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Horizontal lifts of tensor fields to the bundle of volume forms

Abstract. Dhooghe in [Dho] has given the definition and basic properties of a horizontal lift of a vector field to the bundle of volume forms in order to investigate the Thomas connection from the point of view of projective connection. In this paper we present a systematic approach to the horizontal lift of tensor fields to the bundle of volume forms of basic types of tensors with respect to a symmetric linear connection.

1. Basic definitions. Let M be an oriented manifold and let $\mathcal{V}$ be a line bundle of the volume forms over M (see [Dho], [DVV]). We consider two charts (U, x^i) and $(\bar{U}, \bar{x}^i)$ of M, $U \cap \bar{U} \neq \emptyset$, and the volume form $\omega \in \mathcal{V}$, $\omega = v(x) dx^1 \wedge \ldots \wedge dx^n = v(\bar{x}) d\bar{x}^1 \wedge \ldots \wedge d\bar{x}^n$, $v, \bar{v} > 0$. The functions $(v, x^1, \ldots, x^n)$ (resp. $(v, \bar{x}^1, \ldots, \bar{x}^n)$) are called the local coordinates of ω in the chart (U, x^i) (resp. $(\bar{U}, \bar{x}^i)$). In our setting the functions $\bar{x}^i = \bar{x}^i(x)$ are the orientation-preserving transition functions on M. Then the lifted functions on $\mathcal{V}$ are given as

$$\bar{v} = \bar{\mathcal{I}} \cdot v, \qquad \bar{x}^i = \bar{x}^i(x), \tag{1.1}$$

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where $\bar{\mathcal{I}} = \det\left(\frac{\partial \bar{x}^i}{\partial x^j}\right)$ is the Jacobian of the map $\bar{x}^i = \bar{x}^i(x)$. Following [Dho] we introduce a new coordinate system $(x^0, \dots, x^n)$ on $\mathcal{V}$, where $x^0 = \ln v$. The transition functions in terms of these coordinates are

$$\begin{cases} \bar{x}^0 = x^0 + \ln \bar{\mathcal{I}}, \\ \bar{x}^i = \bar{x}^i(x). \end{cases} \tag{1.2}$$

We put $\bar{\mathcal{J}}(x) = \ln \bar{\mathcal{I}}(x)$ and $\mathcal{J}(\bar{x}) = \ln \mathcal{I}(\bar{x})$. Since $\mathcal{I} \cdot \bar{\mathcal{I}} = 1$, we have

$$\frac{\partial \mathcal{J}}{\partial \bar{x}^i} = -\frac{\partial \bar{\mathcal{J}}}{\partial x^j}\frac{\partial x^j}{\partial \bar{x}^i} \tag{1.3}$$

and

$$\frac{\partial \bar{\mathcal{J}}}{\partial x^i} = -\frac{\partial \mathcal{J}}{\partial \bar{x}^j}\frac{\partial \bar{x}^j}{\partial x^i}\ . \tag{1.4}$$

Note that the Jacobian matrix of the transition function (1.2) has the following form

$$\begin{bmatrix} 1 & \dfrac{\partial \bar{\mathcal{J}}}{\partial x^i} \\ 0 & \dfrac{\partial \bar{x}^j}{\partial x^i} \end{bmatrix}. \tag{1.5}$$

Let M be equipped with a linear symmetric connection Γ. We have the following well-known formulas (see e.g. [Sch])

$$\begin{aligned} \bar{\Gamma}^i_{ki} &= \frac{\partial x^m}{\partial \bar{x}^k}\Gamma^i_{mi} + \frac{\partial \mathcal{J}}{\partial \bar{x}^i}, \\ \Gamma^i_{ki} &= \frac{\partial \bar{x}^m}{\partial x^k}\bar{\Gamma}^i_{mi} + \frac{\partial \bar{\mathcal{J}}}{\partial x^i}, \end{aligned} \tag{1.6}$$

where Γ^i_{kj} and $\bar{\Gamma}^i_{kj}$ are the coefficients of Γ in the coordinates (x^i) and $(\bar{x}^i)$, respectively.

2. The horizontal lift. Note that on $\mathcal{V}$ there is a canonical vector field $\frac{\partial}{\partial x^0}$. Moreover, it is easy to check that a 1-form $\eta = dx^0 + \Gamma^k_{ik}dx^i$ is globally defined on $\mathcal{V}$ and $d\eta = \Gamma^k_{ik|j}dx^j \wedge dx^i$, where $\Gamma^k_{ik|j} = \frac{\partial \Gamma^k_{ik}}{\partial x^j}$. We call η the canonical 1-form on $\mathcal{V}$. The vector field $\frac{\partial}{\partial x^0}$ and the 1-form η define the canonical tensor field of type $(1,1)$ on $\mathcal{V}$ by the formula

$$\left(\eta \otimes \frac{\partial}{\partial x^0}\right)(w) = \eta(w)\frac{\partial}{\partial x^0}. \tag{2.1}$$

Theorem 2.1. ([Dho], [DVV]) *Let $v = v^i \frac{\partial}{\partial x^i}$ be a vector field on M. Then*

$$v^H = -v^i \Gamma^k_{ik} \frac{\partial}{\partial x^0} + v^i \frac{\partial}{\partial x^i} \tag{2.2}$$

is a globally defined vector field on $\mathcal{V}$ called the horizontal lift of v.

By a direct calculation we get

Lemma 2.1.

$$\left[v^H, w^H\right] = [v,w]^H + d\eta(v^H, w^H)\frac{\partial}{\partial x^0}. \tag{2.3}$$

Lemma 2.2. *Suppose that $v_1, \dots v_n$ are local vector fields which are linearly independent at each point. Then $\frac{\partial}{\partial x^0}, v_1^H, \dots, v_n^H$ are linearly independent at each point. In particular, each vector field on $\mathcal{V}$ is locally a linear combination of such vector fields.*

Now, we are going to consider the horizontal lift of 1-forms. By straightforward calculations we get

Theorem 2.2. *If $\omega = \omega_i dx^i$ is a 1-form on M then*

$$\omega^H = \left(\omega_i + \Gamma^k_{ik}\right) dx^i + dx^0 \tag{2.4}$$

is a 1-form on $\mathcal{V}$ called the horizontal lift of ω.

Corollary 2.1.

(1) *The horizontal lift of 1-forms is not linear.*
(2) *The canonical 1-form η is the lift of zero 1-form, that is $\eta = 0^H$*
(3) *For any 1-form ω and and any vector field v we have $(\omega(v))^V = \omega^H\left(v^H\right)$, where $(\omega(v))^V$ denotes the vertical lift of the function $\omega(v)$.*

Theorem 2.3. *The horizontal lift ω^H of a 1-form to $\mathcal{V}$ is unique and satisfies*

$$\omega^H\left(v^H\right) = (\omega(v))^V, \qquad \omega^H\left(\frac{\partial}{\partial x^0}\right) = 1. \tag{2.5}$$

Proof. The proof follows directly from Lemma 2.2. □

The proof of the next theorem follows by direct calculations of coordinate transformations of coefficients of a tensor under consideration.

Theorem 2.4. *Let $F = \left(F^i_j\right)$ be a tensor of type $(1,1)$ on M. Then*

$$F^H = \begin{bmatrix} 1 & -F^t_i\Gamma^k_{tk} + \Gamma^k_{ik} \\ & \\ 0 & F^i_j \end{bmatrix} \tag{2.6}$$

defines a tensor of the type $(1,1)$ on $\mathcal{V}$. The tensor F^H is called the horizontal lift of F.

Similarly as in the case of 1-forms we have

Lemma 2.3.

$$(F(v))^H = F^H\left(v^H\right). \tag{2.7}$$

Lemma 2.4. *If F and G are any tensors of type $(1,1)$ on M then*

$$(F \circ G)^H = F^H \circ G^H. \tag{2.8}$$

From Lemma 2.2 we have

Theorem 2.5. *The horizontal lift F^H to $\mathcal{V}$ of a tensor F of type $(1,1)$ is unique and satisfies*

$$F^H\left(v^H\right) = (F(v))^H, \qquad F^H\left(\frac{\partial}{\partial x^0}\right) = \frac{\partial}{\partial x^0}. \tag{2.8}$$

Corollary 2.2.

1) $(I_M)^H = I_{\mathcal{V}}$,
2) $(-I_M)^H = -I_{\mathcal{V}} + 2\eta \otimes \frac{\partial}{\partial x^0}$,
3) $(-F)^H = -\left(F^H\right) + 2\eta \otimes \frac{\partial}{\partial x^0}$

In the next theorem we suppose that the connection Γ is locally volume preserving which means that locally there exists a volume form ω which is parallel with respect to Γ. In this case there exist local coordinate systems such that $\Gamma^k_{ik} = 0$ (see [Dh1], [Dh2], [Sch]).

Note that in this case we have

$$\left[v^H, w^H\right] = [v,w]^H. \tag{2.9}$$

Theorem 2.6. *Let F be a tensor field of type $(1,1)$ on M. Suppose that Γ is a symmetric locally volume preserving linear connection. Then*

$$N_F = 0 \Longleftrightarrow N_{F^H} = 0, \tag{2.10}$$

where N_F (resp. N_{F^H}) denotes the Nijenhuis tensor of F (resp. F^H).

Proof. Observe that $N_{F^H}\left(v^H, \frac{\partial}{\partial x^0}\right) = 0$. Moreover,

$$\begin{aligned} N_{F^H}\left(v^H, w^H\right) &= \left(N_F(v,w)\right)^H \\ &\quad + \Big(d\eta\left(F(v)^H, F(w)^H\right) - d\eta\left(v^H, w^H\right) \\ &\quad - d\eta\left(v^H, F(w)^H\right) - d\eta\left(F(v)^H, y^H\right)\Big)\frac{\partial}{\partial x^0}. \end{aligned} \tag{2.11}$$

But according to our assumptions we have $d\eta = 0$. □

Corollary 2.3.

1) *If F is an almost complex structure on M, that is $F \circ F = -I_M$, then $F^H \circ F^H = -I_{\mathcal{V}} + 2\eta \otimes \frac{\partial}{\partial x^0}$.*
2) *If $F \circ F = I_M$ then $F^H \circ F^H = I_{\mathcal{V}}$.*
3) *If $F^3 + F = 0$ then $\left(F^H\right)^3 + F^H = 2\eta \otimes \frac{\partial}{\partial x^0}$.*
4) *If $F^3 - F = 0$ then $\left(F^H\right)^3 - F^H = 0$.*

Now we are going to describe the horizontal lift of a Riemannian metric.

Theorem 2.7. *Let $g = (g_{ij})$ be a tensor of type $(0,2)$ on M. Then*

$$g^H = \begin{bmatrix} 1 & \Gamma^k_{ik} \\ \Gamma^k_{ik} & g_{ij} + \Gamma^k_{ik}\Gamma^k_{jk} \end{bmatrix} \tag{2.12}$$

is globally defined $(0,2)$-tensor on $\mathcal{V}$. The tensor g^H is called the horizontal lift of g.

Proof. It is enough to check the transformation rule. □

Theorem 2.8. *The tensor g^H is unique on $\mathcal{V}$ and satisfies*

1) $g^H\left(v^H, w^H\right) = \left(g(v,w)\right)^H$,
2) $g^H\left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^0}\right) = 1$,
3) $g^H\left(\frac{\partial}{\partial x^0}, v^H\right) = 0$.

Proof. Conditions 1), 2), 3) follow from definitions and the uniqueness follows from Lemma 2.2. □

For the tensors of type $(2,0)$ we have

Theorem 2.9. *Let* $h = \left(h^{ij}\right)$ *be a tensor of type* $(2,0)$ *on* M. *Then*

$$h^H = \begin{bmatrix} h^{ij}\Gamma^k_{ik}\Gamma^t_{jt} & -h^{ij}\Gamma^k_{jk} \\ -h^{ij}\Gamma^k_{jk} & h^{ij} \end{bmatrix} \tag{2.13}$$

is globally defined $(2,0)$*-tensor on* $\mathcal{V}$. *The tensor* h^H *is called the horizontal lift of* h.

Theorem 2.10. *The tensor* h^H *is the unique* $(2,0)$*-tensor on* $\mathcal{V}$ *such that*

1) $h^H\left(\omega^H,\varphi^H\right) = \left(h(\omega,\varphi)\right)^V$,
2) $h^H(\eta,\eta) = 0$,
3) $h^H\left(\omega^H,\eta\right) = h^H\left(\eta,\omega^H\right) = 0$.

Theorem 2.11. *Let* g *be a Riemannian metric on* M. *Then* g^H *is a Riemannian metric on* $\mathcal{V}$ *and*

$$\left(g^H\right)^{-1} = \left(g^{-1}\right)^H + \frac{\partial}{\partial x^0}\otimes\frac{\partial}{\partial x^0} \tag{2.14}$$

Proof. Due to Theorem 2.8 we know that g^H is nonsingular and positively defined. By multiplication one can check directly that $\left(g^H\right)^{-1}\circ g^H = g^H\circ\left(g^H\right)^{-1} = I_{\mathcal{V}}$. $\square$

We shall consider now the horizontal lift of the tensors of type $(0,p)$ and $(p,0)$. Checking the transformation rule we have the following two theorems.

Theorem 2.12. *Let* $F = \left(f^{i_1 i_2\ldots i_p}\right)$ *be a tensor of type* $(p,0)$ *on* M. *Then* $F^H = \left(h^{\alpha_1\alpha_2\ldots\alpha_p}\right)$, $\alpha_i\in\{0,1,\ldots n\}$, *where*

$$\begin{aligned}
&h^{00\ldots0} = f^{i_1 i_2\ldots i_p}\Gamma^{t_1}_{i_1t_1}\Gamma^{t_2}_{i_2t_2}\ldots\Gamma^{t_p}_{i_pt_p},\\
&h^{0\ldots i_k\ldots0} = -f^{i_1\ldots i_k\ldots i_p}\Gamma^{t_1}_{i_1t_1}\ldots\Gamma^{t_{k-1}}_{i_{k-1}t_{k-1}}\Gamma^{t_{k+1}}_{i_{k+1}t_{k+1}}\ldots\Gamma^{t_p}_{i_pt_p},\\
&h^{0\ldots i_k\ldots i_m\ldots0} =\\
&= f^{i_1\ldots i_k\ldots i_m\ldots i_p}\Gamma^{t_1}_{i_1t_1}\ldots\Gamma^{t_{k-1}}_{i_{k-1}t_{k-1}}\Gamma^{t_{k+1}}_{i_{k+1}t_{k+1}}\ldots\Gamma^{t_{m-1}}_{i_{m-1}t_{m-1}}\Gamma^{t_{m+1}}_{i_{m+1}t_{m+1}}\ldots\Gamma^{t_p}_{i_pt_p},\\
&\qquad\vdots\\
&h^{i_1i_2\ldots i_p} = (-1)^p f^{i_1i_2\ldots i_p}
\end{aligned}$$

is a tensor of type $(p,0)$ *on* $\mathcal{V}$.

Theorem 2.13. *Let* $G = (g_{i_1 i_2 \dots i_p})$ *be a tensor on* M *of type* $(0,p)$. *Then* $G^H = (h_{\alpha_1 \alpha_2 \dots \alpha_p})$, *where*

$$\begin{aligned} &h_{00\dots0} = 1, \\ &h_{0\dots i\dots 0} = \Gamma^k_{ik}, \\ &h_{0\dots i\dots j\dots 0} = \Gamma^t_{it}\Gamma^k_{jk}, \\ &\qquad\qquad \vdots \\ &h_{i_1 i_2\dots i_p} = g_{i_1 i_2 \dots i_p} + \Gamma^{t_1}_{i_1 t_1}\Gamma^{t_2}_{i_2 t_2}\dots\Gamma^{t_p}_{i_p t_p}, \end{aligned}$$

is a tensor of type $(0,p)$ *on* $\mathcal{V}$.

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