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PIER LUIGI PAPINI

Constants and symmetries in Banach spaces

Abstract. In this paper we indicate some connections between some properties of normed spaces and the values of some parameters. We also point out the role of "symmetric" points in minimizing or maximizing quantities involving the numbers $\|x-y\|, \|x+y\|$, x and y being on the unit sphere of the space: in fact, the role of these "symmetries" has been sometimes overlooked.

1. Introduction and notations. The starting point for this paper was the reading of paper [DT], where some constants, already introduced and studied by J. Gao several years ago (see e.g. [Pa]), are considered. We remembered that true and false properties of "symmetries" on the unit sphere had already been considered in the past; also, properties of Gao's constants had been investigated, partly in papers not so well known (like [C]). Moreover, two other constants considered in [BCP], that we denote by A_1 and A_2, look partly similar.

We tried to melt this material, and something new (partly almost evident, partly shaded) took form; in particular, relations among A_1, A_2 and Gao's constants are proved.

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Let $(X, \|\cdot\|)$ be a Banach space, of dimension at least 2, over the real field $\mathbb{R}$. We list the notations we shall use in the following:

$S_X = \{x \in X : \|x\| = 1\}$, we shall simply write S instead of S_X when no confusion can arise;

X^* will denote the dual of X;

given a point $x \in X$, we denote by $E(\pm x)$ the "**equidistant set**" from $x, -x$; i.e:

$$E(\pm x) = \{y \in X : \|x - y\| = \|x + y\|\}.$$

Given X, its modulus of convexity, $\delta(\varepsilon)$, is defined, for $\varepsilon \in [0, 2]$, as

$$\delta(\varepsilon) = \inf\{1 - \frac{\|x + y\|}{2} : \ x, y \in S; \ \|x - y\| \geq \varepsilon\}. \tag{1}$$

We remind that δ is non–decreasing, and continuous in $[0, 2)$. A space is said to be **uniformly nonsquare** when

$$\lim_{\varepsilon \to 2^-} \delta(\varepsilon) > 0. \tag{2}$$

Recall that unifomly nonsquare spaces are reflexive.

2. Inf max, sup min, symmetries. In this section we want to summarize some facts, contained in the literature, relating the minimization or maximization of quantities involving $\|x - y\|, \|x + y\|$, to the condition $\|x - y\| = \|x + y\|$.

The following statement appears in [M, pp. 85-86]:

Proposition 0. *Let $\varepsilon > 0$; for x fixed in the two-dimensional subspace E_2 generated by x and y, $\max_{y \in S(E_2)} \frac{\|x+\varepsilon y\| + \|x - \varepsilon y\|}{2}$ is attained at a y_0 such that $\|x + \varepsilon y_0\| = \|x - \varepsilon y_0\|$, and, in addition, $\max_{y \in S(E_2)} \min\{\|x + \varepsilon y\|, \|x - \varepsilon y\|\}$ is attained also at a y_0 such that $\|x + \varepsilon y_0\| = \|x - \varepsilon y_0\|$.*

The second statement is true, also if max and min are exchanged (cf. Propositions 1 and 2 below). But an example of Poulsen (see [F, p. 125]) shows that also a weaker form of the first statement is wrong: in fact, it may happen that $\max_{x,y \in S(E_2)}(\|x+y\| + \|x-y\|)$ is assumed (only) for pairs x, y such that $\min\{\|x + y\|, \|x - y\|\} < 1$; cf. also Example 4 in Section 3 here.

We recall a result proved in [GL_1, Lemma 2.2], and then again in [BR, Lemma 2]; this will be fundamental throughout the paper. Set, for $x \in S$:

(i) $a(x) = \inf_{y \in S} \max\{\|x - y\|, \|x + y\|\}$

and

(ii) $a'(x) = \sup_{y \in S} \min\{\|x - y\|, \|x + y\|\}$.

Proposition 1. *If X is a two-dimensional space, then for every $x \in S$, there is a unique $y \in S$ such that $\|x-y\| = \|x+y\|$, say $= \alpha(x)$; moreover $\alpha(x) = a(x) = a'(x)$. Also, if $p = \frac{x-y}{\|x-y\|}$, then $\inf\limits_{u \in S} \max\{\|p-u\|, \|p+u\|\} = \frac{2}{\alpha(x)}$.*

Remark. The uniqueness of y in Proposition 1 must be understood in the sense that there is a unique such pair $y, -y$. Also, the statement does not exclude that the value $a(x) = a'(x)$ is attained also for other points (see Example 4 below).

Clearly, in any finite-dimensional space, given $x \in S$ the two numbers defined by (i) and (ii) are assumed at some point in S. In general, the inf in (i) or the sup in (ii) are not necessarily assumed (see Example 3 below); but according to Proposition 1, given $x \in S$, to compute (i) or (ii) it is enough to consider points y in S satisfying $\|x-y\| = \|x+y\|$; for the sake of completeness, we give a proof of this fact. A similar remark will apply to some of the constants considered in Section 5.

Proposition 2. *For any X, given $x \in S$ we have:*

(iii) $a'(x) = \sup\{\|x-y\| : y \in S \cap E(\pm x)\}$;

(iv) $a(x) = \inf\{\|x-y\| : y \in S \cap E(\pm x)\}$.

Proof. We prove (iii) (the proof of (iv) being similar). Let

$$\alpha = \sup\{\min\{\|x-y\|, \|x+y\|\} : y \in S\}; \ \beta = \sup\{\|x-y\| : y \in S \cap E(\pm x)\}.$$

Clearly $\alpha \geq \beta = \sup\{\min\{\|x-y\|, \|x+y\|\} : y \in S \cap E(\pm x)\}$; we must prove the converse inequality. Given $\varepsilon > 0$, take $y' \in S$ such that $\min\{\|x-y'\|, \|x+y'\|\} > \alpha - \varepsilon$. Let Y denote the two-dimensional subspace of X generated by x and y'; take $y'' \in Y \cap S$ such that $\|x-y''\| = \|x+y''\|$. Then, according to Proposition 1, since $y'' \in S \cap E(\pm x)$ and $y' \in Y \cap S$, we obtain: $\beta \geq \|x-y''\| = \min\{\|x-y''\|, \|x+y''\|\} \geq \min\{\|x-y'\|, \|x+y'\|\} > \alpha - \varepsilon$; since $\varepsilon > 0$ is arbitrary, this proves that $\beta \geq \alpha$, thus the equality, which is (iii). $\square$

Given X, set, for $x \in S$:

(v) $b(x) = \frac{1}{2}\sup\{\|x-y\| + \|x+y\| : y \in S\}$

(vi) $b'(x) = \frac{1}{2}\sup\{\|x-y\| + \|x+y\| : y \in S \cap E(\pm x)\}$
$= \sup\{\|x-y\| : y \in S \cap E(\pm x)\}$.

According to Proposition 2, we have:

(vii) $b'(x) = a'(x) = \sup\{\min\{\|x-y\|, \|x+y\|\} : y \in S\}$.

Also, given $x \in S$, we have the following chain of (in)equalities:

$$\text{(j)} \qquad 1 \le a(x) = \inf_{y \in S \cap E(\pm x)} \max\{\|x-y\|, \|x+y\|\}$$
$$= \inf_{y \in S \cap E(\pm x)} \|x-y\| \le \sup_{y \in S \cap E(\pm x)} \|x-y\|$$
$$= \sup_{y \in S \cap E(\pm x)} \min\{\|x-y\|, \|x+y\|\}$$
$$= a'(x) = b'(x) \le b(x) \le 2.$$

The extreme values 1 and 2 in (j), are attained in simple cases (also in two-dimensional spaces): see Example 1 in the next section.

Proposition 1 is not true when $\dim(X) > 2$: in fact, if $\dim(X) \ge 3$, then for some x we can have (see e.g. the simple Example 2 in next section):

$$\text{(jj)} \qquad 1 = \inf_{y \in S} \max\{\|x-y\|, \|x+y\|\} < \sup_{y \in S} \min\{\|x-y\|, \|x+y\|\} = 2.$$

Proposition 3. *Given a space X, for any $x \in S$ we have:*

$$0 \le 2b(x) - 2 \le b'(x) \le b(x) \le 2. \tag{3}$$

Proof. The inequality $b(x) \ge 1$ is trivial; and so is $b'(x) \le b(x) < 2$ (see (j)).

$b'(x) \ge 2b(x) - 2$: let be y such that $\|x-y\| + \|x+y\| > 2b(x) - \varepsilon$; since $\max\{\|x-y\|, \|x+y\|\} \le 2$, this implies $b'(x) \ge \min\{\|x-y\|, \|x+y\|\} > 2b(x) - \varepsilon - 2$: since $\varepsilon > 0$ is arbitrary, this proves that $b'(x) \ge 2b(x) - 2$, which completes the proof. $\square$

Note that it is possible to have $2b(x) - 2 = b'(x) = b(x) = 2$ (for example, when $b'(x) = b(x) = 2$). But in fact, the following is an immediate consequence of (3):

Corollary. *Given $x \in X$, the conditions $b(x) = 2$ and $b'(x) = 2$ are equivalent.*

Also: $2b(x) - 2$ always has a positive lower bound (see [BCP, Proposition 2.5]); more precisely:

$$b(x) \ge \frac{3 + \sqrt{21}}{6} \ (> 5/4) \tag{3'}$$

for any x, in any space X.

3. Some examples. In next examples, $\mathbb{R}_n^\infty$ will indicate the space $\mathbb{R}_n$ with the max norm. Note that, by slightly modifying the norm, we obtain a space with a strictly convex norm: thus situations "almost" similar can occur also for X having a strictly convex norm.

Example 1. This example refers to (j) in Section 2: consider in $X = \mathbb{R}_2^\infty$ the points $x = (1, 0)$ (we get $a(x) = a'(x) = 1$) and $x = (1, 1)$ (we get $b(x) = b'(x) = 2$).

Example 2. This example refers to (jj) in Section 2: consider in $X = \mathbb{R}_3^\infty$ the point $x = (0, 1, 1)$; then we obtain 1 at the left for $y = (1, 0, 0)$; we obtain 2 at the right for $y' = (0, -1, 1)$.

Example 3. Consider the space $X = C[0, 1]$; let $f(t) = t$ (so $f \in S_X$). Take $g_n(t) = \min(t, (3 - 2n)t + \frac{2}{n}(n - 1)^2)$; for $n \geq 2$ we have: $\|f - g_n\| = \|f + g_n\| = 2 - \frac{2}{n}$. This shows that

$$\sup\{\|f-g\|+\|f+g\| : \ g \in S\} = \sup\{\|f-g\|+\|f+g\| : \ g \in S \cap E(\pm f)\} = 2$$

(but the value 2 is not attained).

Example 4. Consider the space $X = \mathbb{R}_2^\infty$; let $x = (1, -1/2)$; we have: $\max_{y \in S} \min\{\|x - y\|, \|x + y\|\} = 3/2$. The value 3/2 is attained not only at points satisfying $\|x - y\| = \|x + y\|$ (like (1/2, 1)): for example, if we take $z = (1, 1)$, then we have $\|x + z\| = 2$; $\|x - z\| = 3/2$. A similar remark applies concerning $\min_{y \in S} \max(\|x - y\|, \|x + y\|) = 3/2$: if $z = (0, 1)$, then $\|x - z\| = 3/2$; $\|x + z\| = 1$.

Moreover, again for $x = (1, -1/2)$: $\min_{y \in S}(\|x-y\|+\|x+y\|) = 2$, which is attained e.g. for $y = (1/2, -1)$; in fact $\|x - y\| = 1/2; \|x + y\| = 3/2$. It is not attained for points y such that $\|x - y\| = \|x + y\| = 1$ (since $\min_{y \in S \cap E(\pm x)}(\|x-y\|+\|x+y\|) = 3$). Also: $\max_{y \in S}(\|x-y\|+\|x+y\|) = 7/2$, which is attained e.g. for $y = (1, 1)$; it is not attained for points y such that $\|x-y\| = \|x+y\| = 7/4$ (since $\max_{y \in S \cap E(\pm x)}(\|x-y\|+\|x+y\|) = 3$).

4. Two more constants. In [BCP] the following constants were studied:

$$A_1(X) = \inf_{x \in S} b(x) = \frac{1}{2} \inf_{x \in S} \sup_{y \in S}(\|x - y\| + \|x + y\|); \tag{4}$$

$$A_2(X) = \sup_{x \in S} b(x) = \frac{1}{2} \sup_{x, y \in S} (\|x - y| + \|x + y\|). \tag{5}$$

Note that $A_2(X) = \rho(1) + 1$, where ρ is the modulus of smoothness of the space X. The constant $A_2(X)$ was also used in [G]; in fact, the constant $r(X)$ considered there is nothing else than $4 \cdot A_2(X)$. The main results of [G] indicate that when the value of A_2 is not too large, then the space has some kind of "normal structure".

The following formula was indicated in [BCP, Proposition 2.2]: in any space X,

(a) $$A_2(X) = \sup\{1 + \frac{\varepsilon}{2} - \delta(\varepsilon) : \varepsilon \in (0, 2)\}.$$

Concerning bounds for $A_1(X)$ and $A_2(X)$, we send to [BCP]. In particular, according to (3′):

(3″) $$A_1(X) \geq \frac{3 + \sqrt{21}}{6}$$

Also (see [B, Theorem 6]):

(b) $A_2(X) < 2$ if and only if X is uniformly nonsquare.

Now we define the following constants:

(4′) $$A'_1(X) = \inf_{x \in S} b'(x) = \inf_{x \in S} \sup\{\|x - y\| : y \in S \cap E(\pm x)\}$$

(5′) $$A'_2(X) = \sup_{x \in S} b'(x) = \sup_{x \in S} \sup\{\|x - y\| : y \in S \cap E(\pm x)\}$$

Of course, in any space X:

(c) $$1 \leq A'_1(X) \leq A_1(X);$$

(d) $$A'_2(X) \leq A_2(X) \leq 2$$

and $A'_1(X) \leq A'_2(X)$ and $A_1(X) \leq A_2(X)$.

Proposition 4. *For a space X, $A'_2(X) = 2 \Leftrightarrow A_2(X) = 2 \Leftrightarrow X$ is not uniformly nonsquare.*

Proof. From (3) we obtain

$$0 \leq 2A_2(X) - 2 \leq A'_2(X) \leq A_2(X) \leq 2.$$

Proposition 4 is an immediate consequence of these inequalities and (b). □

We raise the following

Problem. *Is the inequality $A_1(X) \leq A'_2(X)$ true in general?*

Note that the inequality is true in a space X, if for some $x_0 \in S$, $\sup_{y \in S}(\|x_0 + y\| + \|x_0 - y\|)$ is assumed at a point $y \in E(\pm x_0)$: in fact, in this case we obtain $A'_2(X) \geq b'(x_0) = b(x_0) \geq A_1(X)$.

5. Old and new constants. Now we want to compare the "new" constants with some other ones, defined around two decades ago by J. Gao (see [GL$_1$]) and considered also elsewhere: see e.g. [Pa], [C] and [GL$_2$]). Recently, these constants have been generalized and studied in [BR].

Set:

$$g(X) = \inf_{x \in S} \inf_{y \in S} \max\{\|x-y\|, \|x+y\|\}; \tag{6}$$

$$G(X) = \sup_{x \in S} \inf_{y \in S} \max\{\|x-y\|, \|x+y\|\}; \tag{7}$$

$$g'(X) = \inf_{x \in S} \sup_{y \in S} \min\{\|x-y\|, \|x+y\|\}; \tag{8}$$

$$G'(X) = \sup_{x \in S} \sup_{y \in S} \min\{\|x-y\|, \|x+y\|\}. \tag{9}$$

According to Proposition 1, if $\dim(X) = 2$, then $g(X) = g'(X)(\leq \sqrt{2})$; $G(X) = G'(X)(\geq \sqrt{2})$. Recall that, as known (see [C, Remark 2.3]), we always have

$$g(X) - G'(X) = 2 \tag{10}$$

so

$$G'(X) \geq \sqrt{2}; \tag{11}$$

moreover:

$$\begin{aligned} &\text{if } G'(X) < 2, \text{ then } G'(X) \text{ is the unique solution} \\ &\qquad\qquad \text{of the equation (in } \alpha) \ \delta(\alpha) = 1 - \frac{\alpha}{2}. \end{aligned} \tag{12}$$

Recall that $G'(X) < 2$ if and only if X is uniformly nonsquare.

With respect to inclusion of spaces, we may observe that when we "enlarge" the space, $G'(X)$ does not decrease while $g(X)$ does not increase; thus:

$$G'(X) = \sup\{G'(Y) : \ Y \text{ is a two-dimensional subspace of } X\}$$
$$g(X) = \inf\{g(Y) : \ Y \text{ is a two-dimensional subspace of } X\}.$$

Moreover, taking into account Proposition 1, it is not difficult to see that

$$G(X) \leq \sup\{G(V) : \ V \text{ is a two-dimensional subspace of } X\} = G'(X)$$
$$g'(X) \geq \inf\{g'(V) : \ V \text{ is a two-dimensional subspace of } X\} = g(X).$$

Now we want to show the relations among these constants, and those defined in Section 4.

Proposition 5. *In any space X, we have:*

$$A_2'(X) = G'(X); \quad A_1'(X) = g'(X). \tag{13}$$

Proof. According to (vii), we obtain:

$$\begin{aligned} A_2'(X) &= \sup_{x\in S} b'(x) = \sup_{x\in S}\sup_{y\in S} \min\{\|x-y\|, \|x+y\|\} = G'(X);\\ A_1'(X) &= \inf_{x\in S} b'(x) = \inf_{x\in S}\sup_{y\in S} \min\{\|x-y\|, \|x+y\|\} = g'(X). \quad \square \end{aligned}$$

Therefore, we have the following chain of (in)equalities:

$$g'(X) = A_1'(X) \le A_1(X) \tag{14}$$

$$G'(X) = A_2'(X) \le A_2(X). \tag{15}$$

Also, we have:

$$\begin{aligned} A_1(X) &\ge \inf\{A_1(V) :\ V \text{ is a two-dimensional subspace of } X\};\\ A_2(X) &= \sup\{A_2(V) :\ V \text{ is a two-dimensional subspace of } X\}. \end{aligned}$$

Remark. According to (iv) of Proposition 2, we have:

$$g(X) = \inf_{x\in S} \inf\{\|x-y\| :\ y \in S\cap E(\pm x)\};$$

$$G(X) = \sup_{x\in S} \inf\{\|x-y\| :\ y \in S\cap E(\pm x)\}.$$

Note that the above Remark (for $g(X)$) and Proposition 5, together with ($5'$) (for $G'(X)$) answer a question raised at the end of [DT]; in that paper some results concerning these two constants but already proved in [C], were indicated.

As known, several weakenings of the Jordan-von Neumann condition have been considered, each of them implying that the norm derives from an inner product. The following condition instead does not force a space to be an inner product space, at least when $\dim(X) = 2$:

$$\|x+y\| = \|x-y\| \le \sqrt{2} \quad \text{for all } x, y \in S\cap E(\pm x). \tag{$*$}$$

In fact (see e.g. [B, p. 1078]), examples are known of non–Hilbert spaces satisfying $A_2 = \sqrt{2}$; e.g.:

$$\|x+y\| + \|x-y\| \le 2\sqrt{2} \quad \text{for all } x, y \in S; \tag{$**$}$$

in particular, if $\|x+y\| = \|x-y\|$, this implies $(*)$. Also, according to (15), (11), (10), $A_2 = \sqrt{2}$ implies

$$G'(X) = g(X) = \sqrt{2};$$

so these equalities do not imply that X is an inner product space. In fact (see also [GL$_1$, Prop. 2.8]), in the same example quoted we have $\|x+y\| = \|x-y\| = \sqrt{2}$ for all $x, y \in S \cap E(\pm x)$; $g(X) = g'(X) = G(X) = G'(X) = \sqrt{2}$; $\inf_{y\in S} \max\{\|x-y\|, \|x+y\|\} = \sup_{y\in S} \min\{\|x-y\|, \|x+y\|\} = \sqrt{2}$ for every x.

We do not know of similar examples in spaces X such that $\dim(X) \geq 3$. For a discussion of this, see [BCP, p. 143].

It is known that concerning the constants defined in (6)-(9), only the trivial inequalities

$$g(X) \leq G(X)$$
$$g'(X) = A'_1(X) \leq G'(X) = A'_2(X)$$
$$g(X) \leq g'(X)\ ,\ G(X) \leq G'(X)$$

are true (and the 4 constants are really different from each other): see e.g. [C].

By taking $X = \ell_\infty$ (where $G(X) = 2$; $A_1(X) = 3/2$), we see that the following inequality can be true:

$$A_1(X) < G(X).$$

In general, $A_1(X) \neq A'_1(X)$: in fact, we can have $A'_1(X) = 1$, while $A_1(X) > 5/4$ always (see $(3'')$. In particular (see [BCP], § 6): $A_1(c_0) = 3/2 > 1 = A'_1(c_0)$.

For L_p, $1 \leq p < \infty$, the values of $g(X)$, $g'(X)$, $G(X)$, $G'(X)$ have been given in [GL$_1$, Theorem 3.2]; in particular, $g'(L_p) = \max(2^{1/p}, 2^{1-1/p})$. We have the same values for A_2 (which, according to (a), depends on the modulus of convexity): see [BCP, Section 5]. Since $A_1 \leq A_2$ always, by (14) we also obtain $A_1(L_p) = \max(2^{1/p}, 2^{1-1/p})$.

With regard to Proposition 4, note that $g(X) = 1 \Leftrightarrow X$ is not uniformly nonsquare (see (10)). Uniform nonsquare property cannot be characterized by $g'(X)$ or $G(X)$ (see again the examples in [C]); also (see [BCP, Proposition 6.4]) $A_1(X) = 2 \Rightarrow X$ is not uniformly nonsquare, but not conversely.

According to (12), if X is uniformly nonsquare, then $A_2(X) = A'_2(X)$ if and only if $\delta(A_2(X)) = 1 - \frac{A_2(X)}{2}$. In general $A_2(X) \neq A'_2(X)$: in [Pr] an example is given of a space satisfying $\delta(1) = 0$, $\delta(3/2) > 1/4$, so $A'_2(X) < 3/2 < A_2(X)$. Compare with the Example of Poulsen quoted in Section 2 (see also Section 3 in [BCP]).

Concerning A_2, we always have (see [BCP, Proposition 2.2]): $A_2(X) = A_2(X^*)$. Also: $A_1(\ell_\infty) = 3/2$; $A_1(\ell_1) = 2$, so A_1 can both increase and decrease when passing to the dual. The same is true for the constants $G(X)$ and $g'(X) = A_1'(X)$: see e.g. their values in ℓ_p indicated in [C, Proposition 3.2].

Concerning $g(X)$ and $G'(X)$, the fact that they can be different in X and in X^* is not strange, according to the fact that the moduli of convexity of X and X^* are in general different (see (12)); Example 2 in [KMT] indicates a space X such that $G'(X) \neq G'(X^*)$. Also, according to a result in [Gu], $g(X) \geq 1 + \delta(1/2)$ holds always.

6. Other coefficients? Given X, set for $\varepsilon \in [0, 2]$ (see [BR, p.398])

$$\sigma(\varepsilon) = \sup\left\{1 - \frac{\|x+y\|}{2} :\ x, y \in S;\ \|x-y\| = \varepsilon\right\}.$$

This modulus has already been introduced by Day; we have:

$$\delta(\varepsilon) \leq \sigma(\varepsilon) \leq \varepsilon/2 \ \text{ holds always.}$$

It was proved in Section 5 of [BR, p.422], that $g(X)$ is the unique solution, in β, of the equation

$$\sigma(\beta) = 1 - \beta/2.$$

Since $g(X) = 2/G'(X)$, if $g(X) \neq 1$, then by using (12) we have: $g(X) = \beta \Leftrightarrow \beta = 2(1 - \sigma(\beta)) \Leftrightarrow \delta(2/\beta) = 1 - 1/\beta$.

Note that X is uniformly nonsquare $\Leftrightarrow \lim_{\varepsilon \to 2^-} \delta(\varepsilon) > 0 \Leftrightarrow g(X) = 1 \Leftrightarrow \sigma(1) = 1/2 \Leftrightarrow \sigma(\varepsilon) = \varepsilon/2$ for $0 \leq \varepsilon \leq 1$.

The following coefficient was introduced and studied in [N]:

$$\mathrm{NS}(X) = \sup\{\rho :\ \rho \cdot \min\{\|x\|, \|y\|\} \leq \max\{\|x+y\|, \|x-y\|\}\, \forall_{x,y \in X}\}.$$

Next proposition shows that, in fact, this is not a new parameter (so the results in [N] are reformulations of previous results); this fact was implicit in Proposition 7 of [N].

Proposition 6. *For any space X, we have:*

$$\mathrm{NS}(X) = g(X). \tag{16}$$

Proof. We have:

$$\begin{aligned}
\mathrm{NS}(X) &= \inf\left\{\frac{\max\{\|x+y\|, \|x-y\|\}}{\min\{\|x\|, \|y\|\}} :\ x, y \in X \setminus \{0\}\right\} \\
&\leq \inf\left\{\frac{\max\{\|x+y\|, \|x-y\|\}}{\min\{\|x\|, \|y\|\}} :\ x, y \in S\right\} \\
&= g(X).
\end{aligned}$$

We must prove the reverse inequality. Given $\varepsilon > 0$, take $x, y \in X \setminus \{0\}$ such that $\frac{\max\{\|x+y\|,\|x-y\|\}}{\min\{\|x\|,\|y\|\}} < \mathrm{NS}(X) + \varepsilon$; since $\frac{\max\{\|x+y\|,\|x-y\|\}}{\min\{\|x\|,\|y\|\}}$ does not change when passing from x, y to $\tau x, \tau y$ ($\tau \in \mathbb{R}$), we can assume - eventually exchanging x and y - that $0 < \|x\| \leq 1 = \|y\|$. Set $s = 1/\|x\|$ ($\|sx\| = 1$) and $f(t) = \max\{\|sx + ty\|, \|sx - ty\|\}$ ($t \in \mathbb{R}$); we have $f(0) = 1 = \min\{f(t) : t \in \mathbb{R}\}$, so ($s \geq 1$) $f(s) \geq f(1)$.

Therefore we have:

$$\begin{aligned}\mathrm{NS}(X) + \varepsilon > \frac{\max\{\|x+y\|, \|x-y\|\}}{\|x\|} &= \max\{\|sx+sy\|, \|sx-sy\|\}\\ &\geq \max\{\|sx+y\|, \|sx-y\|\} \geq g(X);\end{aligned}$$

this shows that $\mathrm{NS}(X) \geq g(X)$, thus concluding the proof. □

We end by indicating a few remarks concerning the following constant, called the Jordan-von Neumann constant:

$$\mathrm{C_{NJ}}(X) = \sup\left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \;:\; x, y \in X,\ \text{not both } 0 \right\}. \tag{17}$$

This constant was considered e.g. in [KMT] and compared in Section 3 there with some of Gao's constants; we note that it had already been introduced and used in [Pe]. Clearly, $1 \leq \mathrm{C_{NJ}}(X) \leq 2$ is always true.

Among the main results proved for $\mathrm{C_{NJ}}(X)$, we quote the following:

(e) X is Hilbert if and only if $\mathrm{C_{NJ}}(X) = 1$;

(f) X is uniformly nonsquare if and only if $\mathrm{C_{NJ}}(X) < 2$.

We observe that if we define:

$$\mathrm{C'_{NJ}}(X) = \sup\left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X, \text{not both } 0;\ \|x+y\| = \|x-y\| \right\}, \tag{17'}$$

then (e) and (f) are true also for $\mathrm{C'_{NJ}}(X)$ (to prove (e), use the characterizations of inner product spaces indicated at page 50 of [A]).

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Dipartimento di Matematica
Piazza Porta S. Donato 5
40127 Bologna, Italy
e-mail: papini@dm.unibo.it

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