doi: 10.1515/umcsmath-2015-0022

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXIX, NO. 2, 2015 SECTIO A 69–72

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On regular local operators on smooth maps

Abstract. Let X, Y, Z, W be manifolds and $\pi : Z \to X$ be a surjective submersion. We characterize π-local regular operators $A : C^\infty(X, Y) \to C^\infty(Z, W)$ in terms of the corresponding maps $\tilde{A} : J^\infty(X, Y) \times_X Z \to W$ satisfying the so-called local finite order factorization property.

Let X, Y, Z, W be smooth (i.e. C^∞) manifolds and $\pi : Z \to X$ be a surjective C^∞-submersion. The space of smooth (C^∞) maps $U \to V$ we denote by $C^\infty(U, V)$.

An operator $A : C^\infty(X, Y) \to C^\infty(Z, W)$ is π-local if for any $g_1, g_2 \in C^\infty(X, Y)$ and any $x \in X$ from $germ_x(g_1) = germ_x(g_2)$ it follows $A(g_1)_{|\pi^{-1}(x)} = A(g_2)_{|\pi^{-1}(x)}$.

An operator $A : C^\infty(X, Y) \to C^\infty(Z, W)$ is regular if any C^∞ parametrized system of maps from $C^\infty(X, Y)$ is transformed into a C^∞ parametrized system of maps in $C^\infty(Z, W)$, i.e. if it satisfies the implication: if $g : X \times \mathbf{R} \to Y$ is of class C^∞, then so is $Z \times \mathbf{R} \ni (z, t) \to A(g_t)(z) \in W$, where $g_t = g(-, t)$.

Let $J^r(X, Y)$ be the space of r-jets of maps $X \to Y$. $J^s(X, Y)$ is a finite dimensional manifold if s is finite. $J^\infty(X, Y)$ has the inverse limit topology from $\cdots \to J^s(X, Y) \to J^{s-1}(X, Y) \to \cdots \to J^0(X, Y)$. Let $\pi_r : J^\infty(X, Y) \to J^r(X, Y)$ be the jet projection.

2010 *Mathematics Subject Classification.* 53A55.
Key words and phrases. Local regular operator, jet.

We say that a map $\tilde{A} : J^\infty(X,Y) \times_X Z \to W$ satisfies the local finite order factorization property if for any $(\kappa, z) \in J^\infty(X,Y) \times_X Z$ there exist an open neighborhood $U \subset J^\infty(X,Y) \times_X Z$ of (κ, z), a finite number r and a C^∞ (in the classical sense) map $\tilde{A}^r : (\pi_r \times id_Z)(U) \to W$ such that $\tilde{A} = \tilde{A}^r \circ (\pi_r \times id_Z)$ on U. (We see that $(\pi_r \times id_Z)(U)$ is an open subset in finite dimensional manifold $J^r(X,Y) \times_X Z$.)

The main result is the following theorem.

Theorem 1. *Let X, Y, Z, W be C^∞-manifolds and $\pi : Z \to X$ a surjective C^∞-submersion. There is a bijection between the π-local regular operators $A : C^\infty(X,Y) \to C^\infty(Z,W)$ and the maps $\tilde{A} : J^\infty(X,Y) \times_X Z \to W$ with the local finite order factorization property. Precisely, the correspondence is given by $A(g)(z) = \tilde{A}(j^\infty g(\pi(z)), z)$, $g \in C^\infty(X,Y)$, $z \in Z$.*

Proof of Theorem 1. Since operators are local, for the simplicity of considerations we will assume $X = \mathbf{R}^m$ and $Y = \mathbf{R}^n$.

From Corollary 19.8 in [1] it follows:

Lemma 1. *Any π-local operator A as above is of infinite order, i.e. if $g_1, g_2 \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)$, $x \in \mathbf{R}^n$, then from $j^\infty g_1(x) = j^\infty g_2(x)$ it follows $A(g_1)_{|\pi^{-1}(x)} = A(g_2)_{|\pi^{-1}(x)}$.*

From Lemma 19.11 in [1] it follows:

Lemma 2. *Let $A : C^\infty(\mathbf{R}^m, \mathbf{R}^n) \to C^\infty(Z,W)$ be a π-local operator. Let $z_o \in Z$ be a point, $x_o := \pi(z_o)$, $f \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)$. Let $\epsilon : \mathbf{R}^m \setminus \{x_o\} \to \mathbf{R}$, $\epsilon(x) = \exp(-|x - x_o|^{-1})$. There are a neighborhood V of the point $z_o \in Z$ and a natural number r such that for every $z \in V \setminus \pi^{-1}(x_o)$ and all maps $g_1, g_2 \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)$ satisfying $|\partial^\alpha(g_i - f)(\pi(z))| \le \epsilon(\pi(z))$, $i = 1, 2$, $0 \le |\alpha| \le r$, the condition $j^r g_1(\pi(z)) = j^r g_2(\pi(z))$ implies $A(g_1)(z) = A(g_2)(z)$.*

Similarly as in [2], any regular π-local operator $A : C^\infty(\mathbf{R}^m, \mathbf{R}^n) \to C^\infty(Z,W)$ defines a $\pi \times id_{\mathbf{R}}$-local operator $A^{<>} : C^\infty(\mathbf{R}^m \times \mathbf{R}, \mathbf{R}^n) \to C^\infty(Z \times \mathbf{R}, W)$, $A^{<>}(g)(z,t) := A(g_t)(z)$, where $g_t : \mathbf{R}^m \to \mathbf{R}^n$, $g_t(x) = g(x,t)$.

Applying Lemma 2 to the above operator $A^{<>}$ (defined by A) and treating maps $h : \mathbf{R}^m \to \mathbf{R}^n$ as maps $h : \mathbf{R}^m \times \mathbf{R} \to \mathbf{R}^n$ being independent with respect to the last argument we get:

Lemma 3. *Let $A : C^\infty(\mathbf{R}^m, \mathbf{R}^n) \to C^\infty(Z,W)$ be a regular π-local operator. Let $z_o \in Z$ be a point, $x_o := \pi(z_o)$, $f \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)$. Let $\tilde{\epsilon} : \mathbf{R}^{m+1} \setminus \{(x_o, 0)\} \to \mathbf{R}$, $\tilde{\epsilon}(x,t) := \exp(-|(x - x_o, t)|^{-1})$. There are a neighborhood $\tilde{V}$ of $z_o \in Z$, a real number $t_o > 0$ and a natural number $\tilde{r}$ such that for every $z \in \tilde{V}$ and all maps $g_1, g_2 \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)$ satisfying $|\partial^\alpha(g_i - f)(\pi(z))| \le \tilde{\epsilon}(\pi(z), t_o)$, $i = 1, 2$, $0 \le |\alpha| \le \tilde{r}$, the condition $j^{\tilde{r}} g_1(\pi(z)) = j^{\tilde{r}} g_2(\pi(z))$ implies $A(g_1)(z) = A(g_2)(z)$.*

We see that $t_o \leq |(\pi(z) - x_o, t_o)|$ for any z. Then $2\eta_o := \tilde{\epsilon}(x_o, t_o) \leq \tilde{\epsilon}(\pi(z), t_o)$. So, from Lemma 3, we have:

Lemma 4. *Let $A : C^\infty(\mathbf{R}^m, \mathbf{R}^n) \to C^\infty(Z, W)$ be a regular π-local operator. Let $z_o \in Z$ be a point, $x_o := \pi(z_o)$, $f \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)$. There are a neighborhood $\tilde{V}$ of $z_o \in Z$, a real number $\eta_o > 0$ and a natural number $\tilde{r}$ such that for every $z \in \tilde{V}$ and all maps $g_1, g_2 \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)$ satisfying $|\partial^\alpha(g_i - f)(\pi(z))| \leq 2\eta_o$, $i = 1, 2$, $0 \leq |\alpha| \leq \tilde{r}$, the condition $j^{\tilde{r}} g_1(\pi(z)) = j^{\tilde{r}} g_2(\pi(z))$ implies $A(g_1)(z) = A(g_2)(z)$.*

Taking (eventually) smaller $\tilde{V}$ such that $|\partial^\alpha f(\pi(z)) - \partial^\alpha f(\pi(z_o))| \leq \eta_o$ for $z \in \tilde{V}$, $0 \leq |\alpha| \leq \tilde{r}$, we get:

Lemma 5. *Let $A : C^\infty(\mathbf{R}^m, \mathbf{R}^n) \to C^\infty(Z, W)$ be a regular π-local operator. Let $z_o \in Z$ be a point, $x_o := \pi(z_o)$, $f \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)$. There are a neighborhood $\tilde{V}$ of $z_o \in Z$, a real number $\eta_o > 0$ and a natural number $\tilde{r}$ such that for all $z \in \tilde{V}$ and for all $g_1, g_2 \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)$ satisfying $|\partial^\alpha g_i(\pi(z)) - \partial^\alpha f(\pi(z_o))| < \eta_o$, $i = 1, 2$, $0 \leq |\alpha| \leq \tilde{r}$, the condition $j^{\tilde{r}} g_1(\pi(z)) = j^{\tilde{r}} g_2(\pi(z))$ implies $A(g_1)(z) = A(g_2)(z)$.*

Thus Lemma 5 can be reformulated as follows.

Lemma 6. *Let $A : C^\infty(\mathbf{R}^m, \mathbf{R}^n) \to C^\infty(Z, W)$ be a regular π-local operator. Let $z_o \in Z$ be a point, $x_o := \pi(z_o)$, $f \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)$, $\kappa_o := j^\infty f(\pi(z_o))$. There are a natural number r and an open neighborhood $V \subset J^r(\mathbf{R}^m, \mathbf{R}^n) \times_{\mathbf{R}^m} Z$ of $(\pi_r(\kappa_o), z_o)$ such that for any $g_1, g_2 \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)$ and z with $(j^r g_i(\pi(z)), z) \in V$, $i = 1, 2$, the condition $j^r g_1(\pi(z)) = j^r g_2(\pi(z))$ implies $A(g_1)(z) = A(g_2)(z)$.*

Any map $\tilde{A} : J^\infty(\mathbf{R}^m, \mathbf{R}^n) \times_{\mathbf{R}^m} Z \to W$ satisfying the local finite order factorization property defines a regular π-local operator $A : C^\infty(\mathbf{R}^m, \mathbf{R}^n) \to C^\infty(Z, W)$. Namely, we have

Example 1. Let $\tilde{A} : J^\infty(\mathbf{R}^m, \mathbf{R}^n) \times_{\mathbf{R}^m} Z \to W$ be a map satisfying the local finite order factorization property. Define an operator $A : C^\infty(\mathbf{R}^m, \mathbf{R}^n) \to W^Z$ by

$$A(f)(z) := \tilde{A}(j^\infty f(\pi(z)), z) \ .$$

Clearly, A is π-local. Consider a smoothly parametrized family of maps $f_t \in C^\infty(\mathbf{R}^n, \mathbf{R}^n)$, $t_o \in \mathbf{R}$ and $z_o \in Z$. By the local finite order factorization property, there are natural number r, an open neighborhood U^r of $(j^r f_{t_o}(\pi(z_o)), z_o)$ in $J^r(\mathbf{R}^m, \mathbf{R}^n) \times_{\mathbf{R}^m} Z$ and a smooth map $\tilde{A}^r : U^r \to W$ such that $A(f_t)(z) = \tilde{A}^r(j^r f_t(\pi(z)), z)$ for (t, z) from some neighborhood of (t_o, z_o). That is why, A has values in $C^\infty(Z, W)$ and it is regular.

Conversely, we have:

Example 2. Let $A : C^\infty(\mathbf{R}^m, \mathbf{R}^n) \to C^\infty(Z, W)$ be a regular π-local operator. We have a function $\tilde{A} : J^\infty(\mathbf{R}^m, \mathbf{R}^n) \times_{\mathbf{R}^m} Z \to W$ by

$$\tilde{A}(\kappa, z) := A(g)(z) \ ,$$

where $\kappa = j^\infty g(\pi(z))$, $g \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)$. (By Lemma 1, the definition is independent of the choice of g.)

Lemma 7. *$\tilde{A}$ satisfies the local finite order factorization property.*

Proof. Consider $(\kappa_o, z_o) \in J^\infty(\mathbf{R}^m, \mathbf{R}^n) \times_{\mathbf{R}^m} Z$, $x_o = \pi(z_o)$. Choose $f \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)$ such that $\kappa_o = j^\infty f(\pi(z_o))$. Let r and V be as in Lemma 6 for z_o, x_o, f as above. Put $U := (\pi_r \times id_Z)^{-1}(V)$. Define $\tilde{A}^r : V = (\pi_r \times id_Z)(U) \to W$ by

$$\tilde{A}^r(\rho, z) := A(g)(z) \ ,$$

where $\rho = j^r g(\pi(z))$, $g \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)$. (By Lemma 6, the definition is independent of the choice of g.) For any smooth curve γ in V, $\gamma(t) = (\rho_t, z_t) \in V$, $t \in \mathbf{R}$, there is a smoothly parametrized family $g_t \in C^\infty(\mathbf{R}^m, \mathbf{R}^n)$ with $\rho_t = j^r g_t(\pi(z_t))$. Then $\tilde{A}^r \circ \gamma(t) = A(g_t)(z_t)$. Then the regularity of A implies $\tilde{A}^r \circ \gamma$ is of C^∞ (for any smooth curve γ in V). Then $\tilde{A}^r$ is of C^∞ because of the well-known Boman theorem. Clearly $\tilde{A} = \tilde{A}^r \circ (\pi_r \times id_Z)$ on U. □

Summing up, we have proved Theorem 1. □

References

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Received November 9, 2015