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# On the modulus of continuity for starlike mappings

*Dedicated to our friend Jan Krzyż*

ABSTRACT. For a conformal mapping of the unit disk onto a starlike domain with boundary in a given annulus we derive an estimate for the modulus of continuity of the boundary correspondence function. The result is in some sense asymptotically sharp.

**1. Introduction and results.** Let $\Gamma$ be a Jordan curve starshaped w.r. to $w = 0$ and lying in $\{w : 1 \le |w| \le R\}$ for some $R > 1$, let $G := \operatorname{int}\Gamma$, and let $f$ be a conformal map of the unit disk $\mathbb{D}$ in the $z$-plane onto $G$ with $f(0) = 0$, extended continuously to $\overline{\mathbb{D}}$. Finally let $\arg f(e^{i\tau}) = \vartheta(\tau)$ which increases continuously with $\tau$. We are interested in the modulus of continuity of this function:

$$\omega_\Gamma(\delta) := \max\{|\vartheta(\tau) - \vartheta(\sigma)| : |\tau - \sigma| \le \delta\} \qquad (\delta > 0), \tag{1.1}$$

which was recently investigated by Stylianopoulos and Wegert [4]. They have shown that

$$\omega_\Gamma(\delta) \le (6 + \pi \log R)\frac{1}{|\log \delta|} \qquad \text{if} \quad 0 < \delta < \tfrac{1}{4}. \tag{1.2}$$

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We shall complement this estimate by a two-sided estimate which shows that we can omit "6" in (1.2) if "1" is replaced by "$1+\varepsilon$", provided that $\delta$ is sufficiently small.

To state our result we introduce

$$\omega(\delta) := \sup\{\omega_\Gamma(\delta) : \Gamma \text{ as above}\} \qquad \text{for } \delta > 0. \tag{1.3}$$

This quantity can be expressed by a certain harmonic measure, and this will give the following two-sided estimate for $\omega(\delta)$.

**Theorem.** *For given $R > 1$ and given $\varepsilon > 0$ there is $\delta_0 = \delta_0(R,\varepsilon) > 0$ such that for $\delta < \delta_0$*
(1.4)
$$\frac{\pi \log R}{|\log \delta|}\Big[1 + \frac{1-\varepsilon}{|\log\delta|}\log|\log\delta|\Big] < \omega(\delta) < \frac{\pi \log R}{|\log \delta|}\Big[1 + \frac{\pi+\varepsilon}{|\log\delta|}\log|\log\delta|\Big].$$

**Corollary.** *Under the conditions of this theorem, we have*

$$\frac{\pi \log R}{|\log \delta|} < \omega(\delta) < \frac{\pi \log R}{|\log \delta|}(1+\varepsilon) \qquad \textit{for } \delta < \delta_1(R,\varepsilon). \tag{1.5}$$

Our proof gives, in principle, the possibility to derive a concrete $\delta_0(R,\varepsilon)$ and $\delta_1(R,\varepsilon)$. But we will omit the lengthy calculations.

**Remarks. 1.** For every fixed $\varepsilon > 0$ there is no $R_0(\varepsilon) > 1$ such that the right–hand side of (1.4), (1.5) holds for all $R < R_0(\varepsilon)$ *and* all sufficiently small $\delta > 0$.
**2.** The theorem can immediately be generalized to quasiconformal mappings. We only have to write such a mapping as the composition of a quasiconformal mapping of the unit disk onto itself (with the well-known Hölder continuity) and a conformal mapping.

**2. Reduction of the problem.** For a given curve $\Gamma$ we take $\sigma$, $\tau$ with $|\tau - \sigma| \le \delta$ and try to estimate $|\vartheta(\tau) - \vartheta(\sigma)|$. Obviously, we can assume without loss of generality that

$$\sigma = 0, \quad 0 < \tau \le \delta \ \text{ and } \ \vartheta(0) = 0, \text{ i.e. } \arg f(1) = 0,$$

so that we have to estimate $\vartheta(\tau)$. Since $\vartheta(\tau)$ increases with $\tau$, it suffices to estimate $\vartheta(\delta)$. Let $\Gamma_\delta$ be the subarc of $\Gamma$ with $0 \le \vartheta \le \vartheta(\delta)$ which is the image of $\gamma_\delta := \{z = e^{i\varphi} : 0 \le \varphi \le \delta\}$ under the mapping $f$.

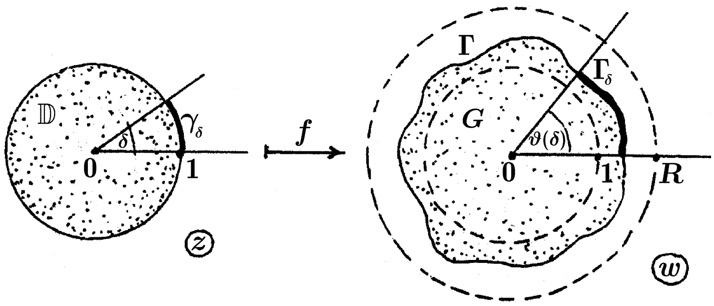


FIG. 1

Because of its conformal invariance, the harmonic measure of $\Gamma_\delta$ with respect to $w = 0$ is

$$\frac{\delta}{2\pi} = \omega(0, \Gamma_\delta, G). \tag{2.1}$$

We now replace $G$ by

$$G' := \mathbb{D} \cup \{w : |w| < R,\ 0 < \arg w < \vartheta(\delta)\} \tag{2.2}$$

and $\Gamma_\delta$ by the circular arc $\Gamma'_\delta$ (Fig. 2). Now $\partial G'$ consists of $\Gamma'_\delta$, of a part of the unit circle, and both are connected by two radial line segments. In other words, we push $\Gamma_\delta$ outwards to the circle $\{w : |w| = R\}$ to get $\Gamma'_\delta$, while we push $\Gamma \setminus \Gamma_\delta$ inwards to the unit circle.

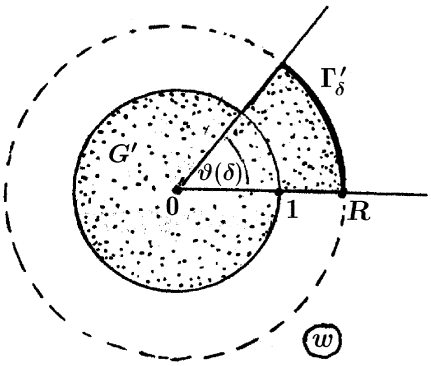


FIG. 2

Now it is readily seen that the functions $\omega(w, \Gamma_\delta, G)$ and $\omega(w, \Gamma'_\delta, G')$ are harmonic in $g := G \cap G'$ and that

$$\omega(w, \Gamma_\delta, G) \geq \omega(w, \Gamma'_\delta, G') \qquad \text{for} \quad w \in \partial g.$$

By the maximum principle this holds also for $w=0$, and (2.1) gives

$$\frac{\delta}{2\pi} = \omega(0,\Gamma_\delta,G) \geq \omega(0,\Gamma'_\delta,G'), \tag{2.3}$$

with equality if and only if $\Gamma_\delta = \Gamma'_\delta$ and $G = G'$. The right–hand side depends only on $\vartheta(\delta)$ and no longer on the shape of $\Gamma_\delta$, and we get from (2.3)

$$\delta \geq 2\pi\,\omega(0,\Gamma'_\delta,G') =: h(\vartheta(\delta)) \qquad \text{for all } \delta \text{ with } 0<\delta<2\pi. \tag{2.4}$$

Notice that $h(\vartheta)$ increases with $\vartheta$, so that (2.4) implies $\vartheta(\delta) \leq h^{-1}(\delta)$. This gives our upper estimate of the desired type

$$\omega_\Gamma(\delta) \leq h^{-1}(\delta) \qquad \text{for all } \delta \text{ with } 0<\delta<2\pi. \tag{2.5}$$

Although $\partial G' =: \Gamma'$ is not starshaped w.r. to $w=0$, we can approximate $\Gamma'$ arbitrarily close by bending the straight line pieces of $\Gamma'$ slightly to get a starshaped Jordan curve which has modulus of continuity near that of $\Gamma'$. This argument shows that the *upper bound in* (2.5) *is best possible, i.e. cannot be decreased.* In other words

$$\omega(\delta) = \sup\{\omega_\Gamma(\delta) : \Gamma\} = h^{-1}(\delta). \tag{2.6}$$

**3. A lemma for harmonic measure.** We now have to estimate in (2.4) the harmonic measure $h(\vartheta(\delta))$. For simplicity we write in what follows $h(\vartheta)$ instead of $h(\vartheta(\delta))$. The main tool for estimating $h(\vartheta)$ is a connection with a conformal module for which we then can use known estimates.

**a.** For this purpose we use the following scheme.

With the square root transformation $\sqrt{w}$ we obtain from the two–sheeted $G'$ a quadrilateral in the plane $\sqrt{w}$ which is symmetric with respect to 0 whose opposite sides are circular arcs $\sqrt{R}\cdots\sqrt{R}e^{i\vartheta/2}$ and $-\sqrt{R}\cdots -\sqrt{R}e^{i\vartheta/2}$.

With the in the scheme prescribed Riemann mapping $W = W(w)$ our harmonic measure obviously satisfies

$$h(\vartheta) = \frac{\theta}{\pi}, \tag{3.1}$$

where $e^{2i\theta}$ is the image of $Re^{i\vartheta}$.

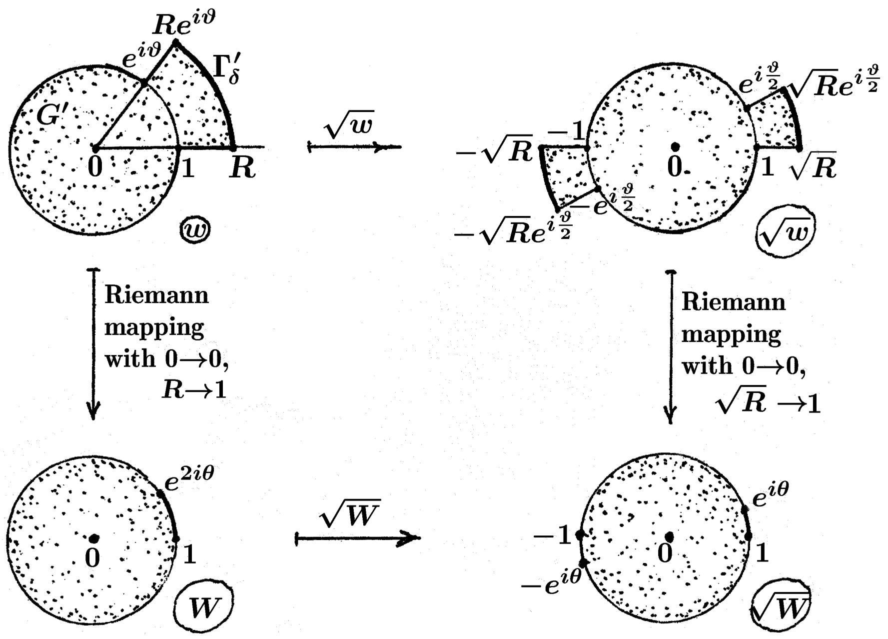


FIG. 3

Now the relation between the conformal module $\mathfrak{M}$ of the quadrilateral in the plane $\sqrt{w}$ (resp. the image in the plane $\sqrt{W}$) and the harmonic measure $h(\vartheta) = \theta/\pi$ is given in the following lemma; see also [3], Theorem 2.75.

**Lemma.** *With the usual notation $\mu(...)$ of the module of Grötzsch's extremal domain (see [2], p. 53) we have*

$$\mathfrak{M} = \frac{\pi}{2\mu(\sin\frac{\theta}{2})} = \frac{K(\sin\frac{\theta}{2})}{K(\cos\frac{\theta}{2})} \tag{3.2}$$

*($K$ = elliptic integral of the first kind).*

**Proof.** Under the Möbius transformation $\zeta(\mathfrak{w})$ defined by

$$\zeta = -i\frac{1-\mathfrak{w}}{1+\mathfrak{w}}$$

the unit disk in the $\mathfrak{w} = \sqrt{W}$-plane is mapped on the lower $\zeta$-halfplane with

$$1 \to 0,\ -1 \to \infty,\ e^{i\theta} \to -\mathrm{tg}\,\frac{\theta}{2},\ -e^{i\theta} \to \mathrm{ctg}\,\frac{\theta}{2}\ .$$

We now consider the $\zeta$-plane with cuts along the segment $-\mathrm{tg}\,\frac{\theta}{2}\cdots 0$ and the ray $\mathrm{ctg}\,\frac{\theta}{2}\cdots+\infty$ as a Teichmüller extremal domain (see [2], p. 55). The corresponding module $M$ (= logarithm of the quotient $> 1$ of the radii of a conformally equivalent annulus) satisfies

$$M = 2\mu\left(\sin\frac{\theta}{2}\right).$$

Because of conformal invariance $\mathfrak{M}$ is also the conformal module of the lower half of our Teichmüller extremal domain, considered as a quadrilateral with obvious opposite sides. This yields

$$\mathfrak{M} = \frac{\pi}{M} = \frac{\pi}{2\mu(\sin\frac{\theta}{2})}.$$

The lemma is proved. □

**b.** The inequality

$$\log\frac{(1+\sqrt{1-r^2})^2}{r} < \mu(r) < \log\frac{4}{r} \tag{3.3}$$

(see [2], p. 61, in our case $r = \sin\frac{\theta}{2}$) gives us with (3.2)

$$\frac{\pi/2}{\log\dfrac{4}{\sin\frac{\theta}{2}}} < \mathfrak{M} < \frac{\pi/2}{\log\dfrac{(1+\cos\frac{\theta}{2})^2}{\sin\frac{\theta}{2}}},$$

or with $\sin\frac{\theta}{2} > \frac{\theta}{2}\left(1-\frac{\theta^2}{24}\right)$ and $\sin\frac{\theta}{2} < \frac{\theta}{2}$, $\cos\frac{\theta}{2} > 1-\frac{\theta}{\pi}$

$$\frac{\pi/2}{\log\dfrac{8}{\theta\left(1-\frac{\theta^2}{24}\right)}} < \mathfrak{M} < \frac{\pi/2}{\log\dfrac{2\left(2-\frac{\theta}{\pi}\right)^2}{\theta}}. \tag{3.4}$$

**4. Proof of (1.4) (left–hand side).** Our aim is the connection between $\vartheta$ and the harmonic measure $h = \frac{\theta}{\pi}$. Because we have by the lemma a connection between $\mathfrak{M}$ and $\theta$, we need only a connection between $\vartheta$ and $\mathfrak{M}$. So we have to consider $\mathfrak{M}$ as a function of $\vartheta$.

After a conformal mapping of the plane $\sqrt{w}$ under the logarithm we can reduce this problem with the Schwarz-Christoffel formula to the discussion of an elliptic integral (for this reason it is enough to study a quarter of the domain in the plane $\sqrt{w}$ which is a pentagon).

But we prefer to estimate $\mathfrak{M}$ in what follows in both directions using module estimates.

An upper estimate follows easily by using classical inequalities for the comparison of modules. We introduce three quadrilaterals $\mathfrak{V}_1$, $\mathfrak{V}_2$, $\mathfrak{V}_3$ with modules $\mod \mathfrak{V}_k$ as parts of our whole quadrilateral with the module $\mathfrak{M}$.

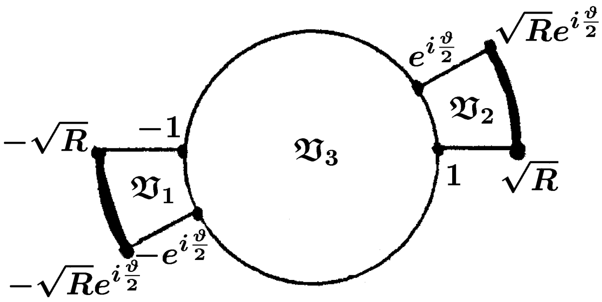


FIG. 4

Namely, define the quadrilaterals $\mathfrak{V}_k$ as follows

$\mathfrak{V}_1$: opposite sides = segments $-\sqrt{R} \cdots -1$ and $-\sqrt{R}e^{i\vartheta/2} \cdots -e^{i\vartheta/2}$ other sides = circular arcs with radii 1 and $\sqrt{R}$ (see Fig. 4),

$\mathfrak{V}_2$: opposite sides = segments $1 \cdots \sqrt{R}$ and $e^{i\vartheta/2} \cdots \sqrt{R}e^{i\vartheta/2}$ other sides = circular arcs with radii 1 and $\sqrt{R}$,

$\mathfrak{V}_3$: opposite sides = arcs $-1 \cdots e^{i\vartheta/2}$ and $-e^{i\vartheta/2} \cdots 1$ on the unit circle, other sides = remaining arcs on the unit circle.

Then we have immediately

$$\sum_{k=1}^{3} \mod \mathfrak{V}_k \le \frac{1}{\mathfrak{M}} . \tag{4.1}$$

Here we have

$$\mod \mathfrak{V}_1 = \mod \mathfrak{V}_2 = \frac{\log R}{\vartheta} .$$

Further we obtain from (3.2) (replacing $\theta$ by $\vartheta/2$ and $\mathfrak{M}$ by $1/\mod \mathfrak{V}_3$)

$$\mod \mathfrak{V}_3 = \frac{2}{\pi}\mu(\sin\frac{\vartheta}{4}).$$

The resulting inequality

$$\frac{2\log R}{\vartheta} + \frac{2}{\pi}\mu(\sin\frac{\vartheta}{4}) \le \frac{1}{\mathfrak{M}}$$

leaves us because of (3.3) with

$$\frac{2\log R}{\vartheta} + \frac{2}{\pi}\log\frac{\left(1+\cos\frac{\vartheta}{4}\right)^2}{\sin\frac{\vartheta}{4}} \le \frac{1}{\mathfrak{M}}\ ,$$

$$\frac{2\log R}{\vartheta}\left[1+\frac{\vartheta}{\pi\log R}\Big(\log\frac{16}{\vartheta}-\frac{\vartheta^2}{3}+...\Big)\right] \le \frac{1}{\mathfrak{M}}$$

(... even powers of $\vartheta$ starting with $\vartheta^4$),

$$\vartheta \ge 2\,\mathfrak{M}\log R\cdot\left[1+\frac{\vartheta}{\pi\log R}\Big(\log\frac{16}{\vartheta}-\frac{\vartheta^2}{3}+...\Big)\right]\ ,$$

especially $\vartheta \ge 2\,\mathfrak{M}\log R$ (for small $\vartheta$), therefore

$$\vartheta \ge 2\,\mathfrak{M}\log R\left[1+\frac{2}{\pi}\mathfrak{M}\Big(\log\frac{8}{\mathfrak{M}\log R}-\frac{1}{3}(2\,\mathfrak{M}\log R)^2+...\Big)\right], \tag{4.2}$$

because the function $\vartheta\left(\log\frac{16}{\vartheta}-\frac{\vartheta^2}{3}+...\right)$ is monotonically increasing (for small $\vartheta$).

Now we combine this with

$$\mathfrak{M} \ge \frac{\pi/2}{\left(\log\frac{8}{\theta}\right)\left[1+\frac{\theta^2+\cdot\cdot\,\theta^4+...}{24\log\frac{8}{\theta}}\right]} \tag{4.3}$$

(see (3.4)). Because of

$$\log\frac{8}{\theta} = \left(\log\frac{1}{2\theta}\right)\left[1+\frac{\log 16}{\log\frac{1}{2\theta}}\right]$$

this finally yields

$$\vartheta \ge \frac{\pi\log R}{|\log 2\theta|}\left[1+\frac{1}{|\log 2\theta|}\log|\log 2\theta| + \mathcal{O}\left(\frac{1}{|\log 2\theta|}\right)\right]. \tag{4.4}$$

Because we have considered the extremal situation, (4.4) gives us after replacing $\vartheta$ by $\omega$ and $\theta$ by $\frac{\delta}{2}$ the left–hand side of (1.4).

To *prove the remark* after (1.5), take a fixed small $\theta = \frac{\delta}{2}$ and consider $R \to 1$. Because of (3.2) $\mathfrak{M}$ is also fixed, and therefore

$$\mathfrak{M}\cdot\log\frac{8}{\mathfrak{M}\,\log R} \to +\infty \qquad \text{for } R \to 1.$$

So the remark follows from (4.2).

**5. Proof of (1.4) (right–hand side).** We use a continuous analogue of the classical Grötzsch module estimate for families of curves which depend on a parameter (see [1]):

$$\mathfrak{M} \geq \int \frac{dt}{\int\limits_{\mathfrak{C}(t)} \frac{ds}{a}} . \tag{5.1}$$

Here $\mathfrak{C}(t)$ are sufficiently smooth arcs in the quadrilateral with module $\mathfrak{M}$, which connect the opposite sides. For different values of parameter $t$ the corresponding $\mathfrak{C}(t)$ are disjoint, and the dependence on the parameter $t$ is also sufficiently smooth. It is not necessary that the arcs $\mathfrak{C}(t)$ fill the quadrilateral completely. Moreover, $s$ is the arc–length on the corresponding $\mathfrak{C}(t)$, and $a\,dt$ is the infinitesimal distance between the arcs $\mathfrak{C}(t)$ and $\mathfrak{C}(t+dt)$. In this way a function $a$ is defined at all points situated on a curve $\mathfrak{C}(t)$.

In our case we use the following concrete family of arcs $\mathfrak{C}(t)$.

Instead of the quadrilateral of Fig. 3 in the plane $\sqrt{w}$ we use the following quadrilateral of Fig. 5 which has the same module $\mathfrak{M}$ because of symmetry. Every $\mathfrak{C}(t)$ consists of three segments $\mathfrak{C}_1(t)$, $\mathfrak{C}_2(t)$, $\mathfrak{C}_3(t)$ as shown in Fig. 5 with the given endpoints. The parameter $t$ is defined by the endpoints $e^{it}$ of $\mathfrak{C}_1(t)$, $0 < t < \vartheta/4$. In this way we have for points with the same $t$ a linear correspondence in the arc–length between: a.) the points of the arc $1 \cdots e^{i\vartheta/4}$ of the unit circle, b.) the points of the arc $\sqrt{R} \cdots \sqrt{R}e^{i\vartheta/4}$ of the circle with center 0 and radius $\sqrt{R}$, c.) the points of the segment $\cos\frac{\vartheta}{4} \cdots e^{i\vartheta/4}$, and d.) the points of the segment $0 \cdots i$.

To get an estimate of the right hand side of (5.1) we start with

$$\int\limits_{\mathfrak{C}_1(t)} \frac{ds}{a} = \int\limits_1^{\sqrt{R}} \frac{ds}{s} = \log\sqrt{R}\ . \tag{5.2}$$

For the corresponding part of $\mathfrak{C}_2(t)$ it is enough to have a rough estimate. An elementary geometric consideration of $\mathfrak{C}_2(t)$ gives us the boundedness of the angle $\gamma$ between $\mathfrak{C}_2(t)$ and the real axis. This means

$$\cos\gamma \geq c \tag{5.3}$$

with a universal constant $c > 0$ (for example $c = 0.6$). Because $a$ attains its extremal values on $\mathfrak{C}_2(t)$ at the endpoints, it is enough to estimate this quantity there. At the left–hand endpoint we have

$$a \geq \frac{4}{\vartheta}\sin\frac{\vartheta}{4}\cos\gamma\ ,$$

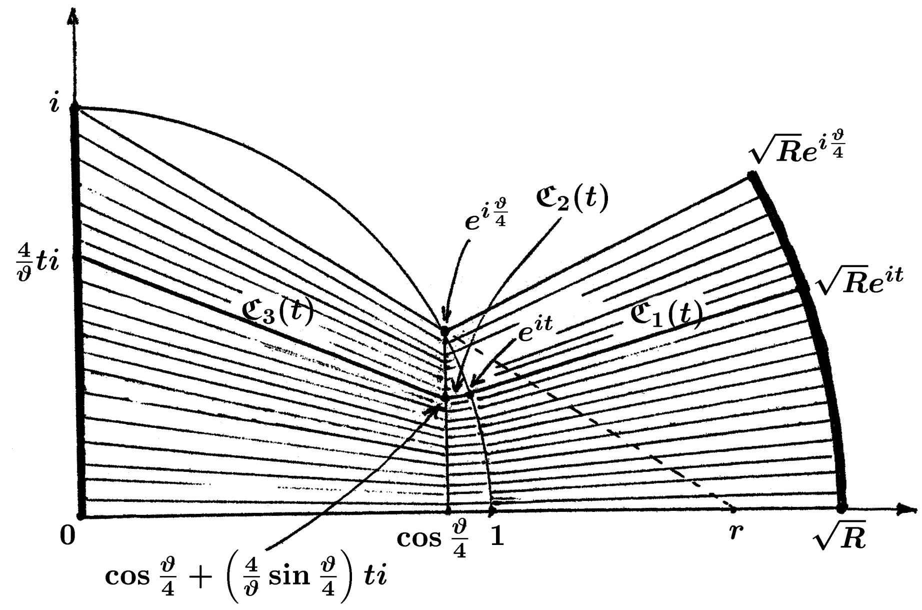


FIG. 5

because $\frac{4}{\vartheta}\sin\frac{\vartheta}{4}\,dt$ is the orthogonal infinitesimal distance there (in the direction of the imaginary axis) between $\mathfrak{C}(t)$ and $\mathfrak{C}(t+dt)$. Mutatis mutandis we have at the right–hand endpoint

$$a \geq \cos\gamma \cdot \frac{d}{dt}\sin t = \cos\gamma\cos t \geq \cos\frac{\vartheta}{4}\cdot\cos\gamma.$$

Therefore we have on the whole segment $\mathfrak{C}_2(t)$:

$$a \geq \cos\frac{\vartheta}{4}\cos\gamma \geq c\cdot\cos\frac{\vartheta}{4}\ .$$

Moreover, the length of $\mathfrak{C}_2(t)$ is less than $\frac{1}{c}(1-\cos\frac{\vartheta}{4})$. Therefore

$$(5.4)\qquad \int\limits_{\mathfrak{C}_2(t)} \frac{ds}{a} \leq \frac{1}{c\cdot\cos\frac{\vartheta}{4}}\cdot\frac{1}{c}\left(1-\cos\frac{\vartheta}{4}\right) \leq \frac{1}{c^2}\frac{\frac{1}{2}\left(\frac{\vartheta}{4}\right)^2}{\cos\frac{\vartheta}{4}} = \frac{\vartheta^2}{32c^2\cdot\cos\frac{\vartheta}{4}}\ .$$

For the last part $\int_{\mathfrak{C}_3(t)}$ in (5.1) let us denote by $a'\,dt$ the orthogonal distance (in the direction of the imaginary axis) between $\mathfrak{C}(t)$ and $\mathfrak{C}(t+dt)$ at the corresponding point. Then obviously $a \geq a'/\sqrt{2}$. Let us further denote by $d\sigma$ the orthogonal projection of every element $ds$ of $\mathfrak{C}_3(t)$ on the real axis. Then $\frac{ds}{d\sigma} < \sqrt{2}$. Because $a'$ is a linear function of $\sigma$, we have

$$a' = \frac{\sigma}{d} \cdot \frac{4}{\vartheta} \sin\frac{\vartheta}{4} \quad \text{with} \quad d = \frac{\cos\frac{\vartheta}{4}\sin\frac{\vartheta}{4}}{1-\sin\frac{\vartheta}{4}}\ , \tag{5.5}$$

if we set $\sigma = 0$ for the common real point of intersection

$$r = \frac{\cos\frac{\vartheta}{4}}{1-\sin\frac{\vartheta}{4}}$$

(see Fig. 5) of the prolongations of the $\mathfrak{C}_3(t)$. The value (5.5) for $a'$ follows because we have for $\sigma = d$ (corresponding to the right–hand endpoint of $\mathfrak{C}_3(t)$) the value $a' = \frac{4}{\vartheta}\sin\frac{\vartheta}{4}$.

This altogether yields

$$\begin{aligned}\int\limits_{\mathfrak{C}_3(t)} \frac{ds}{a} &\le \sqrt{2} \int\limits_{\mathfrak{C}_3(t)} \frac{ds}{a'} = \sqrt{2}\int\limits_d^r \frac{ds}{d\sigma}\frac{d\sigma}{a'} \le 2\int\limits_d^r \frac{d\sigma}{a'} = 2d\frac{\vartheta}{4}\frac{1}{\sin\frac{\vartheta}{4}}\int\limits_d^r \frac{d\sigma}{\sigma} \\ &< \frac{\vartheta/2}{1-\sin\frac{\vartheta}{4}} \cdot \left|\log\sin\frac{\vartheta}{4}\right| .\end{aligned} \tag{5.6}$$

(If we use instead of the arcs $\mathfrak{C}_3(t)$ a "better" curve family which fills the space under the unit circle, we get with much more lengthy calculations in the result (1.4) a smaller constant than $\pi + \varepsilon$.)

Collecting (5.2), (5.4), (5.6) we obtain from (5.1)

$$\mathfrak{M} \ge \frac{\vartheta}{4}\Big[\log\sqrt{R} + \frac{\vartheta^2}{32c^2\cos\frac{\vartheta}{4}} + \frac{\vartheta/2}{1-\sin\frac{\vartheta}{4}}\Big|\log\sin\frac{\vartheta}{4}\Big|\Big]^{-1} . \tag{5.7}$$

To obtain now an estimate of $\vartheta$ with $\mathfrak{M}$ from above we write (5.7) in the form

$$\vartheta \le 2\,\mathfrak{M}\Big[\log R + \frac{\vartheta^2}{16c^2\cos\frac{\vartheta}{4}} + \frac{\vartheta}{1-\sin\frac{\vartheta}{4}}\Big|\log\sin\frac{\vartheta}{4}\Big|\Big] . \tag{5.8}$$

If we compare the module $\mathfrak{M}$ of our quadrilateral in Fig. 5 with the module of the rectangle with corners 0, $\sqrt{R}$, $i\sin\frac{\vartheta}{4}$, $\sqrt{R}+i\sin\frac{\vartheta}{4}$, we obtain additionally

$$\mathfrak{M} \ge \frac{\sin\frac{\vartheta}{4}}{\sqrt{R}} \ge \frac{\vartheta}{2\pi\sqrt{R}}\,, \qquad \text{or} \qquad \vartheta \le 2\pi\sqrt{R}\,\mathfrak{M}\ .$$

Inserting this in the right–hand side of (5.8) we get

$$\vartheta \le 2\,\mathfrak{M}\Big[\log R + \mathcal{O}\left(\mathfrak{M}\log\frac{1}{\mathfrak{M}}\right)\Big]\ .$$

And inserting this now in the right–hand side of (5.8) we arrive at

$$\vartheta \le 2\,\mathfrak{M}\Big[\log R + 2(\log R)\,\mathfrak{M}\log\frac{1}{\mathfrak{M}} + \mathcal{O}(\mathfrak{M})\Big]. \tag{5.9}$$

Now we have to combine this with the inequality

$$\mathfrak{M} \le \frac{\pi/2}{|\log 2\theta|}\Big[1 + \mathcal{O}\left(\frac{1}{|\log 2\theta|}\right)\Big], \tag{5.10}$$

which follows from (3.4). This leaves us with

$$\vartheta \le \frac{\pi\log R}{|\log 2\theta|}\Big[1 + \frac{\pi}{|\log 2\theta|}\log|\log 2\theta|\Big(1 + \mathcal{O}\left(\frac{1}{\log|\log 2\theta|}\right)\Big)\Big]. \tag{5.11}$$

Because we have considered the extremal situation, (5.11) gives us finally the right–hand side of (1.4) after replacing $\vartheta$ by $\omega$ and $2\theta$ by $\delta$.

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