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Subordination chains and Loewner differential equations in several complex variables

Abstract. Let B be the unit ball of $\mathbb{C}^n$ with respect to the Euclidean norm and $f(z,t)$ be a Loewner chain. In this paper we study certain properties of $f(z,t)$ and we obtain a sufficient condition for the transition mapping associated to $f(z,t)$ to satisfy the Loewner differential equation.

1. Introduction and preliminaries. Let $\mathbb{C}^n$ be the space of n complex variables $z = (z_1, \dots, z_n)'$ with the usual inner product $\langle z, w\rangle = \sum_{j=1}^{n} z_j \overline{w}_j$ and the Euclidean norm $\|z\| = \langle z, z\rangle^{1/2}$, $z \in \mathbb{C}^n$. The symbol $'$ means transpose of vectors and matrices. Let $B_r = \{z \in \mathbb{C}^n : \ \|z\| < r\}$ and let $B = B_1$ be the unit ball in $\mathbb{C}^n$. The closed ball $\{z \in \mathbb{C}^n : \ \|z\| \le r\}$ is denoted by $\overline{B}_r$. In the case of one variable B_r is denoted by U_r and the unit disc U_1 by U. If G is an open set in $\mathbb{C}^n$, let $H(G)$ be the set of holomorphic maps from G into $\mathbb{C}^n$.

By $L(\mathbb{C}^n, \mathbb{C}^m)$ we denote the space of continuous linear operators from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the standard operator norm. Let I be the identity in $L(\mathbb{C}^n, \mathbb{C}^n)$.

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We say that $f \in H(B)$ is locally biholomorphic on B if f has a local holomorphic inverse at each point in B.

If $f, g \in H(B)$, we say that f is subordinate to g if there is a Schwarz mapping v (i.e. $v \in H(B)$, $v(0) = 0$ and $\|v(z)\| < 1$, $z \in B$) such that $f(z) = g(v(z))$, $z \in B$. We shall write $f \prec g$ to mean that f is subordinate to g.

Definition 1.1. The mapping $f : B \times [0, \infty) \to \mathbb{C}^n$ is called a Loewner chain (or a subordination chain) if

(i) $f(\cdot, t)$ is holomorphic and univalent on B, $t \geq 0$;

(ii) $f(0, t) = 0$, $Df(0, t) = e^t I$, $t \geq 0$;

(iii) $f(\cdot, s) \prec f(\cdot, t)$ for $0 \leq s \leq t < \infty$;

(iv) $f(z, t)$ is a locally absolutely continuous function of $t \in [0, \infty)$ locally uniformly with respect to $z \in B$.

Note that in the case of one variable, the assumption (iv) is always satisfied, as a consequence of the distortion result for the class of normalized univalent functions on U.

The subordination condition (iii) is equivalent to the condition

$$f(z, s) = f(v(z, s, t), t), \quad z \in B,\ 0 \leq s \leq t < \infty, \tag{1.1}$$

where $v = v(z, s, t)$ is a univalent Schwarz mapping, normalized by $v(0, s, t) = 0$ and $Dv(0, s, t) = e^{s-t} I$.

A key role in our discussion is played by the n-dimensional version of the Carathéodory set:

$$\mathcal{M} = \{h \in H(B) :\ h(0) = 0,\ Dh(0) = I,\ \mathrm{Re}\ \langle h(z), z\rangle \geq 0,\ z \in B\}.$$

Recently the authors proved in [2] the following result.

Lemma 1.2. *Let $p \in \mathcal{M}$. Then for each $r \in (0, 1)$ there is a constant*

$$M = M(r) \leq \frac{4}{(1 - r)^2},$$

which is independent of p, such that $\|p(z)\| \leq M(r)$ for $\|z\| \leq r$.

Also in [1] the present authors have recently proved that the transition mapping $v = v(z, s, t)$ associated to a Loewner chain is locally Lipschitz continuous in $t \in [s, \infty)$ locally uniformly with respect to $z \in B$.

Lemma 1.3. *Let $f(z, t)$ be a Loewner chain and $v = v(z, s, t)$ be the transition mapping associated to $f(z, t)$. Then for all $r \in (0, 1)$ and $0 \leq s \leq t_1 < t_2 < \infty$,*

$$\|v(z, s, t_1) - v(z, s, t_2)\| \leq \frac{4}{(1 - r)^2}(t_2 - t_1), \quad \|z\| \leq r. \tag{1.2}$$

Also for all $r \in (0,1)$ *and* $0 \le s_1 < s_2 \le t < \infty$,

$$\|v(z,s_1,t) - v(z,s_2,t)\| \le \frac{4(1+r)}{(1-r)^3}(s_2 - s_1), \quad \|z\| \le r. \tag{1.3}$$

2. Main results. We begin this section with the following result.

Lemma 2.1. *Let* $f(z,t) = e^t z + \dots$ *be a Loewner chain. Then for each* $r \in (0,1)$ *and* $t_0 > 0$ *there is* $K = K(r,t_0) > 0$*, such that*

$$\|f(z,t) - f(z,s)\| \le K(r,t_0)(t-s), \quad \|z\| \le r,\ 0 \le s \le t \le t_0. \tag{2.1}$$

Thus $f(z,t)$ *is locally Lipschitz in* t*, locally uniformly with respect to* $z \in B$.

Proof. It is easy to see that f is a continuous mapping on $B \times [0,\infty)$, since $f(z,\cdot)$ is locally absolutely continuous function of $t \in [0,\infty)$ locally uniformly with respect to $z \in B$, Hence, for each $r \in (0,1)$ and $t_0 > 0$ there exists $M = M(r,t_0) > 0$ such that

$$\|f(z,t)\| \le M(r,t_0), \quad \|z\| \le r,\ t \in [0,t_0]. \tag{2.2}$$

On the other hand, using the Cauchy integral formula, it is not difficult to prove that there exists $L = L(r,t_0) > 0$ such that

$$\|Df(z,t)\| \le L(r,t_0), \quad \|z\| \le r,\ t \in [0,t_0]. \tag{2.3}$$

Indeed,

$$Df(z,t)(u) = \frac{1}{2\pi i}\int\limits_{|\zeta|=\rho} \frac{f(z+\zeta u,t)}{\zeta^2}d\zeta,$$

for all $u \in \mathbb{C}^n$, $\|u\| = 1$ and $\rho \in (0,1)$, such that $z + \zeta u \in B$, $|\zeta| = \rho$. For example, if $\rho = \dfrac{1-r}{2}$, then $\|z + \zeta u\| \le \dfrac{1+r}{2} < 1$ and the above equality yields that

$$\|Df(z,t)(u)\| \le M\left(\frac{1+r}{2}, t_0\right)\frac{2}{1-r} = L(r,t_0),$$

for $\|z\| \le r < 1$, $\|u\| = 1$ and $t \in [0,t_0]$. Thus (2.3) follows. Moreover, since

$$\|f(z,t) - f(w,t)\| \le \|z-w\|\int_0^1 \|Df((1-\tau)z + \tau w, t)\|d\tau,$$

for $\|z\| \le r$, $\|w\| \le r$, and $t \in [0, t_0]$, we obtain in view of (2.3) that

$$\|f(z,t) - f(w,t)\| \le L(r,t_0)\|z - w\|, \quad \|z\| \le r, \ \|w\| \le r,$$

for all $t \in [0, t_0]$. Taking into account (1.1), (1.2) and the above relation, we deduce that

$$\begin{aligned}\|f(z,s) - f(z,t)\| =& \|f(v(z,s,t),t) - f(z,t)\| \\ &\le L(r,t_0)\|z - v(z,s,t)\| \le \frac{4}{(1-r)^2} L(r,t_0)(t-s) \\ =& \frac{8}{(1-r)^3} M\left(\frac{1+r}{2}, t_0\right)(t-s),\end{aligned}$$

for $\|z\| \le r$, $0 \le s \le t \le t_0$. Here we have used the fact that $\|v(z,s,t)\| \le \|z\| \le r$, since $v(z,s,t)$ is a Schwarz map. This completes the proof. $\square$

Next, we show the following result.

Lemma 2.2. *Let $f(z,t) = e^t z + \dots$ be a Loewner chain and let $v = v(z,s,t)$ be the transition mapping associated to $f(z,t)$. Then the following conditions hold:*

(i) There exists a subset E of $(0,\infty)$ of measure zero such that for each $t \in (0,\infty) \setminus E$, the limit

$$\frac{\partial f}{\partial t}(z,t) = \lim_{h \to 0} \frac{f(z,t+h) - f(z,h)}{h} \tag{2.4}$$

exists uniformly on compact sets in B. Moreover, the mapping $\frac{\partial f}{\partial t}(\cdot,t)$ given by (2.4) is holomorphic on B for each $t \in (0,\infty) \setminus E$.

(ii) For each $s \ge 0$, there exists a subset E' of $[s,\infty)$ of measure zero such that for each $t \in [s,\infty) \setminus E'$ the limit

$$\frac{\partial v}{\partial t}(z,s,t) = \lim_{h \to 0} \frac{v(z,s,t+h) - v(z,s,t)}{h} \tag{2.5}$$

exists uniformly on compact sets in B. Also, the mapping $\frac{\partial v}{\partial t}(\cdot,s,t)$ given by (2.5) is holomorphic on B for each $t \in [s,\infty) \setminus E'$.

(iii) For each $t > 0$, there exists a subset E'' of $(0,t)$ of measure zero such that for each $s \in (0,t] \setminus E''$, the limit

$$\frac{\partial v}{\partial s}(z,s,t) = \lim_{h \to 0} \frac{v(z,s+h,t) - v(z,s,t)}{h}$$

exists uniformly on compact sets in B. The above mapping $\frac{\partial v}{\partial s}(\cdot, s, t)$ is holomorphic on B for each $s \in (0, t] \setminus E''$.

Proof. It suffices to prove the first condition, for the latter and third conditions it would be possible to use similar arguments as to (i).

Since $f(z, t)$ is subordination chain, $f(z, t)$ is a locally absolutely continuous function of t locally uniformly with respect to $z \in B$. Hence the limit

$$\frac{\partial f}{\partial t}(z, t) = \lim_{h \to 0} \frac{f(z, t+h) - f(z, t)}{h}$$

exists for a.e. $t \geq 0$. The exceptional null set depends on z, but we can choose a set $E \subset (0, \infty)$ of measure zero such that $\frac{\partial f}{\partial t}(z, t)$ exists for all $t \in (0, \infty) \setminus E$ and $z \in Q$, where Q is a countable set of uniqueness for the holomorphic functions on B (for example, any countable dense subset of B may be chosen as Q). Next, fix $t \in (0, \infty) \setminus E$. Since $f(z, t)$ is locally Lipschitz in t, the set

$$\left\{ \frac{f(z, t+h) - f(z, t)}{h} \right\}_{0 < |h| < \frac{t}{2}}$$

is locally uniformly bounded on B. In view of Vitali's theorem for holomorphic functions in higher dimensions [3], we conclude that the limit

$$\lim_{m \to \infty} \frac{f(z, t+h_m) - f(z, t)}{h_m}$$

exists uniformly on compact sets for any sequence $\{h_m\}_{m \geq 0}$ such that $\lim_{m \to \infty} h_m = 0$. Since Q is a set of uniqueness and all two such limits coincide on Q, (2.4) follows. Finally, since the limit

$$\frac{\partial f}{\partial t}(z, t) = \lim_{h \to 0} \frac{f(z, t+h) - f(z, t)}{h}$$

exists uniformly on compact sets in B, we deduce that $\frac{\partial f}{\partial t}(\cdot, t)$ is holomorphic on B for $t \in (0, \infty) \setminus E$. $\square$

We are now able to prove that Loewner chains satisfy the generalized Loewner differential equation. A part of this result was also obtained in [2, Theorem 1.10], but here we give another proof.

Theorem 2.3. *Let $f : B \times [0,\infty) \to \mathbb{C}^n$ be a Loewner chain. Then there exists a set $E \subset (0,\infty)$ of measure zero such that for each $t \in (0,\infty) \setminus E$, there exists a mapping $h = h(z,t)$ such that*

(*i*) $h(\cdot,t) \in \mathcal{M}$, $t \in (0,\infty) \setminus E$,

(*ii*) $h(z,\cdot)$ *is measurable on* $[0,\infty)$ *for each* $z \in B$, *and*

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad t \in (0,\infty) \setminus E, \ \forall \ z \in B. \tag{2.6}$$

Proof. Let $v = v(z,s,t)$ be the transition mapping associated to $f(z,t)$. Then

$$f(z,s) = f(v(z,s,t),t), \quad z \in B, \ 0 \le s \le t < \infty,$$

and hence

$$\begin{aligned} f(z,s) - f(z,t) =& f(v(z,s,t),t) - f(z,t) \\ =& Df(z,t)(v(z,s,t) - z) + o(v(z,s,t),z), \end{aligned}$$

where $\dfrac{\|o(w,z)\|}{\|w-z\|} \to 0$ as $\|w - z\| \to 0$.

In view of Lemma 2.2, there is a null set $E \subset (0,\infty)$ such that $\dfrac{\partial f}{\partial t}(\cdot,t)$ exists and is holomorphic on B for each $t \in (0,\infty) \setminus E$. For such $t \in (0,\infty) \setminus E$, we have

$$\begin{aligned} [Df(z,t)]^{-1}\frac{\partial f}{\partial t}(z,t) &= [Df(z,t)]^{-1} \lim_{s \nearrow t} \frac{f(z,s) - f(z,t)}{s-t} \\ &= \lim_{s \nearrow t} \left[\frac{v(z,s,t) - z}{s-t} - [Df(z,t)]^{-1} \frac{o(v(z,s,t),z)}{t-s} \right]. \end{aligned} \tag{2.7}$$

First, we show that for $t \in (0,\infty) \setminus E$ and $z \in B$,

$$\lim_{s \nearrow t} \frac{o(v(z,s,t),z)}{t-s} = 0. \tag{2.8}$$

Indeed,

$$\lim_{s \nearrow t} \frac{o(v(z,s,t),z)}{t-s} = \lim_{s \nearrow t} \frac{o(v(z,s,t),z)}{\|v(z,s,t) - z\|} \cdot \frac{\|v(z,s,t) - z\|}{t-s} = 0,$$

since $\dfrac{o(v(z,s,t),z)}{\|v(z,s,t) - z\|} \to 0$ and $\dfrac{\|v(z,s,t) - z\|}{t-s}$ is bounded in view of (1.2).

Hence, from (2.7) and (2.8) we deduce that for each $t \in (0,\infty) \setminus E$, the limit

$$\lim_{s \nearrow t} \frac{z - v(z,s,t)}{t-s}$$

exists for each $z \in B$. Further, if

$$h(z,t) = \lim_{s \nearrow t} \frac{z - v(z,s,t)}{t-s},$$

for all $z \in B$ and $t \in (0,\infty) \setminus E$, then $h(\cdot,t) \in H(B)$ and in view of the fact that $v = v(\cdot,s,t)$ is a Schwarz mapping, we obtain

$$\mathrm{Re}\ \langle h(z,t), z\rangle = \lim_{s \nearrow t} \frac{1}{t-s}[\|z\|^2 - \mathrm{Re}\ \langle v(z,s,t), z\rangle] \geq 0.$$

Also, since $v(0,s,t) = 0$ and $Dv(0,s,t) = e^{s-t}I$, it is obvious that $h(0,t) = 0$ and $Dh(0,t) = I$ for $t \in (0,\infty) \setminus E$. Therefore, $h(\cdot,t) \in \mathcal{M}$. Moreover, using (2.7), the equality (2.6) now follows.

Finally, we show that $h(z,t)$ is a measurable function of $t \in [0,\infty)$. Indeed, since

$$h(z,t) = \lim_{m\to\infty} m\left(z - v\left(z, t-\frac{1}{m}, t\right)\right),$$

it suffices to prove that for each $m \in \mathbb{N}$, $v\left(z, t-\frac{1}{m}, t\right)$ is a continuous function of t. For this purpose, we observe that

$$\begin{aligned}
&\left\|v\left(z,t-\frac{1}{m},t\right) - v\left(z,\tau-\frac{1}{m},\tau\right)\right\| \\
&\leq \left\|v\left(z,t-\frac{1}{m},t\right) - v\left(z,t-\frac{1}{m},\tau\right)\right\| \\
&+ \left\|v\left(z,t-\frac{1}{m},\tau\right) - v\left(z,\tau-\frac{1}{m},\tau\right)\right\| \leq \frac{8}{(1-\|z\|)^3}|t-\tau|,
\end{aligned}$$

where for the last inequality we have used the relations (1.2) and (1.3). This completes the proof. $\square$

In order to prove that the transition mappings associated to Loewner chains satisfy the Loewner differential equation (compare with [5, Theorem 6.3]), we use the lemma below. For the proof, it suffices to use similar kind of arguments as in [6, p.192-193].

Lemma 2.4. *Let $f : [a,b] \to \mathbb{C}^n$ given by $f(t) = g(h(t),t)$, where $g : B_r \times [a,b] \to \mathbb{C}^n$ and $h : [a,b] \to B_r$. Assume for each $t \in [a,b]$, $g(\cdot,t) \in H(B_r)$ and there exist some constants $M, K > 0$ such that*

$$\|g(z,t) - g(w,t)\| \leq M\|z-w\|$$

and

$$\|g(z,s)-g(z,t)\| \le K|s-t|$$

for all $s,t \in [a,b]$ *and* $z,w \in B_r$.

If h *is absolutely continuous, then* f *is also absolutely continuous and*

$$\frac{df}{dt}(t) = Dg(h(t),t)\frac{dh}{dt}(t) + \frac{\partial g}{\partial t}(h(t),t), \ \text{a.e. } t \in [a,b].$$

Theorem 2.5. *Let* $f : B \times [0,\infty) \to \mathbb{C}^n$ *be a Loewner chain and let* $v = v(z,s,t)$ *be the transition mapping associated to* $f(z,t)$. *Also let* $h = h(z,t)$ *be given by Theorem 2.3. Then for each* $s \ge 0$ *and for almost all* $t \ge s$,

$$\frac{\partial v}{\partial t}(z,s,t) = -h(v(z,s,t),t), \ \forall\ z \in B. \tag{2.9}$$

Proof. Fix $s \ge 0$. In view of Lemmas 2.1 and 2.2 we deduce that for almost all $t \ge s$ there exist the mappings $\frac{\partial f}{\partial t}(\cdot,t)$ and $\frac{\partial v}{\partial t}(\cdot,s,t)$ which are holomorphic on B. Also for any $\tau > s$ and for almost all $t \in [s,\tau]$, there exist the mappings $\frac{\partial v}{\partial t}(\cdot,t,\tau)$ which are holomorphic on B. Moreover,

$$v(z,s,\tau) = v(v(z,s,t),t,\tau), \quad z \in B,\ 0 \le s \le t \le \tau < \infty,$$

(see for example [1]) and

$$f(w,t) = f(v(w,s,t),t), \quad w \in B,\ 0 \le s \le t < \infty.$$

Differentiating the first equality with respect to t and the second equality with respect to t and w, we deduce that

$$0 = \frac{\partial v}{\partial t}(z,s,\tau) = Dv(v(z,s,t),t,\tau)\frac{\partial v}{\partial t}(z,s,\tau) + \frac{\partial v}{\partial t}(v(z,s,t),t,\tau), \tag{2.10}$$

for all $z \in B$ and a.e. $t \in [s,\tau]$, and

$$\begin{aligned} \frac{\partial f}{\partial t}(w,t) &= Df(v(w,t,\tau),\tau)\frac{\partial v}{\partial t}(w,t,\tau) \\ &= Df(w,t)[Dv(w,t,\tau)]^{-1}\frac{\partial v}{\partial t}(w,t,\tau), \end{aligned} \tag{2.11}$$

for all $w \in B$ and a.e. $t \in [s,\tau]$. Next, combining the relations (2.6) and (2.11), we obtain

$$\frac{\partial v}{\partial t}(w,t,\tau) = Dv(w,t,\tau)h(w,t), \quad w \in B,\ \text{a.e. } t \in [s,\tau].$$

Letting $w = v(z, s, t)$ in the above relation and taking into account (2.10), we conclude that

$$0 = Dv(v(z, s, t), t, \tau)\left[\frac{\partial v}{\partial t}(z, s, t) + h(v(z, s, t), t)\right],$$

for all $z \in B$ and a.e. $t \in [s, \tau]$. Since $Dv(v(z, s, t), t, \tau)$ is nonsingular and τ was arbitrarily chosen, (2.9) follows, as desired. This completes the proof. $\square$

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