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D. VAMSHEE KRISHNA[1], B. VENKATESWARLU
and T. RAMREDDY

Third Hankel determinant for starlike and convex functions with respect to symmetric points

Abstract. The objective of this paper is to obtain best possible upper bound to the $H_3(1)$ Hankel determinant for starlike and convex functions with respect to symmetric points, using Toeplitz determinants.

1. Introduction. Let A denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disc $E = \{z : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions. For any two analytic functions g and h respectively with their expansions as $g(z) = \sum_{k=0}^{\infty} a_k z^k$ and $h(z) = \sum_{k=0}^{\infty} b_k z^k$, the Hadamard product or convolution of $g(z)$ and $h(z)$ is defined as the power series

$$(g * h)(z) = \sum_{k=0}^{\infty} a_k b_k z^k.$$

[1]Corresponding author

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The Hankel determinant of f for $q \geq 1$ and $n \geq 1$ was defined by Pommerenke [9] as

$$(1.2) \qquad H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, \quad (a_1 = 1).$$

One can easily observe that the Fekete–Szegő functional is $H_2(1)$. Fekete–Szegő then further generalized the estimate $|a_3 - \mu a_2^2|$ with μ real and $f \in S$. Ali [1] found sharp bounds on the first four coefficients and sharp estimate for the Fekete–Szegő functional $|\gamma_3 - t\gamma_2^2|$, where t is real, for the inverse function of f defined as $f^{-1}(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$, when $f \in \widetilde{ST}(\alpha)$, the class of strongly starlike functions of order α $(0 < \alpha \leq 1)$. Further, sharp bounds for the functional

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = |a_2 a_4 - a_3^2|,$$

when $q = 2$ and $n = 2$, known as the second Hankel determinant, were obtained for various subclasses of univalent and multivalent analytic functions. For our discussion, in this paper, we consider the Hankel determinant in the case of $q = 3$ and $n = 1$, denoted by $H_3(1)$, given by

$$(1.3) \qquad H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}.$$

For $f \in A$, $a_1 = 1$, so that, we have

$$H_3(1) = a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2^2)$$

and by applying triangle inequality, we obtain

$$(1.4) \qquad |H_3(1)| \leq |a_3||a_2 a_4 - a_3^2| + |a_4||a_2 a_3 - a_4| + |a_5||a_3 - a_2^2|.$$

Babalola [2] obtained sharp upper bounds to the functional $|a_2 a_3 - a_4|$ and $|H_3(1)|$ for the familiar subclasses namely starlike and convex functions respectively denoted by ST and CV of S. The sharp upper bounds to the second Hankel determinant $|a_2 a_4 - a_3^2|$ for the classes ST and CV were obtained by Janteng et al. [6].

Motivated by the results obtained by Babalola [2] and recently by Raja and Malik [11] in finding the sharp upper bound to the Hankel determinant $|H_3(1)|$ for certain subclasses of S, in this paper, we obtain an upper bound to the functional $|a_2 a_3 - a_4|$ and hence for $|H_3(1)|$, for the function f given in (1.1), belonging to the classes namely starlike with respect to symmetric points and convex with respect to symmetric points denoted by ST_s and CV_s respectively, defined as follows.

Definition 1.1. A function $f(z) \in A$ is said to be in the class ST_s, if it satisfies the condition

$$\operatorname{Re}\left\{\frac{2zf'(z)}{f(z)-f(-z)}\right\} > 0,\ \forall z \in E. \tag{1.5}$$

The class ST_s was introduced and studied by Sakaguchi [15]. Further, he has shown that the functions in ST_s are close-to-convex and hence are univalent. The concept of starlike functions with respect to symmetric points have been extended to starlike functions with respect to N-symmetric points by Ratanchand [14] and Prithvipalsingh [10]. RamReddy [12] studied the class of close-to-convex functions with respect to N-symmetric points and proved that this class is closed under convolution with convex univalent functions.

Definition 1.2. A function $f(z) \in A$ is said to be in CV_s, if it satisfies the condition

$$\operatorname{Re}\left\{\frac{2\left\{zf'(z)\right\}'}{\left\{f(z)-f(-z)\right\}'}\right\} > 0,\ \forall z \in E. \tag{1.6}$$

The class CV_s was introduced and studied by Das and Singh [3]. From the Definitions 1.1 and 1.2, it is evident that $f \in CV_s$ if and only if $zf' \in ST_s$. Some preliminary lemmas required for proving our results are as follows:

2. Preliminary Results. Let $\mathscr{P}$ denote the class of functions consisting of p, such that

$$p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots = 1 + \sum_{n=1}^{\infty} c_nz^n, \tag{2.1}$$

which are regular in the open unit disc E and satisfy $\operatorname{Re} p(z) > 0$, for any $z \in E$. Here $p(z)$ is called the Carathéodory function [4].

Lemma 2.1 ([8, 16]). *If $p \in \mathscr{P}$, then $|c_k| \leq 2$, for each $k \geq 1$ and the inequality is sharp for the function $\frac{1+z}{1-z}$.*

Lemma 2.2 ([5]). *The power series for $p(z) = 1 + \sum_{n=1}^{\infty} c_nz^n$ given in (2.1) converges in the open unit disc E to a function in $\mathscr{P}$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ c_{-2} & c_{-1} & 2 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, \quad n = 1, 2, 3 \ldots.$$

and $c_{-k} = \bar{c}_k$, are all non-negative. They are strictly positive except for $p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{it_k}z)$, with $\sum_{k=1}^{m} \rho_k = 1$, t_k real and $t_k \neq t_j$, for $k \neq j$, *where* $p_0(z) = \frac{1+z}{1-z}$; *in this case* $D_n > 0$ *for* $n < (m-1)$ *and* $D_n \doteq 0$ *for* $n \geq m$.

This necessary and sufficient condition found in [5] is due to Carathéodory and Toeplitz. We may assume without restriction that $c_1 > 0$. On using Lemma 2.2, for $n = 2$, we have

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \overline{c}_1 & 2 & c_1 \\ \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix} = 8 + 2\,\mathrm{Re}\,\{c_1^2 c_2\} - 2\mid c_2\mid^2 - 4|c_1|^2 \geq 0$$

$$\Leftrightarrow 2c_2 = c_1^2 + x(4 - c_1^2), \tag{2.2}$$

for some x, $|x| \leq 1$. For $n = 3$,

$$D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \overline{c}_1 & 2 & c_1 & c_2 \\ \overline{c}_2 & \overline{c}_1 & 2 & c_1 \\ \overline{c}_3 & \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix} \geq 0$$

and is equivalent to

$$|(4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \leq 2(4 - c_1^2)^2 - 2|(2c_2 - c_1^2)|^2. \tag{2.3}$$

Simplifying the expressions (2.2) and (2.3), we get

$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z, \tag{2.4}$$

with $|z| \leq 1$. In obtaining our results, we refer to the classical method devised by Libera and Złotkiewicz [7] and used by several authors in the literature.

3. Main results.

Theorem 3.1. *If* $f(z) \in ST_s$ *then* $|a_2 a_3 - a_4| \leq \frac{1}{2}$.

Proof. For the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in ST_s$, by virtue of Definition 1.1, there exists an analytic function $p \in \mathscr{P}$ in the unit disc E with $p(0) = 1$ and $\mathrm{Re}\, p(z) > 0$ such that

$$\frac{2zf'(z)}{f(z) - f(-z)} = p(z) \Leftrightarrow 2zf'(z) = [f(z) - f(-z)]\,p(z). \tag{3.1}$$

Replacing $f(z)$, $f'(z)$, $f(-z)$ and $p(z)$ with their equivalent series expressions in (3.1), we have

$$2z\left\{1 + \sum_{n=2}^{\infty} n a_n z^{n-1}\right\} = \left[\left\{z + \sum_{n=2}^{\infty} a_n z^n\right\} - \left\{-z + \sum_{n=2}^{\infty} a_n (-z)^n\right\}\right] \times \left\{1 + \sum_{n=1}^{\infty} c_n z^n\right\}.$$

Upon simplification, we obtain

$$\begin{aligned} &1 + 2a_2z + 3a_3z^2 + 4a_4z^3 + 5a_5z^4 \ldots \\ &= 1 + c_1z + (c_2 + a_3)z^2 + (c_3 + c_1a_3)z^3 + (c_4 + c_2a_3 + a_5)z^4 + \ldots . \end{aligned} \tag{3.2}$$

Equating the coefficients of like powers of z, z^2, z^3 and z^4 respectively in (3.2), after simplifying, we get

$$a_2 = \frac{c_1}{2};\ a_3 = \frac{c_2}{2};\ a_4 = \frac{1}{8}(2c_3 + c_1c_2);\ a_5 = \frac{1}{8}(2c_4 + c_2^2). \tag{3.3}$$

Substituting the values of a_2, a_3 and a_4 from (3.3) in the functional $|a_2a_3-a_4|$ for the function $f \in ST_s$, we obtain

$$|a_2a_3 - a_4| = \frac{1}{8}|c_1c_2 - 2c_3|. \tag{3.4}$$

From Lemma 2.2, substituting the values of c_2 and c_3 from (2.2) and (2.4) respectively, on the right-hand side of the expression (3.4), we have

$$\begin{aligned} \left|c_1c_2 - 2c_3\right| = \Big|&c_1\frac{1}{2}\{c_1^2 + x(4-c_1^2)\} - 2\cdot\frac{1}{4}\{c_1^3 + 2c_1(4-c_1^2)x \\ &- c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z\}\Big|. \end{aligned}$$

Using the facts $|z| < 1$ and $|pa + qb| \le |p||a| + |q||b|$, where p, q, a and b are real numbers, after simplifying, we get

$$2|c_1c_2 - 2c_3| \le |2(4-c_1^2) + c_1(4-c_1^2)|x| + (c_1+2)(4-c_1^2)|x|^2|. \tag{3.5}$$

Since $c_1 = c \in [0, 2]$, noting that $c_1 + a \ge c_1 - a$ where $a \ge 0$, applying triangle inequality and replacing $|x|$ by μ on the right hand side of the above inequality, we have

$$2|c_1c_2 - 2c_3| \le \left\{2 + c\mu + (c-2)\mu^2\right\}(4-c^2) = F(c,\mu), \tag{3.6}$$

for $0 \le \mu = |x| \le 1$, where

$$F(c,\mu) = \left\{2 + c\mu + (c-2)\mu^2\right\}(4-c^2). \tag{3.7}$$

Now, we maximize the function $F(c, \mu)$ on the closed region $[0, 2] \times [0, 1]$. From (3.7), we get

$$\frac{\partial F}{\partial \mu} = \{c + 2(c-2)\mu\}(4-c^2) \tag{3.8}$$

and

$$\frac{\partial F}{\partial c} = \left\{\mu + \mu^2\right\}(4-c^2). \tag{3.9}$$

The only stationary point for the function $F(c, \mu)$ in the region $[0, 2] \times [0, 1]$ for which $\frac{\partial F}{\partial c} = 0$ and $\frac{\partial F}{\partial \mu} = 0$ simultaneously is $(0, 0)$, from the elementary calculus, we observe that the function $F(c,\mu)$ attains maximum value at this point only and from (3.7), it is given by

$$G_{\max} = F(0,0) = 8. \tag{3.10}$$

Simplifying the expressions (3.4) and (3.6) together with (3.10), we obtain

$$|a_2a_3 - a_4| \leq \frac{1}{2}.$$

This completes the proof of our Theorem 3.1. □

Theorem 3.2. *If $f(z) \in ST_s$, then $|a_3 - a_2^2| \leq 1$ and the inequality is sharp for the values $c_1 = c = 0$, $c_2 = 2$ and $x = 1$.*

Proof. Substituting the values a_2 and a_3 from (3.3) into the functional $|a_3 - a_2^2|$, we obtain

$$4|a_3 - a_2^2| = \left|2c_2 - c_1^2\right|. \tag{3.11}$$

Substituting the value of c_2 from (2.2) of Lemma 2.2 on the right-hand side of (3.11), we get

$$\left|2c_2 - c_1^2\right| = \left|\left(4 - c_1^2\right)x\right|. \tag{3.12}$$

Since $c_1 = c \in [0,2]$, replacing $|x|$ by μ on the right hand side of the above expression, we see that

$$\left|2c_2 - c_1^2\right| \leq (4 - c^2)\mu = F(c,\mu), \tag{3.13}$$

for $0 \leq \mu = |x| \leq 1$. Next, we maximize the function $F(c,\mu)$ on the closed region $[0,2] \times [0,1]$. Differentiating $F(c,\mu)$ in (3.13) partially with respect to μ, we obtain

$$\frac{\partial F}{\partial \mu} = (4 - c^2). \tag{3.14}$$

From (3.14), we observe that $\frac{\partial F}{\partial \mu} > 0$, for $0 < \mu < 1$ and $0 < c < 2$. Therefore, $F(c,\mu)$ is an increasing function of μ and hence it cannot have maximum value at any point in the interior of the closed region $[0,2] \times [0,1]$.

Moreover, for fixed $c \in [0,2]$, we have

$$\max_{0 \leq \mu \leq 1} F(c,\mu) = F(c,1) = G(c) = (4 - c^2), \tag{3.15}$$

$$G'(c) = -2c. \tag{3.16}$$

From the expression (3.16), we observe that $G'(c) \leq 0$ for every $c \in [0,2]$. Therefore, $G(c)$ becomes a decreasing function of c, whose maximum value occurs at $c = 0$ only, from (3.15), it is given by

$$G_{\max} = G(0) = 4. \tag{3.17}$$

Simplifying the expressions (3.11), (3.13) along with (3.17), we obtain

$$|a_3 - a_2^2| \leq 1. \tag{3.18}$$

This completes the proof of our Theorem 3.2. □

Theorem 3.3. *If $f(z) \in ST_s$, then $|a_k| \leq 1$, for $k \in \{2, 3, 4, \ldots\}$ and the inequality is sharp.*

Proof. Using the fact that $|c_n| \leq 2$, for $n \in N = \{1, 2, 3, \ldots\}$, with the help of c_2 and c_3 values given in (2.2) and (2.4) respectively, together with the values determined in (3.3), we obtain $|a_k| \leq 1$, for $k \in \{2, 3, 4, \ldots\}$. This completes the proof of our Theorem 3.3. □

Substituting the results of Theorems 3.1, 3.2 , 3.3 and the inequality $|a_2a_4 - a_3^2| \leq 1$ (see [13]) in the inequality (1.4), we obtain the following corollary.

Corollary 3.4. *Let $f(z) \in ST_s$ then $|H_3(1)| \leq \frac{5}{2}$.*

Theorem 3.5. *If $f(z) \in CV_s$ then $|a_2a_3 - a_4| \leq \frac{4}{27}$.*

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in CV_s$, from the Definition 1.2, there exists an analytic function $p \in \mathscr{P}$ in the unit disc E with $p(0) = 1$ and $\operatorname{Re} p(z) > 0$ such that

$$\frac{2\{zf'(z)\}'}{f'(z) + f'(-z)} = p(z) \Leftrightarrow 2\{zf'(z)\}' = \{f'(z) + f'(-z)\}p(z). \tag{3.19}$$

Replacing $f'(z)$, $f''(z)$, $f'(-z)$ and $p(z)$ with their series equivalent expressions in (3.20) and applying the same procedure as described in Theorem 3.1, we get

$$a_2 = \frac{c_1}{4};\ a_3 = \frac{c_2}{6};\ a_4 = \frac{1}{32}(2c_3 + c_1c_2);\ a_5 = \frac{1}{40}(2c_4 + c_2^2). \tag{3.20}$$

Substituting the values of a_2, a_3, and a_4 from (3.21) in $|a_2a_3 - a_4|$ for the function $f \in CV_s$, upon simplification, we obtain

$$|a_2a_3 - a_4| = \frac{1}{96}|c_1c_2 - 6c_3|. \tag{3.21}$$

Applying the same procedure as described in Theorem 3.1, we arrive at

$$2|c_1c_2 - 6c_3| \leq [2c^3 + \{6 + 5c\mu + 3(c-2)\mu^2\}(4 - c^2)] = F(\mu), \tag{3.22}$$

for $0 \leq \mu \leq 1$, where

$$F(\mu) = 2c^3 + \{6 + 5c\mu + 3(c-2)\mu^2\}(4 - c^2). \tag{3.23}$$

Next, we maximize the function $F(\mu)$ on the closed region $[0, 2] \times [0, 1]$. Note that $F'(\mu) \geq F'(1) > 0$. Then there exists $c^* \in [0, 2]$ such that $F'(\mu) > 0$ for $c \in (c^*, 2]$ and $F'(\mu) \leq 0$ otherwise. Then for $c \in [c^*, 2]$, $F(\mu) \leq F(1)$, that is

$$2|c_1c_2 - 6c_3| \leq -6c^3 + 32c = G(c), \tag{3.24}$$

where

$$G(c) = -6c^3 + 32c, \tag{3.25}$$

$$G'(c) = -18c^2 + 32. \tag{3.26}$$

For optimum value of $G(c)$, consider $G'(c) = 0$. From the equation (3.26), we obtain $c = \pm\frac{4}{3}$. Since $c \in [0, 2]$, consider $c = \frac{4}{3}$ (c^*) only. Further, we observe that $F(c, \mu)$ attains the maximum value at the point $[\frac{4}{3}, 1]$ only and from (3.25) it is given by

$$G_{\max} = \frac{256}{9}. \tag{3.27}$$

Simplifying the expressions (3.21), (3.24) along with (3.27), we obtain

$$|a_2a_3 - a_4| \leq \frac{4}{27}. \tag{3.28}$$

This completes the proof of our Theorem 3.5. □

The following results are straightforward verification on applying the same procedure of Theorems 3.2 and 3.3 respectively.

Theorem 3.6. *If $f(z) \in CV_s$, then $|a_3 - a_2^2| \leq \frac{1}{3}$ and the inequality is sharp for the values $c_1 = c = 0$, $c_2 = 2$ and $x = 1$.*

Theorem 3.7. *If $f(z) \in CV_s$, then $|a_k| \leq \frac{1}{k}$, for $k \in \{2, 3, 4, \dots\}$.*

For $f(z) \in CV_s$, using the result $|a_2a_4 - a_3^2| \leq \frac{1}{9}$ (see [13]) along with the results of Theorems 3.5, 3.6, 3.7 in the inequality (1.4), we have the following corollary.

Corollary 3.8. *If $f(z) \in CV_s$ then $|H_3(1)| \leq \frac{19}{135}$.*

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D. Vamshee Krishna
Department of Mathematics
GIT, GITAM University
Visakhapatnam-530 045, A.P.
India
e-mail: vamsheekrishna1972@gmail.com

B. Venkateswarlu
Department of Mathematics
GIT, GITAM University
Visakhapatnam-530 045, A.P.
India
bvlmaths@gmail.com

T. RamReddy
Department of Mathematics
Kakatiya University
Warangal-506 009, T.S.
India
e-mail: reddytr2@gmail.com

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