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Oscillation of third-order delay difference equations with negative damping term

ABSTRACT. The aim of this paper is to investigate the oscillatory and asymptotic behavior of solutions of a third-order delay difference equation. By using comparison theorems, we deduce oscillation of the difference equation from its relation to certain associated first-order delay difference equations or inequalities. Examples are given to illustrate the main results.

1. Introduction. In this paper, we study the oscillatory behavior of solutions of the third-order delay difference equation of the form

$$(1.1) \quad \Delta^3 y_n - p_n \Delta y_{n+1} + q_n f(y_{n-\ell}) = 0, \quad n \geq n_0$$

where $\{p_n\}$ and $\{q_n\}$ are real sequences, $n_0 \in \mathbb{N}_0$, and f is a real-valued continuous function. Throughout this paper, we assume the following conditions without further mention:

- (H₁) $\{p_n\}$ is a nonnegative real sequence and $\{q_n\}$ is a positive real sequence for all $n \geq n_0$;
- (H₂) $\ell \in \mathbb{N}$;
- (H₃) $uf(u) > 0$, f is nondecreasing for $u \neq 0$, and

$$-f(uv) \geq f(uv) \geq f(u)f(v) \quad \text{for } uv > 0.$$

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By a solution of (1.1), we mean a nontrivial real sequence $\{y_n\}$, defined for all $n \geq n_0 - \ell$, and satisfying (1.1) for all $n \geq n_0$. A nontrivial solution $\{y_n\}$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise. Any difference equation is called nonoscillatory (oscillatory) if all its solutions are nonoscillatory (oscillatory).

In recent years, there is a great interest in studying the oscillatory and asymptotic behavior of solutions of several classes of third-order functional difference equations, see [3]–[8], [10]–[13], [15]–[17] and the references cited therein. In [5], [10], [12], [13], the authors considered the difference equation

$$(1.2) \quad \Delta(a_n \Delta(b_n \Delta y_n)) + p_n \Delta y_{n+1} + q_n f(y_{n-\ell}) = 0, \quad n \geq n_0,$$

and established some new sufficient conditions for the oscillation and asymptotic behavior of solutions of (1.2). Very recently, in [6], the authors discussed the oscillatory and asymptotic behavior of solutions of the equation

$$(1.3) \quad \Delta(a_n \Delta(b_n (\Delta y_n)^\alpha)) + p_n (\Delta y_{n+1})^\alpha + q_n f(y_{n-\ell}) = 0, \quad n \geq n_0.$$

In (1.2) and (1.3), the authors assumed the coefficient sequence of the damping term is positive, and therefore, in this paper, we investigate the oscillatory and asymptotic behavior of solutions of (1.1) by assuming the coefficient of the damping term is negative. Thus the results obtained in this paper are new and complement those in [5], [6], [10], [12], [13].

In Section 2, we present some preliminary results, and in Section 3, we obtain some sufficient conditions for the oscillation of all solutions of (1.1). Examples are provided in Section 4 in order to illustrate the importance of the main results.

2. Preliminary Results. In this section, we present some preliminary results, which are used to prove our main theorems. The main theorems, presented in Section 3, relate the properties of solutions of third-order delay difference equations of the form (1.1) to those of solutions of an auxiliary second-order linear difference equation of the form

$$(2.1) \quad \Delta^2 z_n - p_n z_{n+1} = 0, \quad n \geq n_0.$$

The first result is based on an equivalent representation for the linear difference operator

$$(2.2) \quad L(y_n) := \Delta^3 y_n - p_n \Delta y_{n+1}$$

in terms of a positive solution $\{z_n\}$ of (2.1). We prove that under the assumption (H_1) , (2.1) is nonoscillatory.

Lemma 2.1. *Let $\{p_n\}$ be a nonnegative real sequence for all $n \geq n_0$. Then (2.1) is nonoscillatory.*

Proof. Equation (2.1) can be written as

$$z_{n+2} + z_n = (p_n + 2)z_{n+1}, \quad n \geq n_0,$$

and we have

$$\frac{1}{(p_n + 2)(p_{n+1} + 2)} \leq \frac{1}{4}, \quad n \geq n_0.$$

Hence by applying [2, Theorem 1.8.9], we see that (2.1) is nonoscillatory. \square

Lemma 2.2. *Assume that (2.1) has a positive solution $\{z_n\}$. Then the operator (2.2) can be represented as*

$$L(y_n) = \frac{\Delta \left(z_n z_{n+1} \Delta \left(\frac{\Delta y_n}{z_n} \right) \right)}{z_{n+1}}.$$

Proof. It is easy to see that

$$\begin{aligned} L(y_n) &= \Delta^3 y_n - p_n \Delta y_{n+1} \\ &= \frac{z_{n+1} \Delta^3 y_n + (\Delta z_n)(\Delta^2 y_n) - (\Delta z_n)(\Delta^2 y_n) - (\Delta^2 z_n)(\Delta y_{n+1})}{z_{n+1}} \\ &= \frac{\Delta (z_n \Delta^2 y_n - (\Delta z_n)(\Delta y_n))}{z_{n+1}} \\ &= \frac{\Delta \left(z_n z_{n+1} \Delta \left(\frac{\Delta y_n}{z_n} \right) \right)}{z_{n+1}}, \end{aligned}$$

completing the proof. \square

Lemma 2.3. *If $\{z_n\}$ is a positive solution of (2.1), then (1.1) can be written in the form*

$$(2.3) \quad \Delta \left(z_n z_{n+1} \Delta \left(\frac{\Delta y_n}{z_n} \right) \right) + q_n z_{n+1} f(y_{n-\ell}) = 0.$$

Proof. The proof follows from Lemma 2.2. \square

For our discussion, it is convenient to have (2.3) in canonical form, that is, we require in the sequel that

$$(2.4) \quad \sum_{n=n_0}^{\infty} z_n = \infty$$

and

$$(2.5) \quad \sum_{n=n_0}^{\infty} \frac{1}{z_n z_{n+1}} = \infty.$$

To prove our main results, we find conditions that guarantee the existence of positive solutions of (2.1). The following result is a special case of the discrete Kneser theorem (see [1]) dealing with the structure of nonoscillatory solutions of (2.1).

Lemma 2.4. *Under the assumption (H_1) , (2.1) has a principal (recessive) solution $\{u_n\}$ satisfying*

$$u_n > 0, \quad \Delta u_n < 0, \quad \Delta^2 u_n \geq 0, \quad n \geq n_0$$

and a nonprincipal (dominant) solution $\{v_n\}$ satisfying

$$v_n > 0, \quad \Delta v_n > 0, \quad \Delta^2 v_n \geq 0, \quad n \geq n_0.$$

Next, we study the behavior of solutions of (1.1) with the help of its equivalent representation (2.3). In view of well-known results in [1], [2], we have the following structure of nonoscillatory solutions of (1.1).

Lemma 2.5. *Let $\{z_n\}$ be a positive solution of (2.1) satisfying (2.5). Then, every positive solution $\{y_n\}$ of (1.1) is inside one of the following two classes:*

- (I) $y_n > 0, \Delta y_n > 0, \Delta \left(\frac{\Delta y_n}{z_n} \right) > 0, \Delta \left(z_n z_{n+1} \Delta \left(\frac{\Delta y_n}{z_n} \right) \right) < 0;$
- (II) $y_n > 0, \Delta y_n < 0, \Delta \left(\frac{\Delta y_n}{z_n} \right) > 0, \Delta \left(z_n z_{n+1} \Delta \left(\frac{\Delta y_n}{z_n} \right) \right) < 0.$

We conclude this section with the following notation. We say that (1.1) has Property (B) provided all its nonoscillatory solutions $\{y_n\}$ satisfy the condition

$$(2.6) \quad y_n \Delta y_n < 0.$$

3. Oscillation Results. In the sequel, it is tacitly assumed that the sequence $\{z_n\}$ is a solution of (2.1) and satisfies (2.5). We begin with the following theorem.

Theorem 3.1. *Let $\{z_n\}$ be a positive solution of (2.1) such that (2.4) and (2.5) hold. If the first-order delay difference equation*

$$(3.1) \quad \Delta x_n + q_n z_{n+1} f \left(\sum_{s=n_1}^{n-\ell-1} z_s \sum_{t=n_1}^{s-1} \frac{1}{z_t z_{t+1}} \right) f(x_{n-\ell}) = 0, \quad n \geq n_1$$

is oscillatory, then (1.1) has Property (B).

Proof. Let $\{y_n\}$ be a positive solution of (1.1) for all $n \geq n_1$, where $n_1 \geq n_0$ is some integer. It follows from Lemma 2.5 that $\{y_n\}$ belongs to either class (I) or class (II) for all $n \geq n_1$. We shall prove that $\{y_n\}$ belongs to class (II). For this, we assume that $\{y_n\}$ belongs to class (I). Define a sequence $\{x_n\}$ by

$$x_n = z_n z_{n+1} \Delta \left(\frac{1}{z_n} \Delta y_n \right), \quad n \geq n_1.$$

Then $\{x_n\}$ is strictly decreasing, and we conclude

$$\begin{aligned} \frac{\Delta y_n}{z_n} &\geq \sum_{s=n_1}^{n-1} \Delta \left(\frac{\Delta y_s}{z_s} \right) = \sum_{s=n_1}^{n-1} \frac{1}{z_s z_{s+1}} \left[z_s z_{s+1} \Delta \left(\frac{\Delta y_s}{z_s} \right) \right] \\ &\geq x_n \sum_{s=n_1}^{n-1} \frac{1}{z_s z_{s+1}}, \end{aligned}$$

i.e.,

$$(3.2) \quad \Delta y_n \geq x_n z_n \sum_{s=n_1}^{n-1} \frac{1}{z_s z_{s+1}}.$$

Summing (3.2) from n_1 to $n-1$ yields

$$y_n \geq \sum_{s=n_1}^{n-1} x_s z_s \sum_{t=n_1}^{s-1} \frac{1}{z_t z_{t+1}} \geq x_n \sum_{s=n_1}^{n-1} z_s \sum_{t=n_1}^{s-1} \frac{1}{z_t z_{t+1}}.$$

Hence

$$(3.3) \quad y_{n-\ell} \geq x_{n-\ell} \sum_{s=n_1}^{n-\ell-1} z_s \sum_{t=n_1}^{s-1} \frac{1}{z_t z_{t+1}}.$$

Combining (3.3) with (2.3) and using (H₃), we obtain

$$-\Delta x_n = z_{n+1} q_n f(y_{n-\ell}) \geq z_{n+1} q_n f \left(\sum_{s=n_1}^{n-\ell-1} z_s \sum_{t=n_1}^{s-1} \frac{1}{z_t z_{t+1}} \right) f(x_{n-\ell}).$$

Therefore, $\{x_n\}$ is a positive solution of the delay difference inequality

$$\Delta x_n + z_{n+1} q_n f \left(\sum_{s=n_1}^{n-\ell-1} z_s \sum_{t=n_1}^{s-1} \frac{1}{z_t z_{t+1}} \right) f(x_{n-\ell}) \leq 0.$$

It follows from [17, Lemma 2.7] that (3.1) also has a positive solution, which is a contradiction. Hence, $\{y_n\}$ belongs to class (II), and the first two inequalities in class (II) imply Property (B) of (1.1). This completes the proof. \square

Applying known oscillation criteria to (3.1), we immediately obtain criteria for Property (B) of (1.1), see for example [9], [14].

Corollary 3.2. *Let $\{z_n\}$ be a positive solution of (2.1) satisfying (2.4) and (2.5). Let $f(u) = u$. If*

$$(3.4) \quad \liminf_{n \rightarrow \infty} \sum_{s=n-\ell}^{n-1} q_s z_{s+1} \sum_{t=n_1}^{s-\ell-1} z_t \sum_{j=n_1}^{t-1} \frac{1}{z_j z_{j+1}} > \left(\frac{\ell}{\ell+1} \right)^{\ell+1},$$

then (1.1) has Property (B).

Corollary 3.3. *Let $\{z_n\}$ be a positive solution of (2.1) satisfying (2.4) and (2.5). Let $f(u) = u^\alpha$, where $\alpha \in (0, 1)$ is a quotient of odd positive integers. If*

$$(3.5) \quad \sum_{n=n_1}^{\infty} q_n z_{n+1} \left(\sum_{s=n_1}^{n-\ell-1} z_s \sum_{t=n_1}^{s-1} \frac{1}{z_t z_{t+1}} \right)^\alpha = \infty,$$

then (1.1) has Property (B).

Corollary 3.4. *Let $\{z_n\}$ be a positive solution of (2.1) satisfying (2.4) and (2.5). Let $f(u) = u^\alpha$, where $\alpha > 1$ is a quotient of odd positive integers. If there exists $\lambda > \frac{\log \alpha}{\ell}$ such that*

$$(3.6) \quad \liminf_{n \rightarrow \infty} q_n z_{n+1} \left(\sum_{s=n_1}^{n-\ell-1} z_s \sum_{t=n_1}^{s-1} \frac{1}{z_t z_{t+1}} \right)^\alpha \exp(-e^{\lambda n}) > 0,$$

then (1.1) has Property (B).

Next, we are interested in finding criteria for the oscillation of all solutions of (1.1). Note that, by Theorem 3.1, if (3.1) is oscillatory, then (1.1) has Property (B). Therefore, it follows from Lemma 2.5 that by eliminating the possibility for nonoscillatory solutions belonging to class (II), one obtains sufficient conditions for the oscillation of all solutions of (1.1).

Theorem 3.5. *Let $\{z_n\}$ be a positive solution of (2.1) such that (2.4) and (2.5) hold. If there exists $k \in \mathbb{N}$ with $\ell > 2k$ such that both (3.1) and*

$$(3.7) \quad \Delta x_n + z_n f(x_{n-\ell+2k}) \sum_{s=n}^{n+k} \frac{1}{z_s z_{s+1}} \sum_{t=s}^{s+k} q_t z_{t+1} = 0$$

are oscillatory, then every solution of (1.1) is oscillatory.

Proof. Let $\{y_n\}$ be an eventually positive solution of (1.1). Then, by Lemma 2.5, $\{y_n\}$ belongs to either class (I) or (II). From Theorem 3.1, oscillation of (3.1) excludes the possibility of nonoscillatory solutions belonging to class (I). Therefore, $\{y_n\}$ belongs to class (II). Summation of (2.3) from n to $n+k$ yields

$$z_n z_{n+1} \Delta \left(\frac{\Delta y_n}{z_n} \right) \geq \sum_{s=n}^{n+k} q_s z_{s+1} f(y_{s-\ell}) \geq f(y_{n-\ell+k}) \sum_{s=n}^{n+k} q_s z_{s+1},$$

i.e.,

$$(3.8) \quad \Delta \left(\frac{\Delta y_n}{z_n} \right) \geq \frac{f(y_{n-\ell+k})}{z_n z_{n+1}} \sum_{s=n}^{n+k} q_s z_{s+1}.$$

Summing (3.8) from n to $n + k$, we obtain

$$-\frac{\Delta y_n}{z_n} \geq f(y_{n-\ell+2k}) \sum_{s=n}^{n+k} \frac{1}{z_s z_{s+1}} \sum_{t=s}^{s+k} q_t z_{t+1}.$$

Again summing the last inequality from n to ∞ yields

$$(3.9) \quad y_n \geq \sum_{u=n}^{\infty} z_u f(y_{u-\ell+2k}) \sum_{s=u}^{u+k} \frac{1}{z_s z_{s+1}} \sum_{t=s}^{s+k} q_t z_{t+1}.$$

Let us denote by x_n the right-hand side of (3.9). Clearly, $y_n \geq x_n > 0$, and one can easily see that $\{x_n\}$ is a positive solution of the delay difference inequality

$$\Delta x_n + z_n f(x_{n-\ell+2k}) \sum_{s=n}^{n+k} \frac{1}{z_s z_{s+1}} \sum_{t=s}^{s+k} q_t z_{t+1} \leq 0.$$

But by [17, Lemma 2.7], (3.7) also has a positive solution, which is a contradiction. Thus, we conclude that every solution of (1.1) is oscillatory. This completes the proof. \square

Again applying known oscillation criteria to both (3.1) and (3.7), we obtain explicit conditions that ensure oscillation of all solutions of (1.1), see for example [9], [14].

Corollary 3.6. *Let $\{z_n\}$ be a positive solution of (2.1) such that (2.4) and (2.5) hold. Let $f(u) = u$ and assume there exists $k \in \mathbb{N}$ with $\ell > 2k$. If (3.4) and*

$$(3.10) \quad \liminf_{n \rightarrow \infty} \sum_{s=n-\ell+2k}^{n-1} z_s \sum_{t=s}^{s+k} \frac{1}{z_t z_{t+1}} \sum_{j=t}^{t+k} q_j z_{j+1} > \left(\frac{\ell - 2k}{\ell - 2k + 1} \right)^{\ell - 2k + 1}$$

hold, then every solution of (1.1) is oscillatory.

Corollary 3.7. *Let $\{z_n\}$ be a positive solution of (2.1) such that (2.4) and (2.5) hold. Let $f(u) = u^\alpha$, where $\alpha \in (0, 1)$ is a ratio of odd positive integers. Assume that there exists $k \in \mathbb{N}$ with $\ell > 2k$. If (3.5) and*

$$(3.11) \quad \sum_{n=n_1}^{\infty} z_n \sum_{s=n}^{n+k} \frac{1}{z_s z_{s+1}} \sum_{t=s}^{s+k} q_t z_{t+1} = \infty$$

hold, then every solution of (1.1) is oscillatory.

Corollary 3.8. *Let $\{z_n\}$ be a positive solution of (2.1) such that (2.4) and (2.5) hold. Let $f(u) = u^\alpha$, where $\alpha > 1$ is a ratio of odd positive integers.*

Assume that there exists $k \in \mathbb{N}$ such that $\ell > 2k$. If (3.6) holds and if there exists $\mu > \frac{\log \alpha}{\ell - 2k}$ such that

$$(3.12) \quad \liminf_{n \rightarrow \infty} z_n \exp(-e^{\mu n}) \sum_{s=n}^{n+k} \frac{1}{z_s z_{s+1}} \sum_{t=s}^{s+k} q_t z_{t+1} > 0,$$

then every solution of (1.1) is oscillatory.

4. Examples. In this section, we present some examples in order to illustrate the main results.

Example 4.1. Consider the third-order delay difference equation

$$(4.1) \quad \Delta^3 y_n - \frac{2}{n(n+2)} \Delta y_{n+1} + \frac{3}{n+1} y_{n-2} = 0, \quad n \in \mathbb{N}.$$

Here,

$$p_n = \frac{2}{n(n+2)}, \quad q_n = \frac{3}{n+1}, \quad \ell = 2.$$

Now, (2.1) takes the form

$$(4.2) \quad \Delta^2 z_n - \frac{2}{n(n+2)} z_{n+1} = 0, \quad n \in \mathbb{N}.$$

Since $\{p_n\}$ is positive, (4.2) is nonoscillatory and has a positive solution $\{z_n\} = \{1/n\}$, which satisfies (2.4) and (2.5). Also,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \sum_{s=n-2}^{n-1} \frac{3}{(s+1)^2} \sum_{t=1}^{s-3} \frac{1}{t} \sum_{j=1}^{t-1} j(j+1) \\ = \liminf_{n \rightarrow \infty} \sum_{s=n-2}^{n-1} \frac{(s-3)(s-4)(2s-1)}{6(s+1)^2} = \infty. \end{aligned}$$

Hence, (3.4) is satisfied. Therefore, by Corollary 3.2, (4.1) has Property (B).

Example 4.2. Consider the third-order delay difference equation

$$(4.3) \quad \Delta^3 y_n - \frac{2}{n(n+2)^2} \Delta y_{n+1} + (n+1)y_{n-3}^{1/3} = 0, \quad n \in \mathbb{N}.$$

Here,

$$p_n = \frac{2}{n(n+2)^2}, \quad q_n = n+1, \quad \ell = 3, \quad f(u) = u^\alpha, \quad \alpha = \frac{1}{3}.$$

Now, (2.1) takes the form

$$(4.4) \quad \Delta^2 z_n - \frac{2}{n(n+2)^2} z_{n+1} = 0, \quad n \in \mathbb{N}.$$

By Lemma 2.1, (4.4) is nonoscillatory and has a positive solution $\{z_n\} = \{\frac{n+1}{n}\}$, which satisfies (2.4) and (2.5). By taking $k = 1$, we see that $\ell > 2k$

and (3.5) and (3.11) are satisfied. Hence, by Corollary 3.7, every solution of (4.3) is oscillatory.

Example 4.3. Consider the third-order delay difference equation

$$(4.5) \quad \Delta^3 y_n - \frac{2}{n(n+2)^2} \Delta y_{n+1} + \left(8 + \frac{4}{n(n+2)^2}\right) y_{n-4} = 0, \quad n \in \mathbb{N}.$$

Here,

$$p_n = \frac{2}{n(n+2)^2}, \quad q_n = 8 + \frac{4}{n(n+2)^2}, \quad \ell = 4, \quad f(u) = u.$$

Now, (2.1) takes the form (4.4), and it has a positive solution $\{z_n\} = \{\frac{n+1}{n}\}$, which satisfies (2.4) and (2.5). By taking $k = 1$, we see that $\ell > 2k$ and (3.4) and (3.10) are satisfied. Hence, by Corollary 3.6, every solution of (4.5) is oscillatory. In fact, $\{y_n\} = \{(-1)^n\}$ is one such solution of (4.5).

We conclude this paper with the following remarks.

Remark 4.4. In this paper, we study the asymptotic and oscillatory behavior of solutions of third-order delay difference equations with negative damping term. We transform (1.1) into an equation without damping term and then obtain conditions for the oscillation of (1.1). We stress that, contrary to many known results in the literature leading to the conclusion that all solutions are either oscillatory or tend to zero as $n \rightarrow \infty$, our results guarantee oscillation of all solutions of (1.1).

Remark 4.5. Further note that, in all results, the explicit form of one positive solution of (2.1) is needed. However, it is known that it is very difficult to find the explicit form of solutions to a second-order difference equation with variable coefficient. So it is interesting to obtain criteria for the oscillation of (1.1) without involving the solution of (2.1).

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