NIKOS S. STYLIANOPOULOS and ELIAS WEGERT

A uniform estimate for the modulus of continuity of starlike mappings

ABSTRACT. Let $f\colon \mathbb{D} \to \Omega$ be a conformal mapping of the unit disk $\mathbb{D}$ onto a starlike domain G normalized by $f(0) = 0$. In this note we derive the uniform estimate

$$\omega_\varphi(\delta) \le \frac{\pi \log R + 6}{|\log \delta|},$$

for the modulus of continuity $\omega_\varphi(\delta)$ of the boundary correspondence function $\varphi := \arg f|_{\partial \mathbb{D}}$, for all starlike domains G with $\mathbb{D} \subset G \subset R\,\mathbb{D}$. An example shows that this estimate gives the correct order with respect to δ.

1. Introduction and results. Let G be a simply-connected domain in the complex plane which is starlike with respect to the origin. By this we mean that $0 \in G$ and that G contains the line segment $[0, z]$ for all $z \in G$. Also, let A_R $(R > 1)$ denote the closed annulus $\{z\colon 1 \le |z| \le R\}$ and assume that the boundary ∂G of G is contained in A_R. Finally, let $f\colon \mathbb{D} \to G$ be a conformal mapping of the unit disk $\mathbb{D}$ onto G, normalized by $f(0) = 0$. Then, the boundary correspondence function φ, defined by

$$\varphi(t) := \lim_{r \to 1-0} \arg f(re^{it}),$$

is continuous and increasing with respect to the angle t; see e.g. [4, Thm. 3.18 & Prop. 3.19].

We consider the modulus of continuity

$$\omega_\varphi(\delta) := \sup_{|\tau - \sigma| < \delta} |\varphi(\tau) - \varphi(\sigma)|,$$

of the function φ on $[0, 2\pi]$. There are certain applications in numerical conformal mapping, for instance in the study of Theodorsen's integral equation method (see e.g. [1, p. 61 ff]) that require information about the behaviour of the function ω_φ. In this note we derive a uniform estimate for ω_φ, for all simply connected and starlike domains G with boundary on A_R.

Theorem 1. *With the notation introduced above,*

(1.1) $$\omega_\varphi(\delta) \leq \frac{\pi \, \log R + 6}{|\log \delta|},$$

provided that $0 < \delta < 1/4$.

Remark 1. From the outset a simple observation shows that the order $\omega(\delta) = \mathcal{O}(|\log \delta|^{-1})$ is "nearly optimal". To see this, hypothesize, for instance, that the function $\delta \mapsto \omega_\varphi(\delta)/\delta$ is integrable. This would imply that the family of admissible boundary correspondence functions φ is uniformly Dini-continuous. Then, the same would hold true for the corresponding function $v(e^{it}) = \varphi(t) - t$, and a theorem on the mapping properties of the conjugation operator H (see [3, Ch. III, Thm 1.3]) would guarantee the equicontinuity of the conjugate function u. The latter, in turn, would imply the equicontinuity of the absolute value $|f|$ of all admissible conformal maps f onto starlike regions with boundary on A_R which is obviously not true, as the case of disk domains cut along with many, arbitrarily chosen, radial slits shows.

In fact, a slightly more elaborated argument can be employed to show that the estimate in (1.1) is, apart from a constant, sharp with respect to the order of δ:

Proposition 1. *There exist arbitrarily small positive numbers δ and starlike domains G with $\partial G \subset A_R$ so that*

(1.2) $$\omega_\varphi(\delta) \geq \frac{\pi}{2} \frac{\log R}{|\log \delta|}.$$

Remark 2. Gaier and Kühnau in [2] find the extremal domain which maximizes $\omega_\varphi(\delta)$ for fixed $\delta > 0$ and $R > 1$. Using deep techniques from

the theory of harmonic measure they have been able to improve the estimate in (1.1). In particular, they show that, for δ sufficiently small, the constant 6 can be omitted, provided that $|\log \delta|^{-1}$ is replaced by $(1+\varepsilon)|\log \delta|^{-1}$ with $\varepsilon > 0$. In fact, here we also establish a slightly sharper estimate than that in (1.1). More precisely, it will become apparent from the proof of Theorem 1 that

$$(1.3) \qquad \omega_\varphi(\delta) \leq \frac{\pi \, \log R + 6 - \pi \, \log r_0}{|\log \delta|},$$

where r_0 is a constant depending on G, such that $1 \leq r_0 \leq R$.

2. Proofs.
Proof of Theorem 1. With $\delta > 0$, fix angles τ and σ so that $\tau < \sigma < \tau + \delta$, and consider the auxiliary function $w = u + \mathrm{i}\,v$ defined in $\mathbb{D}$ by

$$w(z) := \log \frac{f(z)}{z} - \log r_0,$$

where $r_0 := |f'(0)|$ is the so-called conformal radius of the domain G with respect to 0. We observe that from the assumptions on G and the monotonicity property of conformal radius, we have $1 \leq r_0 \leq R$.

Clearly, w is bounded and holomorphic in $\mathbb{D}$ and satisfies $u(0) = 0$. Therefore, the real and imaginary parts of its boundary function are related by $u = Hv$, where H denotes the Hilbert transform,

$$Hv(\mathrm{e}^{\mathrm{i}t}) := \frac{1}{2\,\pi} \int_{-\pi}^{\pi} v(\mathrm{e}^{\mathrm{i}s}) \cot \frac{s-t}{2} \, ds \; ;$$

see e.g. [3, §3.1]. Moreover, $v(\mathrm{e}^{\mathrm{i}t}) = \varphi(t) - t$, and

$$(2.1) \qquad Hv(\mathrm{e}^{\mathrm{i}t}) = u(\mathrm{e}^{\mathrm{i}t}) \leq \log R - \log r_0.$$

We assume first that

$$(2.2) \qquad \varepsilon := [v(\mathrm{e}^{\mathrm{i}\sigma}) - v(\mathrm{e}^{\mathrm{i}\tau})]/2 > 0.$$

This, in conjunction with the fact that $0 < \varphi(\sigma) - \varphi(\tau) < 2\pi$, gives

$$(2.3) \qquad 0 < \varepsilon < \pi.$$

Observe that v is continuous on $\partial \mathbb{D}$ and hence there exists an $\eta \in (\tau, \sigma)$ such that $v(\mathrm{e}^{\mathrm{i}\eta}) = [v(\mathrm{e}^{\mathrm{i}\tau}) + v(\mathrm{e}^{\mathrm{i}\sigma})]/2$. In order to estimate $Hv(\mathrm{e}^{\mathrm{i}\eta})$, we divide the interval $(-\pi + \eta, \pi + \eta)$ into six subintervals

$$I_1 := (-\pi + \eta, \tau - \varepsilon), \quad I_2 := (\tau - \varepsilon, \tau), \quad I_3 := (\tau, \eta),$$
$$I_4 := (\eta, \sigma), \quad I_5 := (\sigma, \sigma + \varepsilon), \quad I_6 := (\sigma + \varepsilon, \pi + \eta);$$

where we have assumed, without loss of generality, that

(2.4) $$-\pi + \eta < \tau - \varepsilon \quad \text{and} \quad \sigma + \varepsilon < \pi + \eta.$$

This is possible because $\sigma + \varepsilon - (\tau - \varepsilon) = \varphi(\sigma) - \varphi(\tau) < 2\pi$. Since the operator H annihilates the constants, we can further assume that $v(\mathrm{e}^{\mathrm{i}\eta}) = 0$. This, in turn, implies that $v(\mathrm{e}^{\mathrm{i}\sigma}) = \varepsilon$ and $v(\mathrm{e}^{\mathrm{i}\tau}) = -\varepsilon$. Now, taking into account that the mapping $t \mapsto \varphi(t) = v(\mathrm{e}^{\mathrm{i}t}) + t$ is increasing, we obtain the estimates

(2.5) $$v(\mathrm{e}^{\mathrm{i}s}) \leq \tau - \varepsilon - s \quad \text{on } I_1 \cup I_2,$$

(2.6) $$v(\mathrm{e}^{\mathrm{i}s}) \leq \eta - s \qquad \text{on } I_3,$$

(2.7) $$v(\mathrm{e}^{\mathrm{i}s}) \geq \eta - s \qquad \text{on } I_4,$$

(2.8) $$v(\mathrm{e}^{\mathrm{i}s}) \geq \sigma + \varepsilon - s \quad \text{on } I_5 \cup I_6;$$
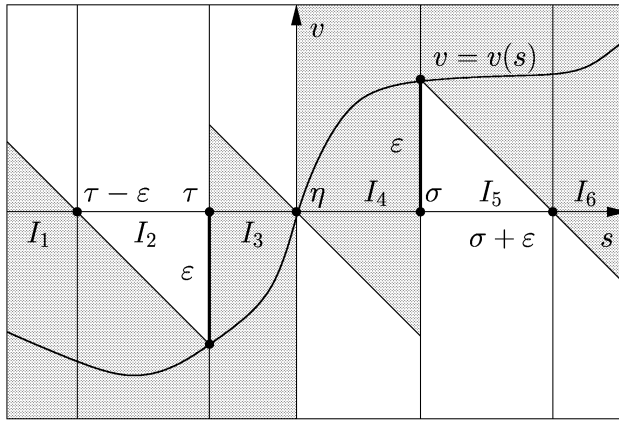
see Figure 1.

FIG. 1

Splitting the Hilbert transform of v into six integrals we have

$$2\pi\, Hv(\mathrm{e}^{\mathrm{i}\eta}) = \int_{-\pi+\eta}^{\pi+\eta} v(\mathrm{e}^{\mathrm{i}s}) \cot \frac{s-\eta}{2}\, ds = \sum_{j=1}^{6} \int_{I_j} v(\mathrm{e}^{\mathrm{i}s}) \cot \frac{s-\eta}{2}\, ds$$

and accordingly observe that by virtue of (2.4) and (2.5)–(2.8), the following inequalities hold:

$$\int_{I_j} v(\mathrm{e}^{\mathrm{i}s}) \cot \frac{s-\eta}{2}\, ds \geq - \int_{\varepsilon}^{\pi} s \cot \frac{s}{2}\, ds, \quad j = 1, 6,$$

$$\int_{I_j} v(\mathrm{e}^{\mathrm{i}s}) \cot \frac{s-\eta}{2}\, ds \geq - \int_{0}^{\delta} s \cot \frac{s}{2}\, ds, \quad j = 3, 4.$$

Furthermore, by (2.4), (2.5) and (2.8),

$$
\int_{I_j} v(\mathrm{e}^{\mathrm{i}s}) \cot \frac{s-\eta}{2}\, ds \geq \int_{\delta}^{\delta+\varepsilon} (\varepsilon - s) \cot \frac{s}{2}\, ds + \delta \int_{\delta}^{\delta+\varepsilon} \cot \frac{s}{2}\, ds
$$

$$
\geq \int_{\delta}^{\delta+\varepsilon} (\varepsilon - s) \cot \frac{s}{2}\, ds, \qquad j = 2, 5\,;
$$

where we made use of the symmetry of $\cot(s/2)$ in $(0, 2\pi)$ and the inequality $\delta + \varepsilon < \pi + 1/4$ (cf. (2.3)).

Consequently,

$$
2\pi\, Hv(\mathrm{e}^{\mathrm{i}\eta}) \geq 2\varepsilon \int_{\delta}^{\delta+\varepsilon} \cot \frac{s}{2}\, ds - 2 \int_{0}^{\pi} s \cot \frac{s}{2}\, ds - 2 \int_{\varepsilon}^{\delta+\varepsilon} s \cot \frac{s}{2}\, ds
$$

$$
\geq 4\varepsilon\, \log \sin \left(\frac{\delta}{2} + \frac{\varepsilon}{2} \right) - 4\varepsilon\, \log \sin \left(\frac{\delta}{2} \right) - 4\pi\, \log 2 - 4\delta
$$

$$
= 4\varepsilon\, \log \left(\cos \frac{\varepsilon}{2} + \cot \frac{\delta}{2} \sin \frac{\varepsilon}{2} \right) - 4\pi\, \log 2 - 4\delta
$$

$$
\geq 4\varepsilon\, \log \left(\cot \frac{\delta}{2} \sin \frac{\varepsilon}{2} \right) - 4\pi\, \log 2 - 4\delta
$$

$$
\geq 4\varepsilon\, |\log \delta| + 4\varepsilon\, \log \left(1.98 \sin \frac{\varepsilon}{2} \right) - 4\pi\, \log 2 - 4\delta
$$

$$
> 4\varepsilon\, |\log \delta| - 11.22\,;
$$

where we used that $\delta \cot(\delta/2) > 1.98$ if $0 < \delta < 1/4$. Comparing the last inequality with (2.1) we get

$$
2\varepsilon |\log \delta| \leq \pi \log R - \pi \log r_0 + 5.61,
$$

and this, with the assumptions on δ and ε, leads to

$$
0 \leq \varphi(\sigma) - \varphi(\tau) = v(\mathrm{e}^{\mathrm{i}\sigma}) - v(\mathrm{e}^{\mathrm{i}\tau}) + \sigma - \tau \leq 2\varepsilon + \delta
$$

$$
\leq (\pi \log R + 5.61 + \delta\, |\log \delta| - \pi \log r_0)\, |\log \delta|^{-1}
$$

$$
\leq \frac{\pi\, \log R + 6 - \pi \log r_0}{|\log \delta|}.
$$

In the complementary case, where $v(\mathrm{e}^{\mathrm{i}\sigma}) \leq v(\mathrm{e}^{\mathrm{i}\tau})$, the assumptions on δ, along with the double inequality $1 \leq r_0 \leq R$, yield at once

$$
0 \leq \varphi(\sigma) - \varphi(\tau) \leq \delta \leq \frac{\pi \log R + 6 - \pi \log r_0}{|\log \delta|},
$$

which completes the proof.   □

We recall our comments in Remark 2, regarding the sharp estimates of Gaier and Kühnau and observe that the elementary approach presented here cannot yield such fine results. The main obstacle comes from the rough estimates used for the integrals over the intervals I_1 and I_6. Here, we estimate v by a linearly decaying function, while it follows from [2] that the function v corresponding to the extremal domain is very close to zero.

Proof of Proposition 1. With $n \in \mathbb{N}$, we set $\alpha := \pi/n$ and let L_k denote the radial rays

$$L_k := \{z : 1 \le |z| < R, \ \arg z = (2k - 1)\alpha\}, \quad k = 1, 2, \ldots, n.$$

We consider the conformal map f of $\mathbb{D}$ onto the slit disk

$$G := \{z : |z| < R\} \setminus (L_1 \cup L_2 \cup \ldots \cup L_n),$$

normalized by $f(0) = 0$ and $f(1) = R$. By symmetry, the restriction $\tilde{f}$ of f to the sector

$$G_\alpha := \{z : |z| < 1, \ -\alpha < \arg z < \alpha\},$$

maps G_α conformally onto the dilated sector RG_α. Clearly, $\tilde{f}$ extends continuously onto the closure of G_α and we use the same notation for this extension. Also, $\tilde{f}(\mathrm{e}^{\pm i\alpha}) = \mathrm{e}^{\pm i\alpha}$ and $\tilde{f}(\mathrm{e}^{\pm i\delta}) = R\mathrm{e}^{\pm i\alpha}$, for some δ with $0 < \delta < \alpha$.

In order to estimate δ in terms of R and α, we represent $\tilde{f}$ as the composition $f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1$ of five conformal maps as follows: The function $f_1(z) := \log z$ maps G_α onto the half strip $S := \{z: \operatorname{Re} z < 0, \ -\alpha < \operatorname{Im} z < \alpha\}$, the function $f_2(z) := \sin(\pi z/2i\alpha)$ maps S onto the upper half plane U. Then $f_3(z) := cz$, with $c := \left[R^{\pi/2\alpha} + R^{-\pi/2\alpha}\right]/2$, maps U onto itself. Further, an appropriately chosen branch of $f_4(z) := (2i\alpha/\pi) \arcsin z$ maps U onto S, and finally S is mapped onto RG_α by $f_5(z) := R\exp z$.

We let $z_6 := R\mathrm{e}^{i\alpha}$ and define $z_5, z_4, \ldots, z_1$, recursively by $f_k(z_k) := z_{k+1}$. Then, $z_1 = \mathrm{e}^{i\delta}$ and

$$z_5 = \mathrm{i}\alpha, \quad z_4 = 1, \quad z_3 = 1/c, \quad z_2 = \mathrm{i}\,d, \quad z_1 = \mathrm{e}^{\mathrm{i}d},$$

where $d = (2\alpha/\pi) \arcsin(1/c)$. Since $\arcsin(1/c) < \pi/(2c)$ and $1/c < 2R^{-\pi/2\alpha}$, we have $d < 2\alpha R^{-\pi/2\alpha}$, and consequently, by comparing the two expressions of z_1 we obtain for $\alpha < 1/2$,

$$\delta = d < R^{-\pi/2\alpha}.$$

The latter inequality is equivalent to

$$\alpha > \frac{\pi}{2} \frac{\log R}{|\log \delta|},$$

and the desired estimate (1.2) follows from this, with sufficiently large n, because $\omega_\varphi(\delta) = \alpha$. $\quad\square$

Added in Proof. During the process of publication of the paper, the authors received a note from Ch. Pommerenke indicating that a similar result to (1.1) can be derived by using Lemma 2 of his paper entitled "On the Green's fundamental domain", Math. Z. **156** (1977), no. 2, 157–164. The above named lemma provides, for any Borel set B of $[0, 2\pi]$, a lower bound for the logarithmic capacity of $\varphi^{-1}(B)$ in terms of the Lebesgue measure of B.

References

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Department of Mathematics and Statistics
University of Cyprus
P.O. Box 20537, CY-1678 Nicosia, Cyprus
e-mail: nikos@ucy.ac.cy

Institut für Angewandte Mathematik I
TU Bergakademie Freiberg
09596 Freiberg, Germany
e-mail: wegert@math.tu-freiberg.de