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On canonical constructions on connections

ABSTRACT. We study how a projectable general connection Γ in a 2-fibred manifold $Y^2 \to Y^1 \to Y^0$ and a general vertical connection Θ in $Y^2 \to Y^1 \to Y^0$ induce a general connection $A(\Gamma, \Theta)$ in $Y^2 \to Y^1$.

Introduction. In Section 1, we introduce the concepts of projectable general connections Γ and general vertical connections Θ in a 2-fibred manifold $Y^2 \to Y^1 \to Y^0$. In Section 2, we construct a general connection $\Sigma(\Gamma, \Theta)$ in $Y^2 \to Y^1$ from a projectable general connection Γ in $Y^2 \to Y^1 \to Y^0$ by means of a general vertical connection Θ in $Y^2 \to Y^1 \to Y^0$. In Section 3 we observe the canonical character of the construction $\Sigma(\Gamma, \Theta)$. In Section 4, we cite the concepts of natural operators. In Section 5, we describe completely the natural operators A transforming tuples (Γ, Θ) as above into general connections $A(\Gamma, \Theta)$ in $Y^2 \to Y^1$. In Section 6, we prove that there is no natural operator C producing general connections $C(\Gamma)$ in $Y^2 \to Y^1$ from projectable general connections Γ in $Y^2 \to Y^1 \to Y^0$. In Section 7, we present a construction of a general connection $\Sigma(\Gamma, \Theta)$ in $Y^2 \to Y^1$ from a system $\Gamma = (\Gamma^2, \Gamma^1)$ of a general connection Γ^2 in $Y^2 \to Y^0$ and a general connection Γ^1 in $Y^1 \to Y^0$ by means of a general vertical connection Θ in $Y^2 \to Y^1 \to Y^0$. In Section 8, we present an application of the obtained result in prolongation of general connections to bundle functors.

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All manifolds considered in the note is Hausdorff, second countable, without boundaries, finite dimensional and smooth (of class C^∞). Maps between manifolds are smooth (infinitely differentiable).

1. Connections. A fibred manifold is a surjective submersion $p: Y \to M$ between manifolds. By [1], an r-th order holonomic connection in $p: Y \to M$ is a section

$$\Gamma: Y \to J^r Y$$

of the holonomic r-jet prolongation $\pi^r_0: J^r Y \to Y$ of $Y \to M$. If $Y \to M$ is a vector bundle and $\Gamma: Y \to J^r Y$ is a vector bundle map, Γ is called a linear r-th order holonomic connection in $Y \to M$. A linear r-th order holonomic connection in the tangent bundle $Y = TM \to M$ of M is called an r-th order linear connection on M. A first order linear connection on M is in fact a classical linear connection on M.

A 1-order holonomic connection $\Gamma: Y \to J^1 Y$ in a fibred manifold $Y \to M$ is called a general connection in $Y \to M$.

We have the following equivalent definitions of general connections in $Y \to M$, see [1].

A general connection in $p: Y \to M$ is a lifting map

$$\Gamma: Y \times_M TM \to TY \ ,$$

i.e. a vector bundle map covering the identity map $id_Y: Y \to Y$ such that

$$Tp \circ \Gamma(y, w) = w$$

for any $y \in Y_x$, $w \in T_x M$, $x \in M$. (More precisely, $\Gamma(y, w) = T_x\sigma(w)$, where $\Gamma(y) = j^1_x\sigma$.)

A general connection in $Y \to M$ is a vector bundle decomposition

$$TY = VY \oplus_Y H^\Gamma$$

of the tangent bundle TY of Y, where VY is the vertical bundle of Y. (More precisely, $H^\Gamma_y = \operatorname{im} T_x\sigma$, where $\Gamma(y) = j^1_x\sigma$.)

A general connection in $Y \to M$ is a vector bundle projection (in direction H^Γ)

$$pr^\Gamma: TY \to VY$$

covering id_Y.

A 2-fibred manifold is a system $Y^2 \to Y^1 \to Y^0$ of two fibred manifolds $Y^2 \to Y^1$ and $Y^1 \to Y^0$.

Let $Y^2 \to Y^1 \to Y^0$ be 2-fibred manifold and

$$p^{ij}: Y^i \to Y^j \ , \ 0 \le j < i \le 2$$

be its projections. Of course, $p^{20} = p^{10} \circ p^{21}$. Let

$$V^{ij} Y^i := ker(Tp^{ij}: TY^i \to TY^j)$$

be the vertical bundle of $p^{ij}: Y^i \to Y^j$, $0 \le j < i \le 2$.

We introduce the following concepts of projectable general connections and of general vertical connections in 2-fibred manifolds $Y^2 \to Y^1 \to Y^0$.

A projectable general connection in $Y^2 \to Y^1 \to Y^0$ is a general connection

$$\Gamma : Y^2 \times_{Y^0} TY^0 \to TY^2$$

in $p^{20} : Y^2 \to Y^0$ such that there is a (unique) general connection

$$\underline{\Gamma} : Y^1 \times_{Y^0} TY^0 \to TY^1$$

in $p^{10} : Y^1 \to Y^0$ satisfying

$$Tp^{21} \circ \Gamma = \underline{\Gamma} \circ (p^{21} \times id_{TY^0}) \ .$$

Connection $\underline{\Gamma}$ is called the underlying connection of Γ.

A general vertical connection in $Y^2 \to Y^1 \to Y^0$ is a vector bundle map

$$\Theta : Y^2 \times_{Y^1} V^{10}Y^1 \to V^{20}Y^2$$

covering the identity map $id_{Y^2} : Y^2 \to Y^2$ such that

$$Tp^{21} \circ \Theta(y^2, v^1) = v^1$$

for any $y^2 \in Y^2_{y^1}$, $y^1 \in Y^1$ and $v^1 \in V^{10}_{y^1}Y^1$.

Equivalently, a general vertical connection in $Y^2 \to Y^1 \to Y^0$ is a smoothly parametrized system $\Theta = (\Theta_x)$ of general connections

$$\Theta_x : Y^2_x \times_{Y^1_x} TY^1_x \to TY^2_x$$

in the fibred manifolds $Y^2_x \to Y^1_x$ for any $x \in Y^0$, where Y^2_x is the fibre of $p^{20} : Y^2 \to Y^0$ over x and Y^1_x is the fibre of $p^{10} : Y^1 \to Y^0$ over x and $Y^2_x \to Y^1_x$ is the restriction of the projection $p^{21} : Y^2 \to Y^1$.

2. A construction. Let Γ be a projectable general connection in $Y^2 \to Y^1 \to Y^0$ with the underlying connection $\underline{\Gamma}$ and Θ be a general vertical connection in $Y^2 \to Y^1 \to Y^0$.

We define a map $\Sigma(\Gamma, \Theta) = \Sigma : Y^2 \times_{Y^1} TY^1 \to TY^2$ by

$$\Sigma(y^2, w^1) := \Theta(y^2, pr^{\underline{\Gamma}}(w^1)) + \Gamma(y^2, Tp^{10}(w^1)) \ ,$$

$y^2 \in Y^2_{y^1}$, $y^1 \in Y^1$, $w^1 \in T_{y^1}Y^1$, where $pr^{\underline{\Gamma}} : TY^1 \to V^{10}Y^1$ is the $\underline{\Gamma}$-projection.

Lemma 1. Σ *is a general connection in* $p^{21} : Y^2 \to Y^1$.

Proof. It is sufficient to verify that $Tp^{21} \circ \Sigma(y^2, w^1) = w^1$. We consider two cases.

(a) Let $w^1 \in V^{10}_{y^1}Y^1$. Then $\Sigma(y^2, w^1) = \Theta(y^2, w^1)$, and then

$$Tp^{21} \circ \Sigma(y^2, w^1) = Tp^{21} \circ \Theta(y^2, w^1) = w^1$$

as Θ is a general vertical connection in $Y^2 \to Y^1 \to Y^0$.

(b) Let $w^1 \in H^{\underline{\Gamma}}_{y^1}Y^1$, the $\underline{\Gamma}$-horizontal space. Denote $w^0 = Tp^{10}(w^1)$. Then $\Sigma(y^2, w^1) = \Gamma(y^2, w^0)$, and then

$$Tp^{21} \circ \Sigma(y^2, w^1) = Tp^{21} \circ \Gamma(y^2, w^0) = \underline{\Gamma}(p^{21}(y^2), w^0) = \underline{\Gamma}(y^1, w^0) \ .$$

Then $w' := Tp^{21} \circ \Sigma(y^2, w^1) \in H^{\underline{\Gamma}}_{y^1}Y^1$, $w^1 \in H^{\underline{\Gamma}}_{y^1}Y^1$ and

$$Tp^{10}(w') = Tp^{10} \circ Tp^{21} \circ \Gamma(y^2, w^0) = Tp^{20} \circ \Gamma(y^2, w^0) = w^0 = Tp^{10}(w^1) \ ,$$

and consequently $w' = w^1$. $\square$

3. Invariance. Let $\tilde{Y}^2 \to \tilde{Y}^1 \to \tilde{Y}^0$ be another 2-fibred manifold with projections $\tilde{p}^{ij} : \tilde{V}^i \to \tilde{V}^j$, $0 \le j < i \le 2$. Let $\tilde{\Gamma}$ be a projectable general connection in $\tilde{Y}^2 \to \tilde{Y}^1 \to \tilde{Y}^0$ and $\tilde{\Theta}$ be a general vertical connection in $\tilde{Y}^2 \to \tilde{Y}^1 \to \tilde{Y}^0$. Let $f = (f^2, f^1, f^0) : (Y^2 \to Y^1 \to Y^0) \to (\tilde{Y}^2 \to \tilde{Y}^1 \to \tilde{Y}^0)$ be a 2-fibred map, i.e. $f^i : Y^i \to \tilde{Y}^i$ for $i = 0, 1, 2$ and $\tilde{p}^{ij} \circ f^i = f^j \circ p^{ij}$ for $0 \le j < i \le 2$.

Lemma 2. *If Γ is f-related with $\tilde{\Gamma}$, (i.e. $Tf^2 \circ \Gamma = \tilde{\Gamma} \circ (f^2 \times_{f^0} Tf^0)$ and then $Tf^1 \circ \underline{\Gamma} = \underline{\tilde{\Gamma}} \circ (f^1 \times_{f^0} Tf^0)$) and Θ is f-related with $\tilde{\Theta}$ (i.e. $V^{20}f^2 \circ \Theta = \tilde{\Theta} \circ (f^2 \times_{f^1} V^{10}f^1)$), then $\Sigma = \Sigma(\Gamma, \Theta)$ is f-related with $\tilde{\Sigma} = \Sigma(\tilde{\Gamma}, \tilde{\Theta})$ (i.e. $Tf^2 \circ \Sigma = \tilde{\Sigma} \circ (f^2 \times_{f^1} Tf^1)$).*

Proof. If $w \in H^{\underline{\Gamma}}Y^1$, then $w = \underline{\Gamma}(y^1, w^0)$ for some $y^1 \in Y^1_{y^0}$ and $w^0 \in Y^0_{y^0}$, and then $Tf^1(w) = \underline{\tilde{\Gamma}}(f^1(y^1), Tf^0(w^0)) \in H^{\underline{\tilde{\Gamma}}}$. Then

$$Tf^1(H^{\underline{\Gamma}}Y^1) \subset H^{\underline{\tilde{\Gamma}}}\tilde{Y}^1 \text{ and (obviously) } Tf^1(V^{10}Y^1) \subset V^{10}\tilde{Y}^1 \ .$$

Consequently, $V^{10}f^1 \circ pr^{\underline{\Gamma}} = pr^{\underline{\tilde{\Gamma}}} \circ Tf^1$. Using this formula and the assumption of the lemma and the formula defining Σ, one can easily verify that

$$Tf^2 \circ \Sigma(y^2, w^1) = \tilde{\Sigma} \circ (f^2(y^2), Tf^1(w^1))$$

for $y^2 \in Y^2_{y^1}$, $w^1 \in T_{y^1}Y^1$, $y^1 \in Y^1$. $\square$

4. Natural operators. The general concept of natural operators can be found in [1]. We need the following partial cases of this general concept.

Let $\mathcal{FM}_{m_0,m_1,m_2}$ be the category of 2-fibred manifolds $Y^2 \to Y^1 \to Y^0$ with $\dim(Y^0) = m_0$, $\dim(Y^1) = m_0 + m_1$, $\dim(Y^2) = m_0 + m_1 + m_2$ and their 2-fibred local diffeomorphisms.

Definition 1. An $\mathcal{FM}_{m_0,m_1,m_2}$-natural operator transforming projectable general connections Γ and general vertical connections Θ in $\mathcal{FM}_{m_0,m_1,m_2}$-objects $Y^2 \to Y^1 \to Y^0$ into general connections $A(\Gamma, \Theta)$ in $Y^2 \to Y^1$ is an $\mathcal{FM}_{m_0,m_1,m_2}$-invariant system A of regular operators (functions)

$$A : Con_{proj}(Y^2 \to Y^1 \to Y^0) \times Con_{vert}(Y^2 \to Y^1 \to Y^0) \to Con(Y^2 \to Y^1)$$

for any $\mathcal{FM}_{m_0,m_1,m_2}$-objects $Y^2 \to Y^1 \to Y^0$, where $Con_{proj}(Y^2 \to Y^1 \to Y^0)$ is the set of projectable general connections in $Y^2 \to Y^1 \to Y^0$, $Con_{vert}(Y^2 \to Y^1 \to Y^0)$ is the set of general vertical connections in $Y^2 \to Y^1 \to Y^0$ and $Con(Y^2 \to Y^1)$ is the set of general connections in $Y^2 \to Y^1$.

The invariance of A means that if $\Gamma \in Con_{proj}(Y^2 \to Y^1 \to Y^0)$ is f-related with $\tilde{\Gamma} \in Con_{proj}(\tilde{Y}^2 \to \tilde{Y}^1 \to \tilde{Y}^0)$ and $\Theta \in Con_{vert}(Y^2 \to Y^1 \to Y^0)$ is f-related with $\tilde{\Theta} \in Con_{vert}(\tilde{Y}^2 \to \tilde{Y}^1 \to \tilde{Y}^0)$ for an $\mathcal{FM}_{m_0,m_1,m_2}$-morphism $f = (f^2, f^1, f^0) : (Y^2 \to Y^1 \to Y^0) \to (\tilde{Y}^2 \to \tilde{Y}^1 \to \tilde{Y}^0)$, then $A(\Gamma, \Theta)$ is f-related with $A(\tilde{\Gamma}, \tilde{\Theta})$.

The regularity of A means that A transforms smoothly parametrized families into smoothly parametrized families.

Because of Lemma 2, the construction $\Sigma(\Gamma, \Theta)$ defines an $\mathcal{FM}_{m_0,m_1,m_2}$-natural operator in the sense of Definition 1. So, to describe all natural operators A in the sense of Definition 1 it is sufficient to describe all natural operators in the sense of the following definition.

Definition 2. An $\mathcal{FM}_{m_0,m_1,m_2}$-natural operator transforming projectable general connections Γ and general vertical connections Θ in $\mathcal{FM}_{m_0,m_1,m_2}$-objects $Y^2 \to Y^1 \to Y^0$ into sections $B(\Gamma, \Theta) : Y^2 \to T^*Y^1 \otimes V^{21}Y^2$ of $T^*Y^1 \otimes V^{21}Y^2 \to Y^2$ is an $\mathcal{FM}_{m_0,m_1,m_2}$-invariant system A of regular operators

$$B : Con_{proj}(Y^2 \to Y^1 \to Y^0) \times Con_{vert}(Y^2 \to Y^1 \to Y^0) \to C^\infty_{Y^2}(T^*Y^1 \otimes V^{21}Y^2)$$

for any $\mathcal{FM}_{m_0,m_1,m_2}$-object $Y^2 \to Y^1 \to Y^0$, where $C^\infty_{Y^2}(T^*Y^1 \otimes V^{21}Y^2)$ is the space of sections of the vector bundle $T^*Y^1 \otimes V^{21}Y^2$ over Y^2 (with respect to the clear projection).

It is obvious that any natural operator A in the sense of Definition 1 is of the form

$$A(\Gamma, \Theta) = \Sigma(\Gamma, \Theta) + B(\Gamma, \Theta)$$

for a uniquely determined (by A) natural operator B in the sense of Definition 2.

A simple example of a natural operator in the sense of Definition 2 is the one B^o defined by

$$B^o(\Gamma, \Theta)(y^2)(w^1) = pr^{\Sigma(\Gamma,\Theta)} \circ \Theta(y^2, pr^{\Gamma}(w^1)) \in V^{21}_{y^2}Y^2$$

for any $\mathcal{FM}_{m_0,m_1,m_2}$-object $Y^2 \to Y^1 \to Y^0$, $\Gamma \in Con_{proj}(Y^2 \to Y^1 \to Y^0)$, $\Theta \in Con_{vert}(Y^2 \to Y^1 \to Y^0)$, $y^2 \in Y^2_{y^1}$, $y^1 \in Y^1$, $w^1 \in T_{y^1}Y^1$, where $pr^{\Sigma(\Gamma,\Theta)} : TY^2 \to V^{21}Y^2$ is the $\Sigma(\Gamma, \Theta)$-projection.

5. A classification. Let $\mathbf{R}^{m_0,m_1,m_2}$ be the trivial $\mathcal{FM}_{m_0,m_1,m_2}$-object $\mathbf{R}^{m_0}\times\mathbf{R}^{m_1}\times\mathbf{R}^{m_2}\to\mathbf{R}^{m_0}\times\mathbf{R}^{m_1}\to\mathbf{R}^{m_0}$ with the usual projections. Let $x^1,...,x^{m_0},y^1,...,y^{m_1},z^1,...,z^{m_2}$ be the usual coordinates on $\mathbf{R}^{m_0,m_1,m_2}$.

Consider a natural operator B in the sense of Definition 2. Because of the invariance of B with respect to 2-fibred manifold charts, B is determined by the linear maps

$$B(\Gamma,\Theta)(0,0,0):T_{(0,0)}(\mathbf{R}^{m_0}\times\mathbf{R}^{m_1})\to V^{21}_{(0,0,0)}(\mathbf{R}^{m_0}\times\mathbf{R}^{m_1}\times\mathbf{R}^{m_2})$$

for all $\Gamma\in Con_{proj}(\mathbf{R}^{m_0,m_1,m_2})$ and all $\Theta\in Con_{vert}(\mathbf{R}^{m_0,m_1,m_2})$ of the forms

$$\Gamma=\Gamma^o+\sum\Gamma^p_i(x,y)dx^i\otimes\frac{\partial}{\partial y^p}+\sum\Gamma^q_i(x,y,z)dx^i\otimes\frac{\partial}{\partial z^q}\ ,$$

$$\Theta=\Theta^o+\sum\Theta^q_p(x,y,z)dy^p\otimes\frac{\partial}{\partial z^q}\ ,$$

where the sums are over $i=1,...,m_0$, $p=1,...,m_1$, $q=1,...,m_2$, and where Γ^o denotes the trivial projectable general connection in $\mathbf{R}^{m_0,m_1,m_2}$ and $\Theta^o=\sum dy^p\otimes\frac{\partial}{\partial y^p}$ denotes the trivial general vertical connection in $\mathbf{R}^{m_0,m_1,m_2}$.

Eventually, using a new 2-fibred manifold chart one can additionally assume $\Gamma^p_i(0,0)=0$ and $\Gamma^q_i(0,0,0)=0$. (More precisely, denote $j^1_0\sigma:=\Gamma(0,0,0)$ and $\sigma(x)=:(x,\tilde\sigma(x),\overline{\sigma}(x))$. We consider the 2-fibred coordinate system $(x,y-\tilde\sigma(x),z-\overline{\sigma}(x))$. In the coordinate system $\Gamma(0,0,0)=\Gamma^o(0,0,0)$.)

Then using the invariance of B with respect to $\mathcal{FM}_{m_0,m_1,m_2}$-map $\frac{1}{t}\,id$ for $t>0$ and then putting $t\to0$, we can assume $\Gamma=\Gamma^o$ and $\Theta^q_p(x,y,z)=\Theta^q_p(0,0,0)=const$. Consequently, B is determined by the maps

$$B\Big(\Gamma^o,\Theta^o+\sum\Theta^q_pdy^p\otimes\frac{\partial}{\partial z^q}\Big)(0,0,0):\mathbf{R}^{m_0}\times\mathbf{R}^{m_1}\to\mathbf{R}^{m_2}$$

for all $\Theta^q_p\in\mathbf{R}$, $p=1,...,m_1$, $q=1,...,m_2$.

Using the invariance of B with respect to $t\,id_{\mathbf{R}^{m_0}}\times id_{\mathbf{R}^{m_1}}\times id_{\mathbf{R}^{m_2}}$ and then putting $t\to0$, we deduce that $B(\Gamma^o,\Theta^o+\sum\Theta^q_pdy^p\otimes\frac{\partial}{\partial z^q})(0,0,0)$ do not depend on elements from $\mathbf{R}^{m_0}$. Consequently, B is determined by the map $\Phi:\mathbf{R}^{m_1^*}\otimes\mathbf{R}^{m_2}\to\mathbf{R}^{m_1^*}\otimes\mathbf{R}^{m_2}$ given by

$$\Phi((\Theta^q_p))=B\Big(\Gamma^o,\Theta^o+\sum\Theta^q_pdy^p\otimes\frac{\partial}{\partial z^q}\Big)(0,0,0)\in\mathbf{R}^{m_1^*}\otimes\mathbf{R}^{m_2}\ .$$

Using the invariance of B with respect to linear isomorphisms from $\{id_{\mathbf{R}^{m_0}}\}\times GL(m_1)\times GL(m_2)$, we deduce that Φ is $GL(m_1)\times GL(m_2)$-invariant. Consequently, Φ is the constant multiple of the identity. Then the space of all $\mathcal{FM}_{m_0,m_1,m_2}$-natural operators B in the sense of Definition 2 is 1-dimensional. So, any natural operator B in the sense of Definition 2 is the constant multiple of B^o.

Thus we proved the following classification theorem.

Theorem 1. *Any $\mathcal{FM}_{m_0,m_1,m_2}$-natural operator A in the sense of Definition 1 is of the form*

$$A(\Gamma,\Theta)=\Sigma(\Gamma,\Theta)+\tau B^o(\Gamma,\Theta)$$

for a uniquely (by A) real number τ.

6. Why do we use auxiliary a general vertical connection? We prove the following theorem.

Theorem 2. *There is no $\mathcal{FM}_{m_0,m_1,m_2}$-natural operator*

$$C : Con_{proj}(Y^2\to Y^1\to Y^0)\to Con(Y^2\to Y^1)$$

transforming projectable general connections Γ in $\mathcal{FM}_{m_0,m_1,m_2}$-objects $Y^2\to Y^1\to Y^0$ into general connections $C(\Gamma)$ in $Y^2\to Y^1$.

Proof. Suppose that such C exists. Let Γ^o be the trivial projectable general connection in the 2-fibred manifold $\mathbf{R}^{m_0,m_1,m_2}$. Then $C(\Gamma^o)$ is φ-invariant by any $\mathcal{FM}_{m_0,m_1,m_2}$-map φ of the form $\varphi(x_0,x_1,x_2)=(x_0,\varphi_1(x_1),\varphi_2(x_1,x_2))$, $x_0\in\mathbf{R}^{m_0}$, $x_1\in\mathbf{R}^{m_1}$, $x_2\in\mathbf{R}^{m_2}$ (as Γ^o is). Then $j^1_{(0,0)}\sigma:=C(\Gamma^o)(0,0,0)$ is φ-invariant for any φ as above with $\varphi(0,0,0)=(0,0,0)$. Then for $\varphi_1(x_1)=x_1$ and $\varphi_2(x_1,x_2)=x_2+(x_1^1,0,...,0)$ we get $j^1_{(0,0)}(\varphi\circ\sigma)=j^1_{(0,0)}\sigma$, i.e. $j^1_{(0,0)}\eta=0$, where $\eta(x_0,x_1)=(x_0,x_1,x_1^1,0,...,0)$. Contradiction. □

So, to construct canonically a general connection in $Y^2\to Y^1$ from a projectable general connection in $Y^2\to Y^1\to Y^0$ the using of auxiliary objects is unavoidable. In the present note we have used general vertical connections as such auxiliary ones.

7. A generalization. Let $Y^2\to Y^1\to Y^0$ be a 2-fibred manifold.

A projectable general connection Γ in $Y^2\to Y^1\to Y^0$ is in fact a system $\Gamma=(\Gamma,\underline{\Gamma})$ of two general connections in $p^{20}:Y^2\to Y^0$ and $p^{10}:Y^1\to Y^0$ (respectively), and $\underline{\Gamma}$ is determined by Γ.

In this section, we present how to extend the construction of $\Sigma(\Gamma,\Theta)$ for $\Gamma=(\Gamma,\underline{\Gamma})$ into a construction $\Sigma(\Gamma,\Theta)$ for $\Gamma=(\Gamma^2,\Gamma^1)$, where $\Gamma^2 : Y^2\times_{Y^0}TY^0\to TY^2$ is a general connection in $p^{20}:Y^2\to Y^0$ and $\Gamma^1 : Y^1\times_{Y^0}TY^0\to TY^1$ is a general connection in $p^{10}:Y^1\to Y^0$.

Let $\Gamma=(\Gamma^2,\Gamma^1)$ and Θ be in question. We define a map $\Sigma(\Gamma,\Theta)=\Sigma : Y^2\times_{Y^1}TY^1\to TY^2$ by

$$\Sigma(y^2,w^1):=\Theta(y^2,pr^{\Gamma^1}(w^1))+\Gamma^2(y^2,w^0)-\Theta(y^2,pr^{\Gamma^1}\circ Tp^{21}\circ\Gamma^2(y^2,w^0))\ ,$$

$$y^2\in Y^2_{y^1},\ y^1\in Y^1,\ w^1\in T_{y^1}Y^1,\ w^0=Tp^{10}(w^1)\ .$$

Lemma 3. *Σ is a general connection in $p^{21}:Y^2\to Y^1$.*

Proof. We are going to prove that $Tp^{21}\circ\Sigma(y^2,w^1)=w^1$. We consider two cases.

(a) Let $w^1 \in V^{10}_{y^1}Y^1$. Then $\Sigma(y^2, w^1) = \Theta(y^2, w^1)$, and next we proceed as in the part (a) of the proof of Lemma 1.

(b) Let $w^1 \in H^{\Gamma^1}_{y^1}Y^1$. Then

$$\Sigma(y^2, w^1) = \Gamma^2(y^2, w^0) - \Theta(y^2, pr^{\Gamma^1} \circ Tp^{21} \circ \Gamma^2(y^2, w^0)) ,$$

and then

$$Tp^{21} \circ \Sigma(y^2, w^1) = Tp^{21} \circ \Gamma^2(y^2, w^0) - pr^{\Gamma^1} \circ Tp^{21} \circ \Gamma^2(y^2, w^0) .$$

So, $w' := Tp^{21} \circ \Sigma(y^2, w^1) \in H^{\Gamma^1}_{y^1}Y^1$ and $w^1 \in H^{\Gamma^1}_{y^1}Y^1 \in H^{\Gamma^1}_{y^1}Y^1$ and

$$Tp^{10}(w') = Tp^{20} \circ \Gamma^2(y^2, w^0) - 0 = w^0 = Tp^{10}(w^1) ,$$

and consequently $w' = w^1$. □

8. An application. We can use the construction $\Sigma(\Gamma, \Theta)$ from the previous section in prolongation of connections to bundle functors.

Namely, let $F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be a bundle functor in the sense of [1] of order r, where $\mathcal{FM}$ is the category of fibred manifolds and fibred maps and $\mathcal{FM}_{m,n}$ is the category of fibred manifolds with m-dimensional bases and n-dimensional fibres and their local fibred diffeomorphisms. Let $p : Y \to M$ be an $\mathcal{FM}_{m,n}$-object. Let Ξ be a general connection in $p : Y \to M$ and λ be an r-th order linear connection on M (i.e. r-th order linear connection in $TM \to M$). Thus we have the F-prolongation $\mathcal{F}(\Xi, \lambda)$ (of Ξ with respect to λ) in the sense of [1, Def. 45.4]. $\mathcal{F}(\Xi, \lambda)$ is a general connection in $FY \to M$. Let λ^1 be an r-th order linear connection in $VY \to Y$. Using the construction $\Sigma(\Gamma, \Theta)$ from the previous section, we can construct a general connection $\mathcal{F}(\Xi, \lambda_1, \lambda)$ in $FY \to Y$ as follows.

Let $Y^2 = FY \to Y^1 = Y \to Y^0 = M$ be the 2-fibred manifold. We have a general vertical connection $\Theta = \Theta(\lambda^1) : Y^2 \times_{Y^1} V^{10}Y^1 \to V^{20}Y^2$ in $Y^2 \to Y^1 \to Y^0$ by

$$\Theta(\lambda^1)(y^2, v^1) := \mathcal{F}X(y^2) \ , \ j^r_{y^1}(X) := \lambda^1(v^1) ,$$

$y^2 \in Y^2_{y^1}$, $y^1 \in Y^1$, $v^1 \in V^{10}_{y^1}Y^1$, where $\mathcal{F}X$ is the flow lift of X with respect to F. Denote $\Gamma = (\mathcal{F}(\Xi, \lambda), \Xi)$. Consequently, we have a general connection $\mathcal{F}(\Xi, \lambda, \lambda^1)$ in $FY \to Y$ by

$$\mathcal{F}(\Xi, \lambda, \lambda^1) := \Sigma(\Gamma, \Theta(\lambda^1)) .$$

Let Ξ and λ be as above and Λ be an r-th order linear connection on Y (i.e. r-th order linear connection in $TY \to Y$). Using the above construction $\mathcal{F}(\Xi, \lambda, \lambda^1)$, we can construct a general connection $\mathcal{F}(\Xi, \lambda, \Lambda)$ in $FY \to Y$ as follows.

We have an r-th order linear connection $\lambda^1 = \lambda^1(\Lambda, \Xi)$ in $VY \to Y$ by

$$\lambda^1(v) = j^r_y(pr^{\Xi} \circ X) \ , \ j^r_y X := \Lambda(v) \ , \ v \in V_yY \ , \ y \in Y ,$$

where $pr^{\Xi} : TY \to VY$ is the Ξ-projection. Then we have a general connection $\mathcal{F}(\Xi, \lambda, \Lambda)$ in $FY \to Y$ by

$$\mathcal{F}(\Xi, \lambda, \Lambda) := \mathcal{F}(\Xi, \lambda, \lambda^1(\Lambda, \Xi)) \ .$$

References

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