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Some remarks on Janiec uniqueness theorem for holomorphic mappings in $\mathbb{C}^n$

Abstract. In this paper the authors complete Janiec generalization of the well known Cartan uniqueness theorem for holomorphic mappings in some domains of $\mathbb{C}^n$.

Let $D \subset \mathbb{C}^n$ be a bounded complete Reinhardt domain with the centre at the origin. For instance D can be the open unit polydisc

$$\left\{ z = (z_1, \dots, z_n) \in \mathbb{C}^n \ : \ \max_{\nu} \ |z_\nu| \ < 1 \right\}$$

or it can be the following domain

$$B^{\alpha_1, \dots, \alpha_n} = \left\{ z = (z_1, \dots, z_n) \in \mathbb{C}^n : \sum_{\nu=1}^{n} | z_\nu |^{\alpha_\nu} < 1 \right\}, \tag{1}$$

where $\alpha_1, \dots, \alpha_n$ are arbitrarily fixed positive real numbers.

If $g = (g_1, \dots, g_n) : D \longrightarrow \mathbb{C}^n$ is a holomorphic mapping, then for each integer $\nu \in \{1, \dots, n\}$ we have the following expansion

$$g_\nu(z) = \sum_{m=0}^{\infty} P_\nu^m(z), \tag{2}$$

where $P_\nu^m : \mathbb{C}^n \to \mathbb{C}$ are homogeneous polynomials of $m - th$ degree and the above series (2) uniformly converges on each compact subset of D.

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E. Janiec proved the following

Theorem 1 ([1, Thm. 3]). *Let $j_1, \dots, j_n$ be arbitrarily fixed positive integers and $g = (g_1, \dots, g_n)$ be a holomorphic mapping of $B^{\alpha_1,\dots,\alpha_n}$ into $B^{\alpha_1/j_1,\dots,\alpha_n/j_n}$. If for each $\nu = 1, \dots, n$ in the expansion (2) the polynomials $P_\nu^0, P_\nu^1, \dots, P_\nu^{j_\nu}$ fulfil*

(3) $$P_\nu^0 = P_\nu^1 = \dots = P_\nu^{j_\nu - 1} = 0$$

and

(4) $$P_\nu^{j_\nu}(z_1, \dots, z_n) = z_\nu^{j_\nu} ,$$

then $g(z_1, \dots, z_n) = \left(z_1^{j_1}, \dots, z_n^{j_n}\right)$.

Of course, for $j_1 = \dots = j_n = 1$ this result follows directly from the well-known theorem of Cartan. In this case g maps holomorphically $B^{\alpha_1,\dots,\alpha_n}$ into itself and is normalized by the conditions : $g(0) = 0$ and $Dg(0) = I$, so consequently $g(z) = z$.

Let us observe that if $D = B^{\alpha_1,\dots,\alpha_n}$ is replaced by the open polydisc, then assumption (3) is unnecessary, (see [1, Thm.1]). The question arises whether in Theorem 1 assumption (3) can be omitted. E. Janiec showed ([1, Thm.4]) that the answer is affirmative under an additional assumption that f is holomorphic also on the boundary of the domain $B^{\alpha_1,\dots,\alpha_n}$. We will show that the answer is positive without additional assumption (3), also when we put $j_1 = \dots = j_n = j \geqq 1$ and replace the domains $B^{\alpha_1,\dots,\alpha_n}$ by the following particular domains

$$B^j = \left\{ z = (z_1, \dots, z_n) \in \mathbb{C}^n \ : \ \sum_{\nu=1}^{n} | z_\nu |^{2j} < 1 \right\}.$$

Let us observe that the sets B^j are open unit balls with adequate norms and $B = B^1$ is the open unit ball with euclidean norm.

Namely, our main result is the following

Theorem 2. *Let j be an arbitrarily fixed positive integer and let $f : B^j \to B$ be a holomorphic mapping of the form*

(5) $$f(z) = \sum_{m=0}^{\infty} P^m(z),$$

where $P^m : \mathbb{C}^n \to \mathbb{C}^n$ are homogeneous polynomial mappings of $m-th$ degree ($P^m = (P_1^m, \dots, P_n^m)$).

If

$$P^j(z_1, \dots, z_n) = \left(z_1^j, \dots, z_n^j\right), \tag{6}$$

then $f = P^j$.

In the proof of our theorem we use some properties of j, k-symmetrical functions which are presented in the paper [2]. Now we will give two lemmas.

Let $k \geqq 2$ be an arbitrarily fixed integer and $\varepsilon = \exp(2\pi i/k)$. A subset U of $\mathbb{C}^n$ will be called k-symmetrical set if $\varepsilon U = U$. Let us observe that the domains B^j are $k-$symmetrical sets with every k. For every integer j and a $k-$symmetrical set $U \subset \mathbb{C}^n$ a mapping $f : U \to \mathbb{C}^n$ will be called (j,k)-symmetrical if $f(\varepsilon z) = \varepsilon^j f(z)$, $z \in U$.

Lemma 1 ([2], Thm.1, Thm.2). *For every mapping $f : U \to \mathbb{C}^n$ there exesists exactly one sequence $f^0, f^1, \dots, f^{k-1}$ of $(j,k)-$symmetrical mappings f^j, $j = 0, 1, \dots, k-1$ such that*

$$f = \sum_{j=0}^{k-1} f^j \tag{7}$$

and

$$\sum_{j=0}^{k-1} \left\| f^j(z) \right\|^2 = k^{-1} \sum_{l=0}^{k-1} \left\| f(\varepsilon^l z) \right\|^2 , \quad z \in U . \tag{8}$$

Moreover, for $j = 0, 1, \dots, k-1$

$$f^j(z) = k^{-1} \sum_{l=0}^{k-1} \varepsilon^{-jl} f(\varepsilon^l z) , \quad z \in U . \tag{9}$$

In view of the uniqueness of the partition (7) the mappings f^j will be called (j,k)-symmetrical parts of the mapping f.

The next lemma is similar to Corollary 6 from [2].

Lemma 2. *Let us fix arbitrary k and j ($k \geqq 2$ and $j \in \{1, \dots, k-1\}$). If the mapping $f : B^j \to B$ is holomorphic and the $(j,k)-$symmetrical part f^j of f has the form*

$$f^j(z_1, \dots, z_n) = \left(z_1^j, \dots, z_n^j\right), \tag{10}$$

then $f = f^j$.

Proof. In view of Lemma 1 it is sufficient to show that the $(l,k)-$symmetrical parts f^l of f vanish if $l \in \{0, 1, \dots, k-1\}$ and $l \neq j$. To demonstrate this, first we observe that by the assumptions and (8) we have

$$\sum_{m=0}^{k-1} \|f^m(z)\|^2 < 1, \quad z \in B^j.$$

From this and (10) we obtain in turn

$$\|f^l(z)\|^2 < 1 - \|f^j(z)\|^2 = 1 - \left\|\left(z_1^j, \dots, z_n^j\right)\right\|^2 = 1 - \sum_{\nu=1}^{n} | z_\nu |^{2j} .$$

Since $\overline{rB^j} \subset B^j$ for every $r \in (0,1)$,

$$\max_{\partial(rB^j)} \|f^l(z)\|^2 \leqq \max_{\partial(rB^j)} \left(1 - \sum_{\nu=1}^{n} | z_\nu |^{2j}\right) = 1 - r^{2j}.$$

On the other hand $\max_{\overline{rB^j}} \|f^l(z)\| = \max_{\partial(rB^j)} \|f^l(z)\|$, because of the maximum principle for the euclidean norm of holomorphic mapping.

Thus, for every $r \in (0,1)$

$$0 \leqq \max_{\overline{rB^j}} \|f^l(z)\| \leqq \sqrt{1 - r^{2j}}\,.$$

Now observe that the family of sets $\left\{\overline{rB^j}\right\}_{r\in(0,1)}$ increases, so the quantity $\max_{\overline{rB^j}} \|f^l(z)\|$ is a nondecreasing function of $r \in (0,1)$. From this and the above inequality and the fact that $\sqrt{1 - r^{2j}}$ is a decreasing function of $r \in (0,1)$, we obtain that for every $r \in (0,1)$ $\max_{\overline{rB^j}} \|f^l(z)\| = 0$. Therefore $\|f^l(z)\| = 0$ for every $z \in B^j$, because B^j is the union of the family $\left\{\overline{rB^j}\right\}_{r\in(0,1)}$ ∎

Now we give the proof of Theorem 2.

Proof. Let us take an arbitrary $k > j$. We will show that the mapping $g = f^j$ fulfils the assumptions of Theorem 1 with $\alpha_1 = \dots = \alpha_n = 2j$ and $j_1 = \dots = j_n = j$.

In fact, observe first that f^j maps holomorphically B^j into B; this follows directly from the assumptions on f and from formula (9). On the other hand by (5) and (9) f^j has the following expansion $f^j(z) = \sum_{s=0}^{\infty} P^{j+sk}(z)$. Thus by (6) also (3) and (4) hold.

Therefore, applying the assertion of Theorem 1 to the mapping $g = f^j$, we obtain the relation (10). Now it is sufficient to apply Lemma 2. ∎

Now we will generalize Theorem 2, similarly as E. Janiec, (compare [1, Thm.5]). For a complex non-singular square matrix $A = [a_{\mu\nu}]_{n\times n}$ (of $n-th$ degree) denote by $\det A$ the determinant of A and by $A_\nu(w)$ the matrix formed out of the matrix A by replacing its $\nu-th$ column with the column $w = (w_1, ..., w_n)$. Then the set

$$D^1 = \left\{ w = (w_1, \dots, w_n) \in \mathbb{C}^n \ : \ \sum_{\nu=1}^{n} \left| \frac{\det A_\nu(w)}{\det A} \right|^2 < 1 \right\}$$

is a complete Reinhardt domain with the centre at the origin.

Under the above notations we have

Theorem 3. *Let j be an arbitrarily fixed positive integer and $F : B^j \to D^1$ be a holomorphic mapping of the form*

$$F(z) = \sum_{m=0}^{\infty} Q^m(z),$$

where $Q^m : \mathbb{C}^n \to \mathbb{C}^n$ are homogeneous polynomial mappings of $m-th$ degree. If $Q^j(z_1, \dots, z_n) = A\left(z_1^j, \dots, z_n^j\right)$, then $F = Q^j$.

Proof. It is sufficient to apply Theorem 2 to the mapping $f = A^{-1}F$.

In fact, the mapping f is holomorphic and from the Cramer theorem it follows that it maps B^j into B. On the other hand, f has expansion (5) where $P^m = A^{-1}Q^m$ and P^j is defined in (6). Therefore all assumptions of Theorem 2 are fulfiled. Now we conclude that $f = P^j$, so $F = Q^j$. ∎

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