doi: 10.1515/umcsmath-2015-0021

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXIX, NO. 2, 2015 SECTIO A 61–68

RACHID ELHARTI and MOHAMED MABROUK

Vector space isomorphisms of non-unital reduced Banach $*$-algebras

Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be two non-unital reduced Banach $*$-algebras and $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a vector space isomorphism. The two following statement holds: If ϕ is a $*$-isomorphism, then ϕ is isometric (with respect to the C^*-norms), bipositive and ϕ maps some approximate identity of $\mathcal{A}$ onto an approximate identity of $\mathcal{B}$. Conversely, any two of the later three properties imply that ϕ is a $*$-isomorphism. Finally, we show that a unital and self-adjoint spectral isometry between semi-simple Hermitian Banach algebras is an $*$-isomorphism.

1. Preliminaries. Our objective under this heading is to describe the basic concepts of reduced Banach $*$-algebras and to try and synthesize some results that are pertinent to the purposes of our paper.

A Banach $*$-algebra is a Banach algebra over the complex field (with a norm denoted by $\|.\|$) together with a fixed involution denoted by $*$. A Banach $*$-algebra is called Hermitian if and only if the spectrum of each self-adjoint element $h = h^*$ in $\mathcal{A}$ is contained in the real line. A $*$-representation of a Banach $*$-algebra $\mathcal{A}$ is an algebra homeomorphism π of $\mathcal{A}$ into the algebra $B(H)$ of all bounded operators on some Hilbert space H. On any Banach $*$-algebra $\mathcal{A}$, there is a maximum C^*-pseudo-norm $\gamma_{\mathcal{A}}$ which satisfies

$$\gamma_{\mathcal{A}}(a) = \sup\{\|\pi(a)\| : \pi \text{ is a } *\text{-representation of } \mathcal{A}\} \tag{1.1}$$

2010 *Mathematics Subject Classification.* 15A86, 46L05, 47A12, 47B49.

Key words and phrases. Reduced Banach algebras, preserving the spectrum.

which is called the Gefland–Naimark pseudo-norm. The algebra $\mathcal{A}$ is said to be reduced if $\gamma_{\mathcal{A}}$ is a norm. That is, if $\gamma_{\mathcal{A}}$ is well defined and $\{a \in \mathcal{A} : \gamma_{\mathcal{A}}(a) = 0\} = \{0\}$. The class of reduced $*$-algebras incorporates a wide class of Banach $*$-algebras. Indeed, any Hermitian and semi-simple Banach $*$-algebra is reduced (including C^*-algebras as a very special case). An example of a reduced Banach algebra which is not hermitian is the algebra of all complex-valued continuously differentiable mappings on $[0,1]$ with pointwise definition of addition, scalar multiplication, product, and the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$, where $\|f\|_\infty = \sup_{t\in[0,1]} |f(t)|$. One more interesting example is the group algebra $L^1(G)$, for some locally compact group G. It is worth mentioning that $L^1(G)$ is Hermitian when G is commutative, but not so in the general case.

In the remainder of this paper, all algebras considered are assumed to be reduced. Therefore, the completion $\hat{A}$ of $\mathcal{A}$ with respect to the C^*-norm $\gamma_{\mathcal{A}}$ is a C^*-algebra. At this juncture, we are to denote by $\mathcal{A}_+$ the set of positive elements as $\mathcal{A}_+ = \{\sum_{k=1}^n aa^* : a \in \mathcal{A},\ n \in \mathbb{N}\}$. Clearly, the following inclusion holds: $\mathcal{A}_s := \{h^2 : h = h^* \in \mathcal{A}\} \subset \mathcal{A}_+$. In general the inclusion is strict, but if $\mathcal{A}$ is Hermitian or a C^*-algebra, then $\mathcal{A}_s = \mathcal{A}_+$.

On a Banach $*$-algebra $\mathcal{A}$, a linear functional $p \in \mathcal{A}^*$ (where $\mathcal{A}^*$ is the topological dual of $\mathcal{A}$ with respect to the norm $\|.\|$) is positive if $p(\mathcal{A}_+) \subset \mathbb{R}_+$ (denoted $p \geq 0$) and a state if $p \geq 0$ and $\|p\| = 1$. The set of all states of $\mathcal{A}$ is denoted by $S_{\mathcal{A}}$. A linear mapping $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ between two reduced Banach $*$-algebras is said to be positive if $\phi(\mathcal{A}_+) \subset \mathcal{B}_+$. Recall also that ϕ is called *unital* if $\phi(1) = 1$, and it is said to be a *Jordan homomorphism* if $\phi(a^2) = \phi(a)^2$ for all $a \in \mathcal{A}$. Equivalently, the map ϕ is a Jordan homomorphism if and only if $\phi(ab + ba) = \phi(a)\phi(b) + \phi(b)\phi(a)$ for all a and b in $\mathcal{A}$. We also recall that the map ϕ is said to be *self-adjoint* provided that $\phi(a^*) = \phi(a)^*$ for all $a \in \mathcal{A}$. Self-adjoint Jordan homomorphisms are called *Jordan $*$-homomorphisms*, and by a Jordan $*$-isomorphism, we mean a bijective $*$-homomorphism.

2. Main results. In [6], Kadisson showed that every Jordan $*$-isomorphism between two unital C^*-algebras is isometric and bipositive and unital. Furthermore, the presence of any combination of two of the latter three properties implies that ϕ is a $*$-isomorphism. These results have been generalized for non-unital C^*-algebras in [10]. The first aim of this paper is to show that the same result holds for non-unital reduced Banach $*$-algebras with bounded approximate identities.

Recall that a bounded approximate identify of an Banach $*$-algebra $\mathcal{A}$ with respect to the norm $\|.\|$ is a net $(e_\alpha)_{\alpha\in\Lambda}$ in $\mathcal{A}$ such that $\sup_\alpha e_\alpha < \infty$ and $\lim_\alpha (\|a - ae_\alpha\| + \|a - e_\alpha a\|) = 0$, for every $a \in \mathcal{A}$. We state the following:

Theorem 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be reduced Banach $*$-algebras having bounded approximate identities relative to the norm $\|.\|$ and $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a vector space isomorphism. If ϕ is a Jordan $*$-isomorphism, then ϕ is isometric (with respect to the C^*-norms), bipositive and ϕ maps some approximate identity of $\mathcal{A}$ (relative to the norm $\gamma_{\mathcal{A}}$) onto an approximate identity of $\mathcal{B}$ (relative to the norm $\gamma_{\mathcal{B}}$).*

Conversely, the presence of any combination of two of the latter three properties implies that ϕ is a Jordan $*$-isomorphism.

To prove the main theorem, we need the following lemmas. The first lemma is devoted to the existence of a bounded approximate identity relative to the norm $\gamma_{\mathcal{A}}$ such that its image by an $*$-isomorphism is a bounded approximate identity for $\mathcal{B}$. It is worth observing that this lemma does not require the existence of a bounded approximate identity relative to the norm $\|.\|$.

Lemma 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two reduced Banach $*$-algebras. Let $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a Jordan $*$-isomorphism. There exists an approximate identity $(u_j)_{j\in J}$ in $\mathcal{A}$ such that its image $(\phi u_j)_{j\in J}$ is an approximate identity for $\mathcal{B}$.*

Proof. Since ϕ is a Jordan $*$-isomorphism between two reduced algebras, then it is contractive relative to $\gamma_{\mathcal{A}}$ and $\gamma_{\mathcal{B}}$ (see [8], Proposition 10.1.4). Extend ϕ by continuity to Jordan $*$-isomorphism $\hat{\phi} : \hat{\mathcal{A}} \longrightarrow \hat{\mathcal{B}}$ of ϕ between the two C^*-algebras $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$. According to [10, Lemma 2.3], there exists an approximate identity $(h_\beta)_{\beta\in\Lambda}$ in $\hat{\mathcal{A}}$ such that $(\hat{\phi}h_\beta)_{\beta\in\Lambda}$ is an approximate identity for $\hat{\mathcal{B}}$. At this level, we proceed as in [8, Proposition 10.1.13]. Since every element in $\hat{\mathcal{A}}$ is a limit of a sequence in $\mathcal{A}$, then, for all $\beta \in \Lambda$, there exist $n \in \mathbb{N}$ and $e_n^\beta \in \mathcal{A}$ satisfying $\gamma_{\mathcal{A}}(e_n^\beta - h_\beta) \leq \frac{1}{n}$. Consequently, we might safely assume that e_n^β is self-adjoint and $\gamma_{\mathcal{A}}(e_n^\beta) \leq 1$.

Now, define $u_j = e_n^\beta$ and $J = \Lambda \times \mathbb{N}$ ordered by defining $j_1 = (\beta_1, n_1) \geq j_2 = (\beta_2, n_2)$ to mean $\beta_1 \geq \beta_2$ and $n_1 \geq n_2$. It is easy to notice that u_j is an approximate identity of $\mathcal{A}$. Similarly, by using the fact that $\hat{\phi}$ is a contraction, the net $(\phi u_j)_{j\in J}$ satisfies $\gamma_{\mathcal{B}}(\phi u_j - \hat{\phi}h_\beta) \leq \frac{1}{n}$ and $\gamma_{\mathcal{B}}(\phi u_j) \leq 1$. It follows also that $(\phi u_j)_{j\in J}$ is an approximate identity for $\mathcal{B}$. $\square$

We shall need also the following lemma, [3, Proposition 2.1], which shows that if $(e_\alpha)_{\alpha\in\Lambda}$ is a bounded approximate identity of a normed algebra $\mathcal{A}$, then it is also a bounded approximate identity for its completion $\hat{\mathcal{A}}$. We give its proof for the sake of completeness.

Lemma 2.3. *Let $(\mathcal{A}, \gamma_{\mathcal{A}})$ be a normed algebra and denote by $\hat{\mathcal{A}}$ its completion with respect to the norm $\gamma_{\mathcal{A}}$. Then every bounded approximate identity $(e_\alpha)_{\alpha\in\Lambda}$ of $\mathcal{A}$ is also a bounded approximate identity of $\hat{\mathcal{A}}$.*

Proof. Let $a \in \hat{\mathcal{A}}$ and $(a_n) \subset \mathcal{A}$ such that $\lim_{n\to\infty} \gamma_{\mathcal{A}}(a_n - a) = 0$. For any $n \in \mathbb{N}$, we have

$$\begin{aligned}\gamma_{\mathcal{A}}(e_\alpha a - a) &\leq \gamma_{\mathcal{A}}(e_\alpha a - e_\alpha a_n) + \gamma_{\mathcal{A}}(e_\alpha a_n - a_n) + \gamma_{\mathcal{A}}(a_n - a)\\ &\leq \gamma_{\mathcal{A}}(e_\alpha)\ \gamma_{\mathcal{A}}(a - a_n) + \gamma_{\mathcal{A}}(e_\alpha a_n - a_n) + \gamma_{\mathcal{A}}(a_n - a).\end{aligned}$$

Using the fact that $\lim_{n\to\infty} \gamma_{\mathcal{A}}(a_n - a) = \lim_{\alpha} \gamma_{\mathcal{A}}(e_\alpha a_n - a_n) = 0$, and the boundedness of (e_α), we can find an integer $n \in \mathbb{N}$ and $\beta \in \Lambda$ such that $\gamma_{\mathcal{A}}(e_\alpha a - a) < \epsilon$, whenever $\alpha \geq \beta$. This shows that $\lim_\alpha \gamma_{\mathcal{A}}(e_\alpha a - a) = 0$. In a similar way, we can also show that $\lim_\alpha \gamma_{\mathcal{A}}(a e_\alpha - a) = 0$. This completes the proof. □

Now we show that every positive mapping ϕ between two reduced Banach $*$-algebras is bounded with respect to the C^*-norms. We begin with the following:

Lemma 2.4. *Let $\mathcal{A}$ be a reduced Banach $*$-algebra with bounded approximate identity $\{e_\alpha\}$ (with respect to the norm $\|.\|$) and $p : \mathcal{A} \longrightarrow \mathbb{C}$ be a linear form. If p is positive, then it is bounded relative to the norm $\gamma_{\mathcal{A}}$ and $\|p\|_* \leq \sup_\alpha p(e_\alpha e_\alpha^*)$, (here $\|p\|_*$ denotes the norm of p relative to the C^*-norm $\gamma_{\mathcal{A}}$).*

Proof. Let p be a positive linear form. Firstly, notice that p is continuous with respect to the norm $\|.\|$ and hermitian (i.e. $p(x^*) = \overline{p(x)}$ for any $x \in \mathcal{A}$), (see [4, Corollary 27.5]). Without loss of generality, assume that $p \neq 0$, since $p \equiv 0$ is certainly bounded. Suppose first that $\mathcal{A}$ is unital. We distinguish two cases. If p is a state, then from the Gelfand–Naimark–Segal theorem (see [4, Theorem 27.2]), there exists a cyclic $*$-representation π of $\mathcal{A}$ on a Hilbert space H, with cyclic vector ξ of norm 1 in H so that $p(a) = (\pi(a)\xi, \xi)$. It follows from the Cauchy–Schwartz inequality that

$$\begin{aligned}|p(a)| &\leq \|\pi(a)\xi\|\|\xi\|\\ &\leq \|\pi(a)\|\|\xi\|^2 = \|\pi(a)\|.\end{aligned}$$

From Equation (1.1), we see that $\|\pi(a)\| \leq \gamma_{\mathcal{A}}(a)$, which implies the boundedness of p with respect to $\gamma_{\mathcal{A}}$ and $\|p\|_* \leq 1 = p(1)$. If p is positive, let $q = p(1)^{-1}p$. It is obvious that q is a state. Then q is bounded from above, hence p is bounded and $\|p\|_* \leq p(1)$. Finally, assume that $\mathcal{A}$ is non-unital. Let $p_1(x + \lambda e) = p(x) + \lambda k$ for any $x + \lambda e \in \mathcal{A}_e$ where $\mathcal{A}_e = \mathcal{A} \oplus \mathbb{C}$ is the the unitization of $\mathcal{A}$ and $k = \sup_\alpha p(e_\alpha e_\alpha^*)$. Since p is continuous with respect of the norm $\|.\|$, then [4, Proposition 21.5] implies that $|p(x)|^2 \leq k\ p(xx^*)$, for all $x \in \mathcal{A}$. A similar reasoning as in the proof of [4, Proposition 21.7] shows that p_1 is a positive linear functional of $\mathcal{A}_e$ which coincides with p on $\mathcal{A}$. Therefore, $\|p\|_* \leq \|p_e\|_* \leq p_e(e) = k$. This completes the proof of boundedness of p. □

Lemma 2.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be two reduced Banach $*$-algebras such that $\mathcal{A}$ has a bounded approximate identity relative to the norm $\|.\|$. Then, every positive linear mapping $\phi : (\mathcal{A}, \gamma_{\mathcal{A}}) \longrightarrow (\mathcal{B}, \gamma_{\mathcal{B}})$ is bounded.*

Proof. Let $a \in \mathcal{A}$ with $a = a^*$. By [9, Proposition 1.5.4], we have

$$\gamma_{\mathcal{B}}(\phi(a)) = \sup_{p \in S_{\hat{\mathcal{B}}}} |p \circ \phi(a)|.$$

By Lemma 2.4, $p \circ \phi$ is a bounded and positive linear functional, for any $p \in S_{\hat{\mathcal{B}}}$. Accordingly

$$|p \circ \phi(a)| \leq \|p \circ \phi\|_* \ \gamma_{\mathcal{A}}(a) \ \leq \ \sup_{\alpha} p \circ \phi(e_\alpha e_\alpha^*) \ \gamma_{\mathcal{A}}(a).$$

By keeping in mind that every $p \in S_{\hat{\mathcal{B}}}$ is continuous with respect to $\gamma_{\mathcal{B}}$ and $\|p\|_* = 1$, we obtain

$$|p \circ \phi(e_\alpha e_\alpha^*)| \leq \|p\|_* \gamma_{\mathcal{B}}(\phi(e_\alpha e_\alpha^*)) = \gamma_{\mathcal{B}}(\phi(e_\alpha e_\alpha^*)).$$

Put $\theta = \sup_\alpha \gamma_{\mathcal{A}}(\phi(e_\alpha e_\alpha^*))$ which is a constant independent of p. Hence, the above inequality implies that

$$\gamma_{\mathcal{B}}(\phi(a)) \leq \theta \ \gamma_{\mathcal{A}}(a), \ \text{for any self-adjoint element in } \mathcal{A}.$$

Therefore, ϕ is continuous with respect to the C^*-norms on the set of self-adjoint elements. Since every element $a \in \mathcal{A}$ is a linear combination of two self-adjoint elements, the continuity of the involution and the positivity of ϕ implies that ϕ is continuous. The proof is thus complete. $\square$

Now, we give the proof of Theorem 2.1.

Proof of Theorem 2.1. Suppose ϕ is a Jordan $*$-isomorphism. By Lemma 2.2, ϕ maps some approximate identity of $\mathcal{A}$ onto an approximate identity for $\mathcal{B}$. Since ϕ and ϕ^{-1} are contractive, then $\gamma_{\mathcal{B}}(\phi a) = \gamma_{\mathcal{A}}(a), \forall a \in \mathcal{A}$. Hence, ϕ is isometric. The extension $\hat{\phi}$ of ϕ is also a $*$-isomorphism between the two C^*-algebras $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$. Thus, Theorem 3.1 of [10] may be applied to show that ϕ is bipositive.

To prove the converse, we have three cases:

Case 1: Assume that ϕ is bipositive and maps some approximate identity of $\mathcal{A}$ onto an approximate identity of $\mathcal{B}$. By Lemma 2.5, ϕ is bounded. Extend ϕ by continuity to a bounded vector space isomorphism $\hat{\phi} : \hat{\mathcal{A}} \longrightarrow \hat{\mathcal{B}}$ where $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ are the completions with respect to the C^*-norms of $\mathcal{A}$ and $\mathcal{B}$ respectively. The set $\hat{\mathcal{A}}^+$ of positive elements in a C^*-algebra such as $\hat{\mathcal{A}}$ is closed and $\hat{\mathcal{A}}_+ = \hat{\mathcal{A}}_s$. Hence by continuity $\hat{\phi}$ is bipositive. Now, Lemma 2.3 entails that $\hat{\phi}$ is a bipositive vector space isomorphism which maps some approximate identity of $\hat{\mathcal{A}}$ onto an approximate identity of $\hat{\mathcal{B}}$. According to [10, Theorem 3.1] , we infer that $\hat{\phi}$, and hence ϕ, is a Jordan $*$-isomorphism.

Case 2: If ϕ is bipositive and isometric. Extend ϕ by continuity to a bijective isometry $\hat{\phi} : \hat{\mathcal{A}} \longrightarrow \hat{\mathcal{B}}$. A similar reasoning as in the first case entails that $\hat{\mathcal{A}}$ is also bipositive. Again, by [10, Theorem 3.1], ϕ is a Jordan $*$-isomorphism.
Case 3: If ϕ is isometric and maps an approximate identity of $\mathcal{A}$ into an approximate identity of $\mathcal{B}$. Then, similarly the extension $\hat{\phi}$ of ϕ is isometric and maps an approximate identity of $\hat{\mathcal{A}}$ into an approximate identity of $\hat{\mathcal{B}}$. It yields that ϕ is a Jordan $*$-isomorphism. This concludes the proof of the theorem. $\square$

As an application of Theorem 2.1, we characterize spectral isometries([1]) between semi-simple hermitian Banach $*$-algebras. Before presenting our result, we recall the famous Ford's square root lemma which will be crucial for our purpose.

Lemma 2.6 ([2, 5]). *Let $\mathcal{A}$ be a Banach $*$-algebra with $a \in \mathcal{A}$, $a = a^*$ and $r(a) < 1$. Then, there exists a unique $x \in \mathcal{A}$ with $2x - x^2 = a$, $r(x) < 1$ and $x = x^*$.*

Theorem 2.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be two hermitian semi-simple Banach $*$-algebras and $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a surjective and unital spectral isometry. If ϕ is self-adjoint, then it is a Jordan $*$-isomorphism.*

Proof. Let us first prove that ϕ is a vector space isomorphism. It is enough to show that ϕ is injective. Let $x \in \mathcal{A}$ be such that $\phi(x) = 0$. For $y \in \mathcal{A}$, we obtain $r_{\mathcal{A}}(x + y) = r_{\mathcal{B}}(\phi(x + y)) = r_{\mathcal{B}}(\phi(y)) = r_{\mathcal{A}}(y)$. Hence, by [1, Theorem 5.3.1], x belongs to the radical of $\mathcal{A}$ which is zero. Thus $x = 0$ and ϕ is injective. Now, we show that ϕ is bipositive, that is $\phi(\mathcal{A}_+) = \mathcal{B}_+$. Let $a \in \mathcal{A}$ be such that $\|a\| < 1$. By the spectral mapping theorem, we know that $\sigma(1 - aa^*) \subset \mathbb{R}^+$. In addition, since $\mathcal{A}$ is semi-simple, this fact yields $\|1 - aa^*\| < 1$. Since ϕ is a unital spectral isometry, we have $r_{\mathcal{B}}(\phi(aa^*) - 1) < 1$. By the square root lemma there exists $x \in \mathcal{A}$ satisfying $x = x^*$ and $(1 - x)^2 = \phi(aa^*)$. In this manner, we have showed that $\phi(\mathcal{A}_+) \subset \mathcal{B}_+$. Since ϕ^{-1} is also a unital spectral isometry, by symmetry we obtain $\phi^{-1}(\mathcal{B}_+) \subset \mathcal{A}_+$ or $\mathcal{B}_+ \subset \phi(\mathcal{A}_+)$, which implies that $\phi(\mathcal{A}^+) = \mathcal{B}^+$. Hence, ϕ is unital and bipositive vector space isomorphism. Therefore, by Theorem 2.1 we conclude that ϕ is a Jordan $*$-isomorphism. $\square$

Remark 2.8. It is well known that every C^*-algebra is a Hermitian semi-simple Banach algebras. This makes the above theorem as an improvement of [7, Proposition 2].

Now we prove the following:

[1]Spectral isometry means that $r_{\mathcal{A}}(a) = r_{\mathcal{B}}(Ta), \forall a \in \mathcal{A}$

Corollary 2.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be Hermitian Banach $*$-algebras and $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ be a self-adjoint and unital bijective spectral isometry. Then, ϕ induce a Jordan $*$-isomorphism $\tilde{\phi} : \mathcal{A}/R(\mathcal{A}) \longrightarrow \mathcal{B}/R(\mathcal{B})$ where $R(\mathcal{A})$ and $R(\mathcal{B})$ denote the Jacobson radical of $\mathcal{A}$ and $\mathcal{B}$, respectively.*

Proof. Let us first prove that $\phi(R(\mathcal{A})) = R(\mathcal{B})$. To this end, we make use of the characterization of the radical given by [1, Theorem 5.3.1]. Take $a \in R(\mathcal{A})$ and $y \in \mathcal{B}$ such that $r_{\mathcal{A}}(y) = 0$. Choose $x \in \mathcal{A}$ with $\phi(x) = y$. By hypothesis $r_{\mathcal{A}}(x) = r_{\mathcal{B}}(y) = 0$. Together, these yield

$$r_{\mathcal{B}}(\phi(a) + y) = r_{\mathcal{B}}(\phi(a + x)) = r_{\mathcal{A}}(a + x) = 0.$$

So that $\phi(a) \in R(\mathcal{B})$. Therefore $\phi(R(\mathcal{A})) \subset R(\mathcal{B})$. In the same way, we can show that $\phi^{-1}(R(\mathcal{B})) \subset R(\mathcal{A})$ or equivalently $R(\mathcal{B})) \subset \phi(R(\mathcal{A}))$. Thus, we have showed that $\phi(R(\mathcal{A})) = R(\mathcal{B})$. However, here the $*$-radical, which is the intersection of the kernels of all $*$-representations of $\mathcal{A}$, coincides with the radical by [4, Corollary 33.13]. Hence by [4, Proposition 32.9], we have $\mathcal{A}_1 = \mathcal{A}/R(\mathcal{A})$ and $\mathcal{B}_1 = \mathcal{B}/R(\mathcal{B})$ are two unital semi-simple Hermitian Banach algebras. Again, by [1, Theorem 3.1.5], we have $\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{A}_1}(\bar{a})$ for the coset $\bar{a}$ of $a \in \mathcal{A}$ in $\mathcal{A}_1$ and $\sigma_{\mathcal{B}}(b) = \sigma_{\mathcal{B}_1}(\bar{b})$ for all $b \in \mathcal{B}$. Now since, $\phi(R(\mathcal{A})) = R(\mathcal{B})$ the mapping $\widetilde{\phi} : \mathcal{A}_1 \longrightarrow \mathcal{B}_1$ given by $\widetilde{\phi}(\bar{a}) = \overline{T(a)}$ for every $\bar{a} \in \mathcal{A}_1$ is well defined. It is also clear that $\tilde{\phi}$ is a bijective self-adjoint unital spectral isometry. Theorem 2.7 implies that $\tilde{\phi}$ is a Jordan $*$-isomorphism. □

References

[1] Aupetit, B., *Spectrum-preserving linear mappings between Banach algebras or Jordan–Banach algebras*, J. Lond. Math. Soc. **62** (2000), 917–924.

[2] Bonsall, F. F., Stirling, D. S. G., *Square roots in Banach $*$-algebras*, Glasg. Math. J. **13** (1972), 74–74.

[3] Dixon, P. G., *Approximate identities in normed algebras*, Proc. Lond. Math. Soc. **26** (3) (1973), 485–496.

[4] Doran R. S., Belfi, V. A., *Characterizations of C^*-algebras. The Gelfand–Naimark Theorems*, Marcel Dekker, New York, 1986.

[5] Ford, J. W. M., *A square root lemma for Banach (*)-algebras*, J. Lond. Math. Soc. **42** (1) (1967), 521–522.

[6] Kadisson, R. V., *Isometries of operator algebras*, Ann. of Math. **54** (2) (1951), 325–338.

[7] Martin, M., *Towards a non-selfadjoint version of Kadison's theorem*, Ann. Math. Inform. **32** (2005), 87–94.

[8] Palmer, T. W., *Banach Algebras and the General Theory of $*$-Algebras. $*$-Algebras, Vol. II*, Cambridge University Press, Cambridge, 2001.

[9] Sakai, S., *C^*-algebras and W^*-algebras*, Springer-Verlag, New York–Berlin, 1971.

[10] Ylinen, K., *Vector space isomorphisms of C^*-algebras*, Studia Math. **46** (1973), 31–34.

Rachid ElHarti
Department of Mathematics
Faculty of Applied Sciences
Umm Al-qura University
21955 Makkah
Saudi Arabia

Department of Mathematics
and Computer Sciences
Faculty of Sciences and Techniques
University Hassan I, BP 577. Settat
Morocco (Permanent address)
e-mail: relharti@gmail.com

Mohamed Mabrouk
Department of Mathematics
Faculty of Applied Sciences
Umm Al-qura University
21955 Makkah
Saudi Arabia

Department of Mathematics
Faculty of Sciences of Gabès
University of Gabès, Cité Erriadh
6072 Zrig, Gabès
Tunisia (Permanent address)
e-mail: msmabrouk@uqu.edu.sa

Received September 6, 2014