2 does not apply since $0 \in \partial V$. However, Lange has recently proved the following results for the sequences in Van Vleck's theorem:

Theorem 2.8 continued. ([25, Thm 2.1])

D. *If* $\sum |b_n| = \infty$, *then* $\{F_n\}$ *converges uniformly in the halfplane* $\overline{H} := \{z :$ $\operatorname{Re} z \geq 0\}$, *and*

$$\text{diam } F_n(\overline{H}) \leq \frac{1}{\operatorname{Re} b_1} \prod_{k=2}^{n} P_k^{-1} \tag{2.5}$$

where $P_k := 1 + |b_k F_{k-1}^{-1}(\infty)| \cos(\frac{\pi}{2} - \varepsilon)$.

E. $\prod_{k=2}^{\infty} P_k$ *and* $\sum |b_k|$ *converge and diverge together.*

Hence the convergence in part C is extended to uniform convergence of $\{F_n\}$ in the half-plane $\overline{H}$ which contains V. Since $f(\overline{H}) \subseteq \overline{H}$ for all $f \in \mathcal{F}$, regardless of the value of $\varepsilon > 0$, it is tempting to analyze what happens if we extend $\mathcal{F}$ to contain all functions $f(z) = 1/(b + z)$ with $b \in \overline{H} \setminus \{\infty\}$. This was also done by Lange. By using a result [14, p.1193] by Gragg and Warner he proved the following generalization of Van Vleck's result.

Theorem 2.9. ([25, Thm 2.4]) *Let* V *be the half-plane*

$$H := \{z : \operatorname{Re} z > 0\}$$

and let $\mathcal{F}$ *be the family of linear fractional transformations* $f(z) = 1/(b+z)$ *mapping* V *into* V. *Let* $\{f_n\} \subseteq \mathcal{F}$ *and let* F_n *be given by* (1.1). *Then the following hold:*

A. *The family* $\mathcal{F}$ *is characterized by* $b \in \overline{V} \setminus \{\infty\}$.

B. *If* $\sum \sqrt{\phi_k} = \infty$, *where* $\phi_k := \operatorname{Re}(b_k) \cdot \operatorname{Re}(b_{k-1})$ *and* $b_k := 1/f_k(0)$, *then* $\{F_n\}$ *converges uniformly in* $\overline{V}$ *and*

$$\text{diam } F_n(\overline{V}) \leq \frac{2}{\phi_1} \prod_{k=2}^{n} \frac{1 - 2\sqrt{\phi_k}}{\sqrt{4 + \phi_k} + \sqrt{\phi_k}} \leq \frac{2}{\phi_1 \prod_{k=2}^{n}(1 + \sqrt{\phi_k})}. \tag{2.6}$$

V **is a circular domain and** $f(z) = (az + c)/(b + dz)$ **in** $\mathcal{F}$.

Without loss of generality we assume that the circular domain V is the open unit disk U. It is also customary to require that $ad - bc = 1$, which can be done without loss of generality. We shall, however, not do so here. Of course, Theorem 2.2 is still valid, so if $\mathcal{F} \subseteq \mathcal{M}_\varepsilon(V)$ for some $\varepsilon > 0$, then we know quite a lot about the composite sequence $\{F_n\}$. Actually, it also helps if $f^{-1} \in \mathcal{M}_\varepsilon(\tilde{V})$ for some $\varepsilon > 0$ where $\tilde{V} := \widehat{\mathbb{C}} \setminus V$. (See the section in "Connections between F_n and G_n. Orbits" below.) The following results are examples of what is known:

Theorem 2.10. (The Hillam–Thron Theorem [15], improved in [16]) *Let $0 < k < 1$ be a fixed constant, and let $\mathcal{F}$ be the family of linear fractional transformations f mapping U into U with $|f(\infty)| \leq k$. Let $\{f_n\}$ be a sequence from $\mathcal{F}$, and let $\{F_n\}$ be given by (1.1). Then $\{F_n\}$ converges locally uniformly in U to a constant function.*

This can be slightly improved:

Theorem 2.11. *Let $\mathcal{F}$ be the family of linear fractional transformations mapping U into U, and let $\{f_n\}$ be a sequence from $\mathcal{F}$ and $\{F_n\}$ be given by (1.1). If $\limsup |f_n(\infty)| < 1$, then $\{F_n\}$ converges locally uniformly in U to a constant function.*

Actually, as we shall see, Theorem 2.11 is a corollary of the following rather general result:

Theorem 2.12. ([21], [28, Thm 1.2]) *Let $\mathcal{F}$ be the family of linear fractional transformations mapping U into U, and let $\{f_n\}$ be a sequence from $\mathcal{F}$ and $\{F_n\}$ be given by (1.1). If there exists a sequence $\{z_n\} \subseteq \widehat{\mathbb{C}}$ such that*

$$\text{(2.7)} \qquad \liminf \Big| |z_n| - 1 \Big| > 0 \quad \text{and} \quad \liminf \Big| |f_n(z_n)| - 1 \Big| > 0,$$

and either $\limsup \operatorname{diam} f_n(U) < 2$ or $\liminf \operatorname{diam} f_n^{-1}(U) > 2$, then $\{F_n\}$ converges locally uniformly in U to a constant function.

Proof of Theorem 2.11. Since $f_n(U)$ is a circular disk $\subseteq U \setminus f_n(\infty)$, it is clear that $\limsup \operatorname{diam} f_n(U) < 2$. Moreover, (2.7) holds with the choice $z_n := \infty$ for all n. □

Theorem 2.12 requires that $f_n(U)$ is a proper subset of U from some n on, or that $f_n^{-1}(\widehat{\mathbb{C}} \setminus U)$ is a proper subset of $\widehat{\mathbb{C}} \setminus U$, which are equivalent statements. The uniformity of these conditions are not equivalent, though. Such properties seem to be natural to ask for. But they are not sufficient! The need for conditions of some type similar to (2.7) follows from the following example:

Example 2.13. Let $f_n(z) := a_n(1-z)/(1+a_n(1-z))$ for all n, where $a_n := n(n+2)$. Then $f_n(U) = \{z \in \mathbf{C} : |z - r_n| < r_n\}$ where $r_n := \frac{1}{2}(1 - 1/(1+2a_n))$. In other words, all $f_n(U) \subseteq \mathcal{D} := \{z \in \mathbf{C} : |z - \frac{1}{2}| < \frac{1}{2}\}$. Still, $F_n(z)$ does not converge for any $z \in U$, since $f_n = \varphi \circ s_n \circ \varphi^{-1}$ where $\varphi(z) = z/(1+z)$ and $s_n(z) = a_n/(1+z)$, and $S_n(z) := s_1 \circ s_2 \circ \cdots \circ s_n(z)$ diverges by Theorem 2.1 C.

On the other hand, condition (2.7) is not necessary:

Example 2.14. Let $f_n(z) := n(1-z)/(1+n(1-z))$ for all n, and let φ and $\mathcal{D}$ be as in the previous example. Then $f_n(U) \subseteq \mathcal{D}$, and $f_n = \varphi \circ s_n \circ \varphi^{-1}$ where $s_n(z) = n/(1+z)$ maps the half-plane $V_0 = \varphi^{-1}(U)$ into itself. Hence, $S_n := s_1 \circ s_2 \circ \cdots \circ s_n$ converges uniformly in $\overline{V}_0$ to a constant function by Theorem 2.1 C. Still, $\{f_n(z_n)\}$ has a limit point at 1 for every sequence $\{z_n\}$ which is not converging to 1.

Theorem 2.12 is a very general convergence result. Actually, the parabola theorem is a consequence of Theorem 2.10 which is a corollary of Theorem 2.12, and the convergence in the theorems 2.4, 2.6 and 2.7 can be seen as corollaries of the parabola theorem. Theorems 2.8 and 2.9 with their uniform convergence are different, though. This limit point case, where $\{F_n(\overline{U})\}$ shrinks to a point, is of particular importance. It is not known whether the conditions in Theorem 2.12 imply uniform convergence. The following result gives some sufficient conditions for this to occur. Example 2.14 shows that they are not necessary.

Theorem 2.15. ([28, Thm 1.4]) *Let $\mathcal{F}$ be the family of linear fractional transformations mapping the unit disk U into itself, and let $\{f_n\}$ be a sequence from $\mathcal{F}$ and F_n be given by* (1.1). *If either*

(i) $\sum d_n = \infty$, *where* $d_n := \operatorname{dist}(f_n(U), \partial U)$, *or*

(ii) $\sum \tilde{d}_n = \infty$, *where* $\tilde{d}_n := \operatorname{dist}(\partial f_n^{-1}(U), U)$, *or*

(iii) $\lim_{n\to\infty} r_n \dfrac{|\zeta_n|+1}{|\zeta_n|-1} \prod_{j=1}^{n-1} \kappa_j = 0$, *where*

$$\kappa_j := r_j \frac{|\zeta_j|^2 - 1}{(|\zeta_j - c_{j+1}| - r_{j+1})^2} \leq r_j \frac{|\zeta_j|+1}{|\zeta_j|-1},$$

r_j is the radius of $f_j(U)$ and c_j is its center, and $\zeta_j := f_j^{-1}(\infty)$,

then $F_n(w)$ converges uniformly in $\overline{U}$ to a constant function.

V is a more general subset of $\widehat{\mathbb{C}}$.

Let V be some subset of $\widehat{\mathbb{C}}$, and let $\mathcal{F}$ be a family of linear fractional transformations mapping V into V. Under what conditions will $\{F_n(z)\}$ converge to a constant function in V? As before, we pick $\{f_n\}$ from $\mathcal{F}$ and construct F_n by (1.1).

Evidently $\mathcal{F}$ must be empty or very thin if $V^\circ = \emptyset$. (See for instance [30, Prop 5.5].) Hence we assume here that $V^\circ \neq \emptyset$. Since $f(V) \subseteq V$ only if $f(V^\circ) \subseteq V^\circ$, we do not loose much by requiring V to be open. Finally, since $f(V) \subseteq V$ only if $f(\overline{V}) \subseteq \overline{V}$, we may assume that the boundary of V is "nice"; i.e., $(\overline{V})^\circ = V$.

To get an idea of what to expect, we look a little closer at what we obtained when V was a circular domain.

1. The half-plane V_α in Theorem 2.1 is not picked at random. If V is just any half-plane in $\mathbb{C}$, then the corresponding family $\mathcal{F}$ may be empty. Since $\infty \in \partial V$ when V is a half-plane, we need $0 \in \overline{V}$ and $-1 \notin V^\circ$ to get $\mathcal{F} \neq \emptyset$. Actually, to obtain the contraction we want, we need $0 \in V^\circ$ and $-1 \notin \overline{V}$. That is, V must be a half-plane $\text{Re}(ze^{-i\alpha}) > -p\cos\alpha$ for some $|\alpha| \leq \pi/2$ and $0 < p < 1$. If $|\alpha| = \pi/2$, then V is either the upper or the lower half-plane, and $a/(1+V) \subseteq V$ only if a is real and non-positive. Actually, $a/(1+V) = V$ when $a < 0$. Hence $|\alpha| < \pi/2$ is necessary. It follows then that $f \in \mathcal{F}$ if and only if $f(z) = a/(1+z)$ where

$$(2.8) \qquad |a| - \text{Re}(ze^{-i2\alpha}) \leq 2p(1-p)\cos^2\alpha, \qquad [43].$$

   That is, $\mathcal{F} \subseteq \mathcal{F}_\alpha$.

2. The circular disks V in Theorem 2.4 are the only circular disks which lead to families $\mathcal{F}$ containing more than one function. Since V is bounded, the condition $-1 \notin \overline{V}$ is needed to have $\mathcal{F} \neq \emptyset$. Straightforward computation shows that $\mathcal{F} = \emptyset$ if the center Γ of V has real part $< -\frac{1}{2}$, and that $\mathcal{F}$ only contains the function $u/(1+z)$ if $\text{Re}(\Gamma) = -\frac{1}{2}$.

3. The condition (2.2) in Theorem 2.1 is vital since every continued fraction $K(a_n/1)$ for which (2.2) fails, diverges by the Stern–Stolz theorem [12, p.94]. It is clear that (2.2) can only fail if $a_n \to \infty$. Moreover, $a_n \to \infty$ is only possible if the domain V we started with is unbounded.

4. The condition (2.5) in Theorem 2.12 is milder. It implies that $\{f_n\}$ is not allowed to have a subsequence $\{f_{n_k}\}$, where $f_{n_k}(z)$ is pushed towards a boundary point $c \in \partial U$ for almost all $z \in \widehat{\mathbb{C}}$ as $k \to \infty$. Or, that $\{f_n\}$ does not have a subsequence $\{f_{n_k}\}$ where $f_{n_k}^{-1}(z)$ is pushed towards a boundary point $c \in \partial U$ for almost all $z \in \widehat{\mathbb{C}}$ as $k \to \infty$. In particular, the poles should stay away from ∂U.

In view of this, it seems natural to believe that if V is a bounded set and $\mathcal{F}$ consists of functions $f(z) = a/(1+z)$ mapping V into a proper subset of V, then $\{F_n(z)\}$ converges to a constant function in V since the poles -1 of f are bounded away from $\overline{V}$ when V is bounded. The author suggested this in a talk in Luminy, France in 1989, and it was put in writing in a paper by Ruscheweyh and the author in 1993 [31]. In view of Theorem 2.3, our conjecture can be formulated as follows:

Conjecture. ([31]) *Let $V \neq \emptyset$ be an open, bounded subset of $\mathbb{C}$, and let $\mathcal{F}$ be the family of linear fractional transformations $f(z) = a/(1+z)$ mapping*

V into V. If $\mathcal{F}$ contains more than one function, then $\{F_n\}$ converges locally uniformly in V to a constant.

Possibly, what we have is even uniform convergence in $\overline{V}$? Anyway, this conjecture is still open. What we proved in [31] was a result which excludes all the difficult cases. In the present setting it can be formulated as follows:

Theorem 2.16. ([31, Thm 1.1]) *Let $V \neq \emptyset$ be an open, bounded subset of $\mathbb{C}$, and let the family $\mathcal{F}$ of linear fractional transformations $f(z) = a/(1+z)$ mapping V into V contain at least two functions. Let $\{f_n\}$ be a sequence from $\mathcal{F}$. If $\{f_n\}$ has at least one limit point f with the property that*

$$(2.9) \qquad f(\partial^* V) \cap \partial^* V = \emptyset, \quad \text{where } \partial^* V := \partial V \cap (-1 - \partial V),$$

then $\{F_n(z)\}$ converges uniformly in $\overline{V}$ to a constant function.

The reason why (2.9) excludes all the difficult cases becomes evident by the following observation which was part of the arguments in [31]:

Lemma 2.17. *Let $V \neq \emptyset$ be an open, bounded subset of $\widehat{\mathbb{C}}$, and let $\mathcal{F}$ be the family of linear fractional transformations $f(z) = a/(1+z)$ mapping V into V. Let $\{f_n\} \subseteq \mathcal{F}$ and F_n be given by (1.1). If $\mathcal{F}$ contains more than one function, then $\{F_n^{-1}(\infty)\}$ has all its limit points in $-1 - \overline{V}$.*

According to (2.9) $\{F_n^{-1}(\infty)\}$ has a limit point $\zeta \notin \overline{V}$. So, if $F_{n_k}^{-1}(\infty) \to \zeta$, then $F_{n_k}(z)$ is analytic in some domain V^* containing $\overline{V}$ from some k on, and the result is evident by the nestedness (1.8) and the fact that $F_n'(z) \to 0$ uniformly in V. For completeness we give a proof of Lemma 2.17:

Proof. By Theorems 2.2 and 2.3 we know that $F_n'(z) \to 0$ for $z \in V$. Since $F_n(V) \subseteq V$ we thus can write

$$F_n(z) = \begin{cases} P_n + Q_n/(z - Z_n) & \text{if } Z_n := F_n^{-1}(\infty) \neq \infty, \\ P_n + Q_n z & \text{if } Z_n = \infty, \end{cases}$$

where $\{P_n\}$ has all its limit points in $\overline{V}$ and $0 \neq Q_n \to 0$. This gives

$$(2.10) \qquad F_n^{-1}(z) = \begin{cases} Z_n + Q_n/(z - P_n) & \text{if } Z_n \neq \infty, \\ (z - P_n)/Q_n & \text{if } Z_n = \infty. \end{cases}$$

Since $f(z) = -1 - f^{-1}(-1 - z)$ for all $f \in \mathcal{F}$, it follows that $f^{-1}(-1 - V) \subseteq -1 - V$, and thus $F_n^{-1}(-1 - \overline{V}) \subseteq -1 - \overline{V}$. Hence $Z_n \neq \infty$ from some n on by (2.10), and thus $F_n^{-1}(z_n) - Z_n \to 0$ when $\{z_n\} \subseteq -1 - \overline{V}$ is chosen such that $\liminf |z_n - P_n| > 0$. □

Condition (2.9) is restrictive. It is not satisfied in Theorem 2.1, since there $f(V_\alpha)$ is a circular disk touching $\partial V_\alpha = \partial^* V_\alpha$ whenever $a \in \partial P_\alpha \setminus \{\infty\}$. In an attempt to get rid of condition (2.9), Córdova proved the following result in his thesis:

Theorem 2.18. ([8, Thm 5.1]) *Let V be a Jordan domain with $V = \mathbb{C} \setminus (-1 - \overline{V})$, $0 \in V$, $-\frac{1}{2} \in \partial V$, $\infty \in \partial V$ and a smooth boundary with continuous curvature. Moreover, the following two conditions hold for every $\zeta \in \partial V$:*

(i) $[-\zeta^2/(1 + \partial V)] \cap \partial V = \{\zeta\}$.

(ii) $\kappa(-\zeta^2/(1 + \partial V), \zeta) > \kappa(\partial V, \zeta)$, *where $\kappa(\Gamma, z)$ denotes the signed curvature of a curve Γ at the point $z \in \Gamma$, and the sign is chosen so that $\kappa(\partial V, \zeta) > 0$ if ∂V curves inwards towards V at ζ and is non-positive otherwise.*

Then the following hold.

A. $f(z) = a/(1 + z)$ *maps V into V if and only if $a = -p^2$ where $p \in \overline{V} \cap (-\overline{V}) \setminus \{\infty\}$.*

B. *Let $\{p_n\}$ be a bounded sequence from $\overline{V} \cap (-\overline{V})$, and let $f_n(z) := -p_n^2/(1 + z)$ and F_n be given by (1.1). Then $\{F_n\}$ converges uniformly in $\overline{V}$ to a constant function.*

The important thing here is that $\partial^* V$ defined by (2.9) is identical with ∂V, as in Theorem 2.1, and that $f(z) := -\zeta^2/(1 + z)$ maps V into V whenever $\zeta \in \partial V$. Hence we get convergence, regardless of whether (2.9) holds or not. But the conditions which allow us to remove (2.9) are rather restrictive. They have been weakened considerably in a number of theorems in [30], but also these theorems are rather technical.

Lange [23] has given an example of a region V satisfying Theorem 2.18: In his trancendental strip region result, he has V bounded by the curve

$$z = z_d(t) := -\tfrac{1}{2} - \tfrac{2d}{\pi} \tan^{-1} t + it \quad \text{for } -\infty < t < \infty \tag{2.11}$$

for some real constant d, $-\frac{1}{2} \leq d \leq \frac{1}{2}$. He has also considered Jordan domains $V := V(\alpha)$ with $0 \in V(\alpha)$ and boundary

$$z = z(t) := \begin{cases} -1 + e^{i\alpha}(\frac{1}{2} + i(t-1)) & \text{for } t \geq 1, \\ -1 + \frac{1}{2}e^{i\alpha t} & \text{for } 0 \leq t \leq 1, \\ -\frac{1}{2}e^{-i\alpha t} & \text{for } -1 \leq t \leq 0, \\ e^{i\alpha}(-\frac{1}{2} + i(t+1)) & \text{for } t \leq -1, \end{cases} \tag{2.12}$$

where $0 \leq \alpha < \pi/3$ is a fixed constant. Or, similarly, V is equal to the complex conjugate of $V(\alpha)$. For these he proved [23] that Theorem 2.18 A holds. He also proved that Theorem 2.18 B holds if $\{p_n\} \subseteq \overline{V} \cap (-\overline{V}) \setminus \{\infty\}$ is picked such that $f(V) \subseteq K$ where K is a compact subset of the open domain V. Moreover, he proved that if $\{p_n\} \subseteq \overline{V} \cap (-\overline{V})$ satisfies $\inf \operatorname{dist}(f_n(\partial V), \partial V) > 0$, then $\{F_n(0)\}$ converges if and only if $a_n := -p_n^2$ satisfies (2.2). That is, by Theorem 2.2 we have locally uniform convergence in V in these cases. In [30, Thms 11.3, 11.4] it was proved that if V is the region with boundary (2.11) and $\{p_n\} \subseteq \overline{V} \cap (-\overline{V})$ is bounded, then $\{F_n\}$ converges uniformly in $\overline{V}$, regardless of whether $f_n(\partial V) \cap \partial V = \emptyset$ or not. Moreover, if ∂V is given by (2.12), then $\{F_n\}$ converges locally uniformly in V.

Connections between F_n and G_n. Orbits.

As already pointed out in the introduction, we know that if $\operatorname{diam} F_n(\overline{V}) \to 0$ uniformly with respect to $\{f_n\}$ from $\mathcal{F}$, then also $\operatorname{diam} G_n(\overline{V}) \to 0$ uniformly with respect to $\{f_n\}$ from $\mathcal{F}$. However, this does not normally lead to convergence, since $\{G_n(\overline{V})\}$ are not necessarily nested sets, as $\{F_n(\overline{V})\}$ are by (1.8).

But since we are dealing with linear fractional transformations, we have other types of connections between compositions of the form (1.1) and the form (1.2). If $f(V) \subseteq V$, then $f^{-1}(\tilde{V}) \subseteq \tilde{V}$ where $\tilde{V} := \widehat{\mathbb{C}} \setminus V$. So, if $\mathcal{M}(V)$ denotes the family of linear fractional transformations mapping V into itself, then $f \in \mathcal{M}(V)$ if and only if $f^{-1} \in \mathcal{M}(\tilde{V})$. Moreover, if $\{f_n\} \subseteq \mathcal{M}(V)$ and $F_n := f_1 \circ f_2 \circ \cdots \circ f_n$, then $\{f_n^{-1}\} \subseteq \mathcal{M}(\tilde{V})$ and $G_n := f_n^{-1} \circ f_{n-1}^{-1} \circ \cdots \circ f_1^{-1} = F_n^{-1}$.

In [20] we defined *restrained* sequences $\{T_n\}$ of linear fractional transformations; that is, there exists a pair $\{u_n\}$, $\{v_n\}$ of sequences from $\widehat{\mathbb{C}}$ such that

$$(2.13) \qquad \lim_{n\to\infty} d(T_n(u_n), T_n(v_n)) = 0 \quad \text{and} \quad \liminf_{n\to\infty} d(u_n, v_n) > 0,$$

where $d(\cdot\,,\cdot)$ is the chordal metric on the Riemann sphere $\widehat{\mathbb{C}}$. Clearly, by Theorem 2.2 we are mostly dealing with restrained sequences in this section. We proved in [20, Thm 2.6] that $\{T_n\}$ is restrained if and only if $\{T_n^{-1}\}$ is restrained.

In continued fraction theory the orbits $g_{n-1} = f_n(g_n)$ are called tail sequences for a continued fraction $K(a_n/b_n)$; $f_n(z) = a_n/(b_n + z)$. These sequences play an essential role in the theory. Clearly, $g_n = f_n^{-1}(g_{n-1})$ and $g_0 = F_n(g_n)$. If $\{g_n\}$ is such an orbit for $f_n(z) = (a_n + c_n z)/(b_n + d_n z)$ with all $g_n \neq 0$, then

(2.14)
$$\frac{1}{F_N(z)-g_0}=\sum_{j=1}^{N}\frac{d_j}{b_j+d_jg_j}\prod_{k=1}^{j}P_k+\frac{1}{z-g_N}\prod_{k=1}^{N}P_k$$
$$\text{where } P_k:=\frac{(b_k+d_kg_k)^2}{c_kb_k-a_kd_k} \quad \text{for } N=1,2,3,\ldots; \text{ [17, Thm 2.3].}$$

Moreover, if in addition also $\{g_n^*\}$ is such a tail sequence for $K(a_n/b_n)$ with all $g_n^* \neq 0$, then

(2.15)
$$\frac{F_N(z)-g_0^*}{F_N(z)-g_0}=\frac{z-g_N^*}{z-g_N}\prod_{j=1}^{N}\frac{b_j+d_jg_j}{b_j+d_jg_j^*} \quad \text{for } N=1,2,3,\ldots; \quad \text{[17, Thm 2.3].}$$

This allows us to derive new orbits $\widehat{g}_n := F_n^{-1}(\widehat{g}_0)$ from old ones by solving (2.14) or (2.15) for $z=\widehat{g}_N$. Moreover, we get the other type of orbits, $\bar{g}_{n+1} := F_n(\bar{g}_1)$ from $\{g_n\}$ by setting $z := \bar{g}_1$ in (2.14) or (2.15), and solve for $F_N(z)$.

Linear frational transformations in higher dimensions.
A number of the results mentioned has been extended to linear fractional transformations in higher dimensions in a series of papers by B. Aebischer. Here we just refer to [1], where the emphasis is on stability properties useful in dynamical systems.

3. Analytic selfmappings of U

In this section we present some of the beautiful works by Baker and Rippon. Let first $\mathcal{F}$ consist of analytic and *univalent* selfmappings of U. For this particular class they gave a particularly nice result. It uses the notation $\widehat{A}$ for the complement of the unbounded component of $\mathbb{C}\setminus\overline{A}$ for a bounded set $A\subset\mathbb{C}$. By exploiting the subadditivity of the modulus of ring domains for such functions, they obtained:

Theorem 3.1. ([2, Thm 3.1]) *Let $\{f_n\}$ be a sequence of analytic, univalent self-mappings of U, such that the modulus M_n of the ring domain $U\setminus\widehat{f_n(U)}$ satisfies $\sum M_n=\infty$. Then $\{F_n\}$ given by (1.1) converges uniformly in U to a constant function.*

This corresponds in many ways to Theorem 2.15 with condition (i). Actually, in view of Theorem 2.15 it is tempting to ask if maybe $\{F_n\}$ still converges uniformly in U if M_n is replaced by $d_n := \text{dist}(f_n(U), \partial U)$? Maybe, if so, the univalence is not really needed?

Theorem 3.1 is just one of several nice results by Baker and Rippon. In the next one, the univalence is no longer needed:

Theorem 3.2. ([2, Thm 2.2]) *Let $\mathcal{F}$ be a family of functions f, analytic in U and mapping U into U with the following properties:*

(i) *No sequence of the form $\mu_k := f_{1,k} \circ f_{2,k} \circ \cdots \circ f_{n(k),k}$, all $f_{i,k} \in \mathcal{F}$ converges in U to a constant limit $c \in \partial U$.*

(ii) *No sequence $\{\mu_k\}$ as above, converges to the identity function in U.*

Then $\{F_n\}$ converges locally uniformly in U to a constant function.

The conditions here are not so easy to check. Condition (ii) clearly holds if

$$(3.1) \qquad f(U) \subseteq U \setminus B(z_f, \varepsilon) \quad \text{for some } z_f \in \overline{U} \text{ for all } f \in \mathcal{F}$$

for some constant $\varepsilon > 0$, but this does not imply (i). The danger of having such accumulation points on the boundary, as avoided by (i), is that we do not have control on the boundary. So, in a way, (i) can be compared to condition (2.9) in Theorem 2.16, where $\partial^* V$ was the problem part of $\overline{V}$. Condition (i) is also related to (2.7) which requires that no subsequence of $\{f_n\}$ converges to a constant limit $\in \partial U$.

The advantage of Theorem 3.2 is that we may assumeare allowed $\partial f(U) \cap \partial U \neq \emptyset$ for all $f \in \mathcal{F}$. If we renounce this advantage and require that $\text{dist}(\partial F(U), \partial U) \geq \varepsilon$ for some constant $\varepsilon > 0$, then both (3.1) and condition (i) hold, and we actually get:

Theorem 3.3. ([2, Cor 2.3], [27, Thm 1.2]) *Let $\mathcal{F}$ be a family of functions f analytic in U and mapping U into a compact set $K \subseteq U$. Then $\{F_n\}$ converges uniformly in U to a constant function. The convergence is also uniform with respect to $\{f_n\}$ from $\mathcal{F}$.*

The uniformity in Theorem 3.3 was established in [27], independently of the work by Baker and Rippon. Explicit bounds for diam $F_n(U)$ were also given in [27]. If $\{f_n\}$ is a sequence of analytic self-mappings from U which has a subsequence $\{f_{n_k}\}$ of functions mapping U into K, then $\{F_n\}$ still converges uniformly in U to a constant function by the nestedness (1.8). However, we loose the uniformity with respect to $\{f_n\}$ from $\mathcal{F}$, of course.

Baker and Rippon have also proved a theorem on contraction maps f, i.e., $|f'(z)| \leq 1$. With this kind of condition it is too restrictive to let $V := U$. Baker and Rippon require that V be a convex domain. In return they obtain uniform convergence when they exclude all the difficult cases:

Theorem 3.4. ([2, Thm 4.1]) *Let V be a bounded convex domain and let $\mathcal{F}$ be the family of analytic self-mappings of V with a continuously differentiable extension to $\overline{V}$ such that $|f'(z)| \leq 1$ for all $z \in \overline{V}$ and condition* (ii) *of* Theorem 3.2 *holds with U replaced by V. Then $\{F_n\}$ converges uniformly in $\overline{V}$ to a constant function.*

As a consequence of this they obtained Thron's convergence result for towers of exponentials (1.7):

Theorem 3.5. ([2, Thm 5.1], [42]) *Let $V := B(0,e)$ and let $\mathcal{F}$ be the family of functions $f(z) = \exp(bz)$ with $b \in \overline{B}(0,1/e)$. Let $\{f_n\} \subseteq \mathcal{F}$ and F_n be given by* (1.1)*. Then $f_n(V) \subseteq V$ and $\{F_n\}$ converges uniformly in V to a constant function.*

This means in particular that the tower of exponentials (1.7) converges when all $|b_n| \leq 1/e$. Baker and Rippon extended this result in [2]. In a later paper [4, Thm 7] this was further improved.

A simple corollary of Theorem 3.4 is:

Theorem 3.6. ([3, Thm 1]) *Let V be a bounded convex domain and let $\mathcal{F}$ be a family of analytic self-mappings of V with a continuous extension to $\overline{V}$ such that $|f'(z)| \leq 1$ in V, and there exists a $z_0 \in V$ such that $\sup_{f\in\mathcal{F}} |f'(z_0)| < 1$. Let $\{f_n\} \subseteq \mathcal{F}$ and F_n be given by (1.1). Then $\{F_n\}$ converges uniformly in $\overline{V}$ to a constant function.*

The condition that $|f'(z_0)| \leq 1 - \varepsilon$ for all $f \in \mathcal{F}$ and some $\varepsilon > 0$ is essential. But even if we only have $|f'(z)| < 1$, then an extension of Schwarz' lemma shows that f has a continuous extension to $\overline{V}$ with $|f'(z)| < 1$ for all $z \in \overline{V}$, [5, Thm 1]. The uniform bound for $|f'(z_0)|$ shows that $\mathcal{F}$ is contained in a closed subfamily of $\widehat{\mathcal{F}} := \{f : V \to V \text{ analytic with } |f'(z)| < 1 \text{ in } V\}$. In this language, Beardon [5, Thm 5] also proved Theorem 3.6 in a later paper. He also proved:

Theorem 3.7. ([6, Thm 7]) *Let $V \subseteq \mathbb{C}$ be an open, bounded set whose boundary consists of a finite number of simple polygons, and let $\mathcal{F}$ be a closed subfamily of $\widehat{\mathcal{F}} := \{f : V \to V$ analytic with $|f'(z)| < 1$ in $V\}$. Let $\{f_n\} \subseteq \mathcal{F}$ and F_n be given by* (1.1)*. Then $\{F_n\}$ converges uniformly in $\overline{V}$ to a constant function.*

Beardon also derived a series of estimates for the rate of convergence of $\{F_n\}$ depending on the shape of V.

4. Possibilities of extensions

As mentioned in the introduction, our problem can be posed quite generally. If we furnish U with the hyperbolic metric ρ, then by Schwarz' lemma, analyticity of $f : U \to U$ means that f is either a linear fractional transformation mapping U onto U, or f is a contraction

$$\rho(f(x), f(y)) < \rho(x, y) \quad \text{for } x \neq y, \; x, y \in U. \tag{4.1}$$

So, we have been actually dealing with contraction maps in the metric space (U, ρ). Unfortunately, Theorems 3.4 and 3.6 do not readily generalize to hyperbolic contractions. The problem is (as always) what happens close to the boundary of U or V. If we make sure that we "stay away from the difficult cases" in some sense, the theory can be extended to metric spaces (X, d) without too much trouble. The following is due to Beardon. It is a natural extension of the Denjoy–Wolff theorem:

Theorem 4.1. ([5, Thm 4]) *Let (X, d) be a compact metric space, and let $\mathcal{F}$ be a bounded subfamily of $\widehat{\mathcal{F}} := \{f : X \to X : \; d(f(x), f(y)) \leq d(x, y)$ for all $x, y \in X\}$. Let $\{f_n\} \subseteq \mathcal{F}$ and F_n be given by (1.1). Then either*

A. *$\{F_n\}$ converges uniformly in X to a constant function, or*

B. *$d(f(x), f(y)) = d(x, y)$ for some $f \in \mathcal{F}$ and distinct $x, y \in X$.*

Theorem 4.1 may be compared in many ways to Theorem 3.3. In particular, Theorem 3.3 is a consequence of Theorem 4.1 when we let $(X, d) = (U, \rho)$, where ρ is the hyperbolic metric in U.

For results in pseudometric spaces we refer to [11].

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The author has changed her name from Lisa Jacobsen.

Department of Mathematical Sciences
Norwegian University of Science and Technology,
N–7034 Trondheim, Norway

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