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Almost sure functional limit theorems

Abstract. A general almost sure limit theorem is presented. Then it is applied to obtain almost sure versions of some functional (central) limit theorems.

1. Introduction and a general theorem. Let ζ_n, $n \in \mathbb{N}$, be a sequence of random elements defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Almost sure limit theorems state that

$$(1.1) \qquad \frac{1}{D_n} \sum_{k=1}^{n} d_k \delta_{\zeta_k(\omega)} \Rightarrow \mu\,, \quad \text{as} \quad n \to \infty, \quad \text{for almost every} \quad \omega \in \Omega\,,$$

where δ_x is the point mass at x and $\Rightarrow \mu$ denotes weak convergence to the probability measure μ.

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In the simplest forms of the almost sure central limit theorem (a.s. CLT) $\zeta_n = (X_1 + \dots + X_n)/\sqrt{n}$, where $X_1, X_2, \dots,$ are i.i.d. real random variables with mean 0 and variance 1, $d_k = 1/k$, $D_n = \log n$, and μ is the standard normal law $\mathcal{N}(0,1)$; see Brosamler (1988), Schatte (1988), Lacey and Philipp (1990). Then almost sure versions of several known usual limit theorems were proved, however most of them contained logarithmic average, i.e. $d_k = 1/k$ and $D_n = \log n$; see Berkes (1998) for an overview. Only few papers dealt with more general weights d_k and D_n, see e.g. Atlagh (1993), Rodzik and Rychlik (1994). But recently, several papers are devoted to general forms of the a.s. CLT, e.g. Ibragimov and Lifshits (1999), Berkes and Csáki (2001), Chuprunov and Fazekas (2001a).

In this paper the general result is Theorem 1.1 which is a common extension of the basic results of Berkes and Csáki (2001) and Chuprunov and Fazekas (2001a). Then, in Section 2 we apply this theorem to prove a.s. versions of some functional limit theorems: convergence of Wiener processes, Donsker's theorem, empirical processes, maximum of partial sums processes. In Section 3 we show a modification of our method for dependent variables.

Let (B, ϱ) be a complete separable metric space and ζ_n, $n \in \mathbb{N}$, be a sequence of random elements in B. Let μ_ζ denote the distribution of ζ. Let $\log_+ x = \log x$ if $x \geq 1$ and $\log_+ x = 0$ if $x < 1$.

Theorem 1.1. *Assume that there exist $C > 0$, $\varepsilon > 0$, an increasing sequence of positive numbers c_n with $\lim_{n\to\infty} c_n = \infty$, $c_{n+1}/c_n = \mathrm{O}(1)$ and B-valued random elements ζ_{kl}, $k, l \in \mathbb{N}$, $k < l$, such that the random elements ζ_k and ζ_{kl} are independent for $k < l$ and*

$$\mathbb{E}\{\varrho(\zeta_{kl}, \zeta_l) \wedge 1\} \leq C \left\{ \log_+ \log_+ \left(\frac{c_l}{c_k} \right) \right\}^{-(1+\varepsilon)}, \tag{1.2}$$

for $k < l$. For $0 \leq d_k \leq \log(c_{k+1}/c_k)$ assume that $\sum_{k=1}^\infty d_k = \infty$ and set $D_n = \sum_{k=1}^n d_k$. Then for any probability distribution μ on the Borel σ-algebra of B the following two statements are equivalent

$$\frac{1}{D_n} \sum_{k=1}^n d_k \delta_{\zeta_k(\omega)} \Rightarrow \mu\,, \quad \textit{as} \quad n \to \infty, \quad \textit{for almost every} \quad \omega \in \Omega\,; \tag{1.3}$$

$$\frac{1}{D_n} \sum_{k=1}^n d_k \mu_{\zeta_k} \Rightarrow \mu\,, \quad \textit{as} \quad n \to \infty\,. \tag{1.4}$$

Remark 1.2. *Theorem 1.1 remains valid if condition (1.2) is replaced by the following*

$$\mathbb{E}\{\varrho(\zeta_{kl}, \zeta_l) \wedge 1\} \leq C \left(\frac{c_k}{c_l} \right)^\beta, \tag{1.5}$$

for $k < l$, *where* $\beta > 0$.

Berkes and Csáki (2001) proved general theorems for the real valued case. Our Theorem 1.1 and Remark 1.2 cover Theorems 1-4 of Berkes and Csáki (2001). To see this take $B = \mathbb{R}$, and let $X_1, X_2, \dots,$ be independent $\mathbb{R}$-valued random variables. In Theorem 1.1 let $\zeta_l = f_l(X_1, \dots, X_{n_l})$, $\zeta_{k,l} = f_{k,l}(X_{n_k+1}, \dots, X_{n_l})$ for $k < l$. Then Theorem 1.1 is the same as Theorem 4 of Berkes and Csáki (2001). The extension of the results of Berkes and Csáki (2001) to abstract state space is technically simple, but the scope of possible applications (including a.s. versions of functional limit theorems) become much wider.

Chuprunov and Fazekas (2001a) dealt with the case of metric space valued random elements. If we put $c_l = l$ into (1.5), then our Remark 1.2 with $d_k = 1/k$ and $D_n = \log n$ is the same as Theorem 1 of Chuprunov and Fazekas (2001a). We shall see that the more general weight sequence provides new applications.

We also remark that Ibragimov and Lifshits (1999) gave results for a.s. limit theorems both for real valued and metric space valued sequences. Their theorems are more general than the one in our paper because they did not assume independence in the background. They applied their results for expressions built of independent or weakly dependent random variables. However, when they used their results for the independent case, they turned to the same considerations as included in Theorem 1.1 and in its proof. Moreover, we shall show that our method can easily be extended to weakly dependent variables. We mention that Ibragimov and Lifshits (1999) did not use so general weight sequence as the one in Theorem 1.1.

The importance of condition (1.4) is demonstrated in Berkes, Csáki and Csörgő (1999), when they gave an example where a.s. limit theorem is true, however μ_k does not converge to μ.

To prove Theorem 1.1 and Remark 1.2 we will use the following strong law of large numbers (which is contained and proved (but not stated explicitly) in the proof of Theorem 4 of Berkes and Csáki (2001)). The formulation of Lemma 1.3 is suitable to a.s. CLT for weakly dependent variables.

Lemma 1.3. *Let* ξ_i, $i \in \mathbb{N}$, *be uniformly bounded random variables. Let*

$$T_n = \frac{1}{D_n} \sum\nolimits_{k=1}^{n} d_k \xi_k ,$$

where $\{d_k\}$ *is a nonnegative sequence with* $\sum_{k=1}^{\infty} d_k = \infty$ *and* $D_n = \sum_{k=1}^{n} d_k$.

(a) Assume that there exist $C > 0$, $\varepsilon > 0$, *an increasing sequence of positive numbers* c_n *with* $\lim_{n\to\infty} c_n = \infty$, $c_{n+1}/c_n = \mathrm{O}(1)$ *such that*

$$|\mathbb{E}\{\xi_k \xi_l\}| \le C \left\{ \log_+ \log_+ \left(\frac{c_l}{c_k} \right) \right\}^{-(1+\varepsilon)} , \tag{1.6}$$

for $k < l$. Let $0 \le d_k \le \log(c_{k+1}/c_k)$, $k = 1, 2, \ldots$. Then $D_{n+1}/D_n \to 1$ and

$$\mathbb{E}T_n^2 \le c\big(\log D_n\big)^{-(1+\varepsilon)} \tag{1.7}$$

for n large enough.

(b) Assume that there exists $\varepsilon > 0$ such that (1.7) is satisfied and $D_{n+1}/D_n \to 1$. Then

$$\lim_{n\to\infty} T_n = 0 \quad a.s. \tag{1.8}$$

Proof. (a) Taking into account the proof of Theorem 4 of Berkes and Csáki (2001), we easily get

$$\begin{aligned}\mathbb{E}\Big\{\sum_{k=1}^n d_k\xi_k\Big\}^2 &\le 2\sum_{k=1}^n\sum_{l=k}^n d_k d_l|\mathbb{E}\xi_k\xi_l| \\ &\le 2c\sum_{k=1}^n\sum_{l=k}^n d_k d_l\Big\{\log_+\log_+\Big(\frac{c_l}{c_k}\Big)\Big\}^{-(1+\varepsilon)}.\end{aligned} \tag{1.9}$$

Let n be so large that $D_n \ge 4$.

We divide terms in (1.9) into two classes. First consider pairs (k, l) such that $c_l/c_k \ge \exp\big(D_n^{1/2}\big)$. The contribution of these terms in (1.9) is not greater than

$$2c\sum_{k=1}^n\sum_{l=k}^n d_k d_l\Big(\frac{1}{2}\log D_n\Big)^{-(1+\varepsilon)} \le 2cD_n^2\big(\log D_n\big)^{-(1+\varepsilon)}. \tag{1.10}$$

(Here we did not delete any term from the double sum.)

For the remaining terms we have $c_l/c_k \le \exp\big(D_n^{1/2}\big)$. For a fixed k (and n) let l_k denote the greatest l satisfying this inequality. Since c_{n+1}/c_n is bounded, $M = \sup_n c_{n+1}/c_n$ is finite. Then $\log(c_{l+1}/c_k) = \log(c_{l+1}/c_l) + \log(c_l/c_k) \le \log M + D_n^{1/2}$. Since ξ_n is a bounded sequence, we have $|\mathbb{E}\xi_k\xi_l| \le c$. (Here we use this upper bound, so we do not deal with $\log_+$.) Therefore the contribution of the second class of terms to (1.9) is not greater than

$$\begin{aligned}&2c\sum_{k=1}^n d_k \sum_{\substack{l=k \\ c_l/c_k\le\exp\left(D_n^{1/2}\right)}}^n d_l \\ &\le 2c\sum_{k=1}^n d_k\left(\log\Big(\frac{c_{l_k+1}}{c_{l_k}}\Big) + \log\Big(\frac{c_{l_k}}{c_{l_k-1}}\Big) + \cdots + \log\Big(\frac{c_{k+1}}{c_k}\Big)\right) \\ &\le 2c\sum_{k=1}^n d_k\log\Big(\frac{c_{l_k+1}}{c_k}\Big) \le 2cD_n\left(\log M + D_n^{1/2}\right) \le cD_n^{3/2}.\end{aligned}$$

Now, adding the two contributions

$$\mathbb{E}(D_nT_n)^2 = \mathbb{E}\left(\sum\nolimits_{k=1}^{n} d_k\xi_k\right)^2 \leq cD_n^2\left(\log D_n\right)^{-(1+\varepsilon)} + cD_n^{3/2}$$
$$\leq cD_n^2\left(\log D_n\right)^{-(1+\varepsilon)} .$$

Finally, $D_{n+1}/D_n = 1+d_{n+1}/D_n \to 1$ because d_n is bounded and $D_n \to \infty$.

(b) First we prove that $T_{n_k} \to 0$ a.s. for an appropriate subsequence $\{n_k\}$. Let $\eta > 0$ be so small such that $1 + a = (1+\varepsilon)(1-\eta) > 1$. Then the sequence $\exp\left(k^{1-\eta}\right)$ is increasing and converges to ∞, as $k \to \infty$. Let n_k be the first index such that $D_{n_k} \geq \exp\left(k^{1-\eta}\right)$. So n_k is increasing and converges to ∞, as $k \to \infty$. Then, by (1.7),

$$\mathbb{E}T_{n_k}^2 \leq c\big(\log D_{n_k}\big)^{-(1+\varepsilon)} \leq c\Big(\log\big(\exp\left(k^{1-\eta}\right)\big)\Big)^{-(1+\varepsilon)} = ck^{-(1-\eta)(1+\varepsilon)}$$
$$= ck^{-(1+a)} .$$

Therefore $\sum_{k=1}^{\infty} \mathbb{E}T_{n_k}^2 < \infty$. This implies $\mathbb{E}\Big(\sum_{k=1}^{\infty} T_{n_k}^2\Big) < \infty$, $\sum_{k=1}^{\infty} T_{n_k}^2 < \infty$ a.s. and $T_{n_k} \to 0$ a.s. In this way we obtained a subsequence $\{n_k\}$ such that $T_{n_k} \to 0$ a.s.

Now consider the remaining terms. First we show that $D_{n_{k+1}}/D_{n_k} \to 1$. In fact,

$$1 \leq \frac{D_{n_{k+1}}}{D_{n_k}} = \frac{D_{n_{k+1}}}{D_{n_{k+1}-1}}\frac{D_{n_{k+1}-1}}{D_{n_k}} \leq \frac{D_{n_{k+1}}}{D_{n_{k+1}-1}}\frac{\exp\Big((k+1)^{1-\eta}\Big)}{\exp\Big(k^{1-\eta}\Big)}.$$

Here both fractions converge to 1.

Now let $n_k < n \leq n_{k+1}$. Then

$$|T_n| \leq |T_{n_k}| + \frac{1}{D_n}\sum_{i=n_k+1}^{n} d_i c = |T_{n_k}| + \frac{c}{D_n}(D_n - D_{n_k})$$
$$\leq |T_{n_k}| + c\left(1 - \frac{D_{n_k}}{D_{n_{k+1}}}\right) \to 0\,,$$

a.s. as $n \to \infty$. $\square$

The proof of the next lemma follows from that of Theorem 11.3.3 in Dudley (1989). Let $BL(B)$ be the space of the Lipschitz continuous bounded functions $g : B \to \mathbb{R}$ with $\|g\|_{BL} = \|g\|_\infty + \|g\|_L < \infty$, where $\|g\|_\infty$ is the sup norm and

$$\|g\|_L = \sup_{x\neq y}\frac{|g(x)-g(y)|}{\varrho(x,y)}.$$

Lemma 1.4. *Let μ be a finite Borel measure on B. Then there exists a countable set $M \subset BL(B)$ (depending on μ) such that for any sequence of finite Borel measures μ_n on B, $n \in \mathbb{N}$, we have: $\mu_n \Rightarrow \mu$, $n \to \infty$, if and only if for each $g \in M$*

$$\int_B g(x)d\mu_n(x) \to \int_B g(x)d\mu(x) \quad \text{as } n \to \infty.$$

Proof of Theorem 1.1. (1.4) $\Longrightarrow$ (1.3). Let μ be fixed and M be the countable set of functions from Lemma 1.4 that determines the convergence to μ. Let $g \in M$.

Define the random variables $\xi_k = g(\zeta_k) - \mathbb{E}g(\zeta_k)$, $k \in \mathbb{N}$. Let $K \geq 1$ be a constant with $|g(x)| \leq K$ and $|g(x) - g(y)| \leq K\varrho(x,y)$, $x, y \in B$. Then for $k < l$, using the independence of ζ_{kl} and ξ_k,

(1.11)

$$\begin{aligned}
&\big|\mathbb{E}\{\xi_k\xi_l\}\big| = \big|\mathbb{E}\big(g(\zeta_k) - \mathbb{E}g(\zeta_k)\big)\big(g(\zeta_l) - g(\zeta_{kl}) + g(\zeta_{kl}) - \mathbb{E}g(\zeta_l)\big)\big| \\
&= \big|\mathbb{E}\big(g(\zeta_k) - \mathbb{E}g(\zeta_k)\big)\big(g(\zeta_l) - g(\zeta_{kl})\big) + \mathbb{E}\big(g(\zeta_k) - \mathbb{E}g(\zeta_k)\big)\big(g(\zeta_{kl}) - \mathbb{E}g(\zeta_l)\big)\big| \\
&\leq 2K\mathbb{E}\big|g(\zeta_l) - g(\zeta_{kl})\big| \leq 2K\mathbb{E}\big\{2K\varrho(\zeta_{kl},\zeta_l) \wedge 2K\big\} \\
&\leq 4K^2C\left\{\log_+\log_+\left(\frac{c_l}{c_k}\right)\right\}^{-(1+\varepsilon)}.
\end{aligned}$$

By Lemma 1.3 we obtain

(1.12)

$$\begin{aligned}
&\int_B g(x)d\Big(\frac{1}{D_n}\sum\nolimits_{k=1}^n d_k\delta_{\zeta_k(\omega)}\Big)(x) - \int_B g(x)d\Big(\frac{1}{D_n}\sum\nolimits_{k=1}^n d_k\mu_{\zeta_k}\Big)(x) \\
&= \frac{1}{D_n}\sum\nolimits_{k=1}^n d_k\big(g(\zeta_k(\omega)) - \mathbb{E}g(\zeta_k)\big) = \frac{1}{D_n}\sum\nolimits_{k=1}^n d_k\xi_k(\omega) \to 0\,,
\end{aligned}$$

as $n \to \infty$, for almost all $\omega \in \Omega$. By (1.4) the second term in (1.12) converges to $\int_B g(x)d\mu(x)$. Therefore, since the set M is countable, we have for almost all $\omega \in \Omega$ and all $g \in M$

$$\int_B g(x)d\Big(\frac{1}{D_n}\sum\nolimits_{k=1}^n d_k\delta_{\zeta_k(\omega)}\Big)(x) \to \int_B g(x)d\mu(x)\,,$$

as $n \to \infty$. By Lemma 1.4 this implies (1.4) $\Longrightarrow$ (1.3).

(1.3) $\Longrightarrow$ (1.4). Define the following measures:

$$\mu_n = \frac{1}{D_n}\sum\nolimits_{k=1}^n d_k\mu_{\zeta_k}, \qquad \mu_{n,\omega} = \frac{1}{D_n}\sum\nolimits_{k=1}^n d_k\delta_{\zeta_k(\omega)}\,.$$

Let A be a continuity set of μ: $\mu(\partial A) = 0$. The expectation of $\mu_{n,\omega}(A)$ is $\mu_n(A)$, i.e. $\int_\Omega \mu_{n,\omega}(A)d\mathbb{P}(\omega) = \mu_n(A)$. Now, (1.3) means that

$\lim_{n\to\infty} \mu_{n,\omega}(A) = \mu(A)$, for almost every ω. Take expectation in this relation and use dominated convergence theorem to obtain $\lim_{n\to\infty} \mu_n(A) = \mu(A)$. In this way we obtained (1.4). $\square$

Remark 1.5. *If in Theorem 1.1 condition (1.2) is replaced by*

$$\mathbb{E}\{\varrho(\zeta_{kl}, \zeta_l) \wedge 1\} \le C\left(\frac{c_l}{c_k}\right)^{-\gamma}, \tag{1.13}$$

for $k < l$, where $\gamma > 0$, then one can take $d_k = \log(c_{k+1}/c_k)\exp[(\log c_k)^\alpha]$ with $0 \le \alpha < 1/2$ and the statement remains valid. For the proof see Berkes and Csáki (2001).

Remark 1.6. *If conditions (1.2), (1.5), and (1.13) are valid only for $1 < k_0 \le k < l$, then Theorem 1.1, Remark 1.2, resp. Remark 1.5 remain valid. To prove this one has to apply the statements for $\zeta_{k_0}, \zeta_{k_0+1}, \dots$.*

The above results can be extended to the case when B is not separable and is equipped with a σ-algebra different from the σ-field of the Borel sets.

Remark 1.7. *Let B be a metric space. Let $\mathcal{P}$ be a σ-algebra of subsets of B. Let μ, μ_n, $n = 1, 2, \dots$, be probability measures on the measurable space $(B, \mathcal{P})$. We say that μ_n converges weakly to μ ($\mu_n \Rightarrow \mu$) as $n \to \infty$ if $\int_B f(x)d\mu_n(x) \to \int_B f(x)d\mu(x)$ for each continuous, bounded, measurable function $f : B \to \mathbb{R}$.*

Now, Lemma 1.4 has the following form. Assume that $\mathcal{P}$ contains each closed ball. Suppose that the probability μ is concentrated on a complete, separable subspace of B. Then there exists a countable set $M \subset BL(B)$ (depending on μ) such that for any sequence of probability measures μ_n on $(B, \mathcal{P})$, $n \in \mathbb{N}$, we have: $\mu_n \Rightarrow \mu$, $n \to \infty$, if and only if $\int_B g(x)d\mu_n(x) \to \int_B g(x)d\mu(x)$, $n \to \infty$, for each $g \in M$. One can check this result using the setting of Chapter IV in Pollard (1984).

Theorem 1.1 is also valid in this framework. However, one has to assume that the σ-algebra on B contains each closed ball and μ is concentrated on a complete, separable subspace of B.

2. Applications for independent variables. For a previous version of Theorem 1.1 the following examples were given: Pearson's χ^2–statistic (Chuprunov and Fazekas (2001b)), Poisson functional limit theorem, semistable functional limit theorem, functional limit theorems for sums of independent random variables with replacements (Chuprunov and Fazekas (2001a), see also Chuprunov and Fazekas (1999) for a preliminary form). Here we shall not deal with these ones. Berkes and Csáki (2001) gave the following examples for their general theorem: partial sums, extremes of i.i.d. random variables, maxima of partial sums, empirical distribution functions, U-statistics, local times, return times, and Darling-Erdős type limit theorems. Here we shall give a.s. versions of a few functional limit theorems including functional forms for some of examples in Berkes and Csáki (2001).

Let $\mathrm{I}_A(x)$ denote the indicator function of the set A. The dependence of a stochastic process on the elementary event ω will be denoted by a subscript, e.g. $W_\omega(t)$ or $W_{n,\omega}(t)$.

Example 2.1. *The Wiener process.* This simple example shows that we can construct several weight sequences different from $d_k = 1/k$.

Let $W(t)$, $t \ge 0$, be a standard Wiener process. Let $W^{(s)}(u) = \frac{1}{\sqrt{s}} W(su)$, for $u \in [0,1]$, where $s > 0$ is fixed. Let $1 = i_1 < i_2 < \dots$ be an increasing unbounded sequence of real numbers. Then $W^{(i_n)} \Rightarrow W$, as $n \to \infty$, on $C[0,1]$.

To apply Theorem 1.1, let $\zeta_l(u) = W^{(i_l)}(u) = \frac{1}{\sqrt{i_l}} W(i_l u)$, and for $k < l$ let

$$\zeta_{kl}(u) = \frac{1}{\sqrt{i_l}} \big[W(i_l u) - W(i_k)\big] \mathrm{I}_{(i_k, i_l]}(i_l u), \qquad u \in [0,1].$$

Then ζ_k and ζ_{kl}, $k < l$ are independent. Moreover,

$$\mathbb{E}\varrho(\zeta_{kl}, \zeta_l) = \mathbb{E} \sup_{u \in [0,1]} \big|\zeta_{kl}(u) - \zeta_l(u)\big| = \frac{1}{\sqrt{i_l}} \mathbb{E} \sup_{i_l u \in [0, i_k]} \big|W(i_l u)\big|$$

$$= \frac{1}{\sqrt{i_l}} \mathbb{E} \sup_{t \in [0, i_k]} \big|W(t)\big| \le \frac{1}{\sqrt{i_l}} 2\mathbb{E}\big|W(i_k)\big| \le \frac{1}{\sqrt{i_l}} 2\sqrt{\mathbb{E}W(i_k)^2} = 2\frac{\sqrt{i_k}}{\sqrt{i_l}}.$$

Here we applied Freedman (1971), Lemma (16/c). Now we can choose

$$d_k = 2 \log \left[\frac{\sqrt{i_{k+1}}}{\sqrt{i_k}}\right] = \int_{i_k}^{i_{k+1}} \frac{1}{x} dx,$$

and

$$D_n = \sum_{i=1}^n d_i = \int_1^{i_{n+1}} \frac{1}{x} dx = \log(i_{n+1}).$$

Proposition 2.1. *Assume the notation and conditions as above. Let $\{i_n\}$ be an increasing sequence with $\lim_{n\to\infty} i_n = \infty$ and $i_{n+1}/i_n = \mathrm{O}(1)$. Then*

$$\frac{1}{D_n} \sum_{k=1}^n d_k \delta_{W_\omega^{(i_k)}} \Rightarrow \mu_W, \quad \textit{as} \quad n \to \infty, \quad \textit{for almost every} \quad \omega \in \Omega$$

on $C[0,1]$.

A slightly different form of this proposition was proved in Rodzik and Rychlik (1994), using another method. There it served as a tool to prove a.s. version of the Donsker theorem.

Example 2.2. *The sum of independent variables.* We show that the a.s. version of the Lindeberg type functional limit theorem is a simple consequence of our general result.

Let $\mathcal{F}_1, \mathcal{F}_2, \ldots$ be independent σ-subalgebras. Let $X_{n,i}$ be $\mathcal{F}_i$-measurable for $i = 1, \ldots, m_n$, $n = 1, 2, \ldots$. Here $\{m_n\}$ is an increasing sequence of positive integers converging to ∞. Then, for fixed n, $X_{n,1}, X_{n,2}, \ldots, X_{n,m_n}$ are independent random variables. Assume $\mathbb{E}X_{n,i} = 0$ and $\mathbb{E}X_{n,i}^2 = \sigma_{n,i}^2 \in (0,\infty)$, for all i and n. Set $S_{n,0} = 0$, $S_{n,k} = X_{n,1} + \cdots + X_{n,k}$, $s_{n,k}^2 = \mathbb{E}S_{n,k}^2 = \sigma_{n,1}^2 + \cdots + \sigma_{n,k}^2$, for $k = 1, 2, \ldots, m_n$. Let $s_n = s_{n,m_n}$. Define the random function $Y_n(t)$, $t \in [0,1]$, as follows:

$$Y_n(t) = \frac{S_{n,k}}{s_n} + X_{n,k+1}\frac{ts_n^2 - s_{n,k}^2}{\sigma_{n,k+1}^2 s_n}, \quad \text{if} \quad t \in \left[\frac{s_{n,k}^2}{s_n^2}, \frac{s_{n,k+1}^2}{s_n^2}\right], \tag{2.1}$$

for $k = 0, \ldots, m_n - 1$. Then $Y_n(t) = \frac{S_{n,k}}{s_n}$ if $t = \frac{s_{n,k}^2}{s_n^2}$, for $k = 0, \ldots, m_n$, and $Y_n(t)$ is a broken line joining these points. Therefore $Y_n \in C[0,1]$.

Assume that the Lindeberg condition is satisfied, namely for any $\varepsilon > 0$

$$\lim_{n\to\infty} s_n^{-2} \sum_{i=1}^{m_n} \mathbb{E}X_{n,i}^2 \mathrm{I}_{\{|X_{n,i}| \geq \varepsilon s_n\}} = 0\,. \tag{2.2}$$

Let $W(t)$, $t \in [0,1]$, be a standard Wiener process. Then the generalized Donsker's theorem is valid: $Y_n \Rightarrow W$, as $n \to \infty$, on $C[0,1]$, see Billingsley (1968), Problem in Section 10.

To apply Theorem 1.1, let $\zeta_n = Y_n$ and

$$\zeta_{k,n}(t) = \left[Y_n(t) - \frac{S_{n,m_k}}{s_n}\right] \mathrm{I}_{\left(\frac{s_{n,m_k}^2}{s_n^2}, 1\right]}(t)\,, \qquad t \in [0,1]\,,$$

$k < n$. As $\zeta_{k,n}$ depends only on $X_{n,m_k+1}, \ldots, X_{n,m_n}$, therefore ζ_k and $\zeta_{k,n}$ are independent, $k < n$. Moreover,

$$\begin{aligned}
\mathbb{E}\varrho(\zeta_{k,n}, \zeta_n) &= \mathbb{E} \sup_{u\in[0,1]} \left|\zeta_{k,n}(u) - \zeta_n(u)\right| \\
&= \mathbb{E}\max\left\{|Y_n(t)| \,:\, t \in \left[0, \frac{s_{n,m_k}^2}{s_n^2}\right]\right\} \\
&= \frac{1}{s_n}\mathbb{E}\max\{|S_{n,j}| \,:\, j = 0, \ldots, m_k\} \\
&\leq \frac{1}{s_n}\sqrt{\mathbb{E}\left(\max\{|S_{n,j}| \,:\, j = 0, \ldots, m_k\}\right)^2} \\
&\leq c\frac{1}{s_n}\sqrt{\mathbb{E}S_{n,m_k}^2} = c\frac{s_{n,m_k}}{s_n}\,.
\end{aligned}$$

Here we applied Doob's inequality.

Assume that $\frac{s_{n,m_k}}{s_n} \leq c\frac{c_k}{c_n}$. (This is satisfied when we consider not an array but only a sequence $X_1, X_2, \ldots$.)

Now we can choose

$$d_k = \log\left[\frac{c_{k+1}}{c_k}\right] = \int_{c_k}^{c_{k+1}} \frac{1}{x}dx,$$

and

$$D_n = \sum_{i=1}^{n} d_i = \int_{c_1}^{c_{n+1}} \frac{1}{x}dx = \log(c_{n+1}) - \log(c_1)\,.$$

Proposition 2.2. *Assume the notation and conditions as above. Suppose that c_n is an increasing sequence of positive numbers with $\lim_{n\to\infty} c_n = \infty$ and $c_{n+1}/c_n = \mathrm{O}(1)$. If the Lindeberg condition (2.2) holds then*

$$\frac{1}{D_n}\sum_{k=1}^{n} d_k \delta_{Y_{n,\omega}} \Rightarrow \mu_W \quad \textit{as} \quad n\to\infty, \quad \textit{for almost every} \quad \omega\in\Omega$$

on $C[0,1]$.

We mention that Lesigne (2000), Theorem 2 is an a.s. version of the (non-functional) Lindeberg CLT for arrays. Is is easy to see that our result implies that one.

A version of Proposition 2.2 for sequences $X_1, X_2, \ldots$ was proved in Rodzik and Rychlik (1994) by using another method. Major (2000), Theorem 1 gives the same result as our Proposition 2.2 but for sequences (i.e. not for arrays) and with convergence in $D[0,1]$. There a coupling method is used in the proof.

Example 2.3. *The maximum process of the partial sum process.* Let $X_1, X_2, \ldots$ be independent random variables with partial sum $S_n = \sum_{k=1}^{n} X_k$, $n = 1, 2, \ldots$, $S_0 = 0$. Let S_n^* be the maximum of the partial sums: $S_n^* = \max_{0\leq k\leq n} S_k$, $n = 0, 1, 2, \ldots$. Define the $D[0,1]$-valued maximum process ζ_n by

$$\zeta_n(t) = \frac{1}{b_n} S^*_{[nt]}\,, \quad t\in[0,1]\,, \tag{2.3}$$

where $\{b_n\}$ is a sequence of positive numbers. We want to prove an a.s. limit theorem for ζ_n.

Let $S_{k,n}$ be the increment of the partial sums: $S_{k,n} = X_{k+1} + \cdots + X_n$, for $k < n$. Let $S^*_{k,n}$ be the maximum of these increments: $S^*_{k,n} =$

$\max_{1\le i\le n-k}\{X_{k+1}+\cdots+X_{k+i}\}$, for $k<n$. Denote by $(\cdot)^+$ the positive part of a function. For $k<n$ let

$$\zeta_{k,n}(t)=\begin{cases}0\,, & \text{if}\quad 0\le t<\frac{k+1}{n}\,,\\ \frac{1}{b_n}\left\{S^*_{k,[nt]}\right\}^+\,, & \text{if}\quad \frac{k+1}{n}\le t\le 1\,,\end{cases}\tag{2.4}$$

be a $D[0,1]$-valued process. Then $\zeta_{k,n}$ is independent of ζ_k, $k<n$.

First we remark that for $\frac{k+1}{n}\le t\le 1$

$$S^*_{[nt]}=\max\left\{S^*_k,\,S_k+S^*_{k,[nt]}\right\}=\max\left\{S^*_k,\,S_k+\left\{S^*_{k,[nt]}\right\}^+\right\}.$$

Therefore for $k<n$ we have

$$\begin{aligned}&\zeta_n(t)-\zeta_{k,n}(t)\\ &\quad=\begin{cases}\zeta_n(t)\,, & \text{if}\ \ 0\le t<\frac{k+1}{n}\,,\\ \frac{1}{b_n}\left[\max\left\{S^*_k,\,S_k+\left\{S^*_{k,[nt]}\right\}^+\right\}-\left\{S^*_{k,[nt]}\right\}^+\right]\,, & \text{if}\ \ \frac{k+1}{n}\le t\le 1\,.\end{cases}\end{aligned}$$

Denote the second part of this expression by A. Then

$$A=\frac{1}{b_n}S^*_k\,,\quad\text{if}\quad\left\{S^*_{k,[nt]}\right\}^+=0\,,$$

$$A=\frac{1}{b_n}\max\left\{S^*_k-\left\{S^*_{k,[nt]}\right\}^+,\,S_k\right\}\,,\quad\text{if}\quad\left\{S^*_{k,[nt]}\right\}^+>0\,.$$

Therefore

$$\frac{1}{b_n}S_k\le A\le\frac{1}{b_n}S^*_k\,.$$

Finally, for $k<n$ and $0\le t\le 1$ we have

$$|\zeta_n(t)-\zeta_{k,n}(t)|\le\frac{1}{b_n}\max\{0,\,|S^*_k|,\,|S_k|\}\,.\tag{2.5}$$

The following proposition is a functional version of Theorem D in Berkes and Csáki (2001).

Proposition 2.3. *Assume that $\zeta_n\Rightarrow\zeta$ on $D[0,1]$, as $n\to\infty$. Assume that there exist $K>0$, $\delta>0$, and an increasing sequence of positive numbers b_n with $\lim_{n\to\infty}b_n=\infty$, $b_{n+1}/b_n=\mathrm{O}(1)$ such that*

$$\mathbb{E}\left\{\log_+\log_+\left|\frac{S_n}{b_n}\right|\right\}^{1+\delta}\le K\,,\quad\mathbb{E}\left\{\log_+\log_+\left|\frac{S^*_n}{b_n}\right|\right\}^{1+\delta}\le K\,,\tag{2.6}$$

for $n = 1, 2, \ldots$. *Let* $d_k = \log(b_{k+1}/b_k)$ *and assume that* $\sum_{k=1}^{\infty} d_k = \infty$. *Let* $D_n = \sum_{k=1}^{n} d_k$. *Then in* $D[0,1]$

$$\frac{1}{D_n}\sum_{k=1}^{n} d_k \delta_{\zeta_k(\omega)} \Rightarrow \mu_\zeta \quad \text{as} \quad n\to\infty, \quad \text{for almost every} \quad \omega\in\Omega\,. \tag{2.7}$$

Proof. We use the method of Berkes and Csáki (2001), Theorem A and D. Let $g(x) = 1 + (\log_+ \log_+ x)^{1+\delta}$, $x \geq 0$. This function is continuous and nonzero, therefore $x/g(x)$ is also continuous. Moreover, $x/g(x)$ is increasing for $x \geq x_0 > 0$ and is unbounded. Therefore

$$\frac{x}{g(x)} \leq \frac{y}{g(y)}\,, \quad \text{if} \quad 0\leq x\leq y \quad \text{and} \quad y\geq a_0\,,$$

for some a_0. By (2.5),

$$\begin{aligned}\mathbb{E}\{\varrho(\zeta_{k,n},\zeta_n)\wedge 1\} &\leq \mathbb{E}\big\{\max_{0\leq t\leq 1}|\zeta_{k,n}(t)-\zeta_n(t)|\wedge 1\big\}\\ &\leq \mathbb{E}\left\{\left[\frac{|S_k|}{b_n}\wedge 1\right]+\left[\frac{|S_k^*|}{b_n}\wedge 1\right]\right\}.\end{aligned}$$

Consider

$$\lambda = \frac{b_n}{b_k} \geq \frac{|S_k|}{b_k}\wedge\frac{b_n}{b_k}.$$

Then

$$\begin{aligned}\mathbb{E}\left[\frac{|S_k|}{b_n}\wedge 1\right] &= \frac{b_k}{b_n}\mathbb{E}\left[\frac{|S_k|}{b_k}\wedge\frac{b_n}{b_k}\right] \leq \frac{1}{g(\lambda)}\mathbb{E}g\left[\frac{|S_k|}{b_k}\wedge\lambda\right] \leq \frac{1}{g(\lambda)}\mathbb{E}g\left[\frac{|S_k|}{b_k}\right]\\ &= \frac{1}{1+\left\{\log_+\log_+\frac{b_n}{b_k}\right\}^{1+\delta}}\mathbb{E}\left[1+\left\{\log_+\log_+\left|\frac{S_k}{b_k}\right|\right\}^{1+\delta}\right]\\ &\leq CK\left\{\log_+\log_+\frac{b_n}{b_k}\right\}^{-(1+\delta)}.\end{aligned}$$

The same is true for S_k^* and Theorem 1.1 implies the result. □

Now we specialize our result for i.i.d. variables.

Proposition 2.4. *Let* $X_1, X_2, \ldots$ *be i.i.d. random variables with* $\mathbb{E}X_i = 0$, $\mathbb{E}X_i^2 = \sigma^2 \in (0,\infty)$. *Let the* $D[0,1]$*-valued maximum process* ζ_n *be*

$$\zeta_n(t) = \frac{1}{\sigma\sqrt{n}} S^*_{[nt]}\,, \quad t\in[0,1]\,. \tag{2.8}$$

Then

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\delta_{\zeta_{k,\omega}} \Rightarrow \mu_{W^*}\,, \quad \textit{as} \quad n\to\infty, \quad \textit{for almost every} \quad \omega\in\Omega$$

on $D[0,1]$, *where* W^* *is the maximum process of the Wiener process* W.

Proof. The process

$$X_n(t) = \frac{1}{\sigma\sqrt{n}}S_{[nt]}\,, \quad t\in[0,1]\,,$$

converges to the Wiener process W in $D[0,1]$, see Billingsley (1968), Theorem 16.1. Let $(Mg)(x) = \sup_{0\le t\le x} g(t)$ for $g\in D[0,1]$. Then $M : D[0,1]\to D[0,1]$ is continuous. Therefore $MX_n \Rightarrow MW$ in $D[0,1]$. That is $\zeta_n \Rightarrow W^*$ in $D[0,1]$, where $W^*(x) = \sup_{0\le t\le x} W(t)$ is the maximum process of the Wiener process.

To prove (2.6), we use Doob's inequality:

$$\mathbb{E}\left[\frac{1}{\sigma\sqrt{n}}|S_n^*|\right]^2 \le \mathbb{E}\left[\frac{1}{\sigma\sqrt{n}}|S_n|^*\right]^2 \le c\frac{1}{\sigma^2 n}\mathbb{E}S_n^2 = c\,. \quad \square$$

We remark that for $\mathbb{E}X_i > 0$ the limit of the normalized maximum process converges to the Wiener process (Takahata (1980), Theorem 2). Our a.s. CLT does not concern this case.

Example 2.4. *The empirical process.* Let $X_1, X_2, \ldots$ be i.i.d. random variables with common distribution function F and let

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n}\mathrm{I}_{\{X_i\le x\}}\,,$$

be the empirical distribution function. Suppose

$$\alpha_n(x) = \sqrt{n}\big[F_n(x) - F(x)\big] = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\mathrm{I}_{\{X_i\le x\}} - F(x)\right]\,,$$

is an empirical process.

Let B be a Brownian bridge, and let B_F be defined as $B_F(x) = B(F(x))$, $x\in\mathbb{R}$. Then B_F is a Gaussian process with mean 0 and covariance function $F(r) - F(r)F(s)$ for $r\le s$. It is known that $\alpha_n \Rightarrow B_F$ in $D[-\infty,+\infty]$, the space of cadlag functions endowed with the uniform metric. See Pollard (1984), Section V.2., see also Csörgő and Révész (1981), Theorem 4.3.1*.

However, we consider $D[-\infty,+\infty]$ equipped with the sup norm. Therefore it is not separable. Instead of the Borel sets we consider the projection σ-algebra $\mathcal{P}$ on $D[-\infty,+\infty]$. Then $\mathcal{P}$ contains each closed ball. The limit process is B_F which is concentrated on a complete, separable subspace. This subspace is the image of $C[0,1]$ by the continuous map H:

$$(Hx)(r) = x(F(r)), \qquad r \in [-\infty,+\infty]\,,$$

where $x \in D[0,1]$, see Pollard (1984), Section V.2. Therefore, by Remark 1.7, we can apply Theorem 1.1.

Now let $\zeta_n(x) = \alpha_n(x)$ and for $k < n$ let

$$\zeta_{kn}(x) = \frac{1}{\sqrt{n}} \sum_{i=k+1}^{n} \left[\mathrm{I}_{\{X_i \le x\}} - F(x)\right] \,.$$

Therefore ζ_k and ζ_{kn} are independent if $k < n$.

$$\begin{aligned} \mathbb{E}\varrho(\zeta_{kn},\zeta_n) &= \mathbb{E}\sup_x \left|\zeta_{kn}(x) - \zeta_n(x)\right| \\ &= \frac{\sqrt{k}}{\sqrt{n}} \mathbb{E} \frac{1}{\sqrt{k}} \sup_x \left| \sum_{i=1}^{k} \left[\mathrm{I}_{\{X_i \le x\}} - F(x)\right] \right| \le \frac{\sqrt{k}}{\sqrt{n}} c\,, \end{aligned}$$

because $\mathbb{E}\frac{1}{\sqrt{k}}\sup_x \left|\sum_{i=1}^{k} \left[\mathrm{I}_{\{X_i \le x\}} - F(x)\right]\right|$ is bounded (see Dvoretzky et al. (1956), Lemma 2, see also Theorem 4.1.3 in Csörgő and Révész (1981)).

Proposition 2.5. *Assume the notation and conditions as above. Then*

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \delta_{\alpha_{k,\omega}} \Rightarrow \mu_{B_F} \quad \text{as} \quad n \to \infty, \quad \text{for almost every} \quad \omega \in \Omega$$

on $D[-\infty,+\infty]$.

3. Applications for dependent variables.

Here we show how to apply the previous method if not independent but weakly dependent random variables are in the background. The α-mixing coefficient of the random variables X and Y is defined as

$$\alpha(X,Y) = \alpha\left(\sigma\{X\},\sigma\{Y\}\right) = \sup_{A\in\sigma\{X\},\, B\in\sigma\{Y\}} \left|\mathbb{P}(AB) - \mathbb{P}(A)\mathbb{P}(B)\right|,$$

where $\sigma\{X\}$ is the σ-algebra generated by X. The covariance inequality is

$$|\operatorname{cov}(X,Y)| \le 4\alpha(X,Y)\|X\|_\infty \|Y\|_\infty\,,$$

if X and Y are bounded (see, Lin and Lu (1996)).

Let α_{kl} be the α-mixing coefficient of ζ_k and ζ_{kl}, $k < l$.

Proposition 3.1. *Theorem 1.1, Remark 1.2 and Remark 1.6 remain valid if the condition ζ_k and ζ_{kl} to be independent is replaced by the following: there exist $c > 0$, $\varepsilon > 0$, such that*

$$\sum_{k=1}^{n} \sum_{l=k}^{n} d_k d_l \alpha_{kl} \leq c D_n^2 \left(\log D_n\right)^{-(1+\varepsilon)} . \tag{3.1}$$

Proof. As in (1.11) we obtain

$$\begin{aligned} \left|\mathbb{E}\{\xi_k \xi_l\}\right| &\leq \left|\mathbb{E}\big(g(\zeta_k) - \mathbb{E}g(\zeta_k)\big)\big(g(\zeta_l) - g(\zeta_{kl})\big)\right| \\ &\quad + \left|\mathbb{E}\big(g(\zeta_k) - \mathbb{E}g(\zeta_k)\big)\big(g(\zeta_{kl}) - \mathbb{E}g(\zeta_l)\big)\right| \\ &\leq 4cK^2 \left[\left\{ \log_+ \log_+ \left(\frac{c_l}{c_k} \right) \right\}^{-(1+\varepsilon)} + \alpha_{kl} \right] . \end{aligned} \tag{3.2}$$

By (3.1) and Lemma 1.3 (a) we obtain $\mathbb{E}T_n^2 \leq c\big(\log D_n\big)^{-(1+\varepsilon)}$. By Lemma 1.3 (b), $T_n \to 0$ a.s. The remaining part of the proof is the same as that of Theorem 1.1. $\square$

Since the covariance inequality is satisfied for other types of mixing, Proposition 3.1 is valid for ϱ-, φ-, β-, and ψ-mixing, too.

Now, we present an a.s. CLT for α-mixing sequences. The α-mixing coefficient of the sequence $X_1, X_2, \ldots$ is

$$\alpha(k) = \sup_n \alpha\big(\sigma\{X_1, \ldots, X_n\}, \sigma\{X_{n+k}, X_{n+k+1}, \ldots\}\big) .$$

Proposition 3.2.. *Let $X_1, X_2, \ldots$ be a strictly stationary, α-mixing sequence of real random variables with mixing coefficient $\alpha(k) \leq c/\log k$, $k = 2, 3, \ldots$. Let $\mathbb{E}X_i^2 < \infty$, $\mathbb{E}X_i = 0$. Let $S_n = X_1 + \cdots + X_n$, $\sigma_n^2 = \mathbb{E}S_n^2$, $\zeta_n = S_n/\sigma_n$, $n = 1, 2, \ldots$. Assume that $\sigma_n \to \infty$ and ζ_n satisfies the CLT: $\zeta_n \Rightarrow \mathcal{N}(0,1)$. Then*

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mu_{\zeta_k(\omega)} \Rightarrow \mathcal{N}(0,1) \tag{3.3}$$

as $n \to \infty$, for almost every ω.

Proof. First we remark that the conditions imply $\sigma_n = \sqrt{n}L(n)$, where $L(n)$ is a slowly varying function (Ibragimov and Linnik (1971), Theorem 18.4.1).

Let $\zeta_{kl} = (X_{2k+1} + \cdots + X_l)/\sigma_l$, for $k < l$. Then

$$\mathbb{E}|\zeta_l - \zeta_{kl}| \leq \sqrt{\mathbb{E}(\zeta_l - \zeta_{kl})^2} = \frac{\sqrt{\mathbb{E}S_{2k}^2}}{\sigma_l} = \frac{\sqrt{2k}L(2k)}{\sqrt{l}L(l)} \leq c\frac{\sqrt{k}L(k)}{\sqrt{l}L(l)},$$

by the definition of the slow variation. By the Karamata theorem (see Seneta (1976)) for x large enough

$$L(x) = a(x)\exp\left(\int_B^x \frac{b(t)}{t}dt\right) = a(x)L_0(x)\,,$$

where $a(x)$ is a real function for which $0 < a_1 < a(x) < a_2 < \infty$ and $\lim_{x\to\infty} a(x) = a$ is finite and positive, while $b(x)$ is a continuous function with $\lim_{x\to\infty} b(x) = 0$ and $B > 0$. This implies that

$$\mathbb{E}|\zeta_l - \zeta_{kl}| \leq c\frac{\sqrt{k}L_0(k)}{\sqrt{l}L_0(l)}.$$

Let $c_k = \sqrt{k}L_0(k)$, $k = 1, 2, \ldots$. Then (1.5) is satisfied with $\beta = 1$. Since $\sqrt{x}L_0(x)$ is regularly varying with positive exponent, $\lim_{x\to\infty} \sqrt{x}L_0(x) = \infty$ (Seneta (1976), Section 1.5). Therefore $c_k \to \infty$.

Now, we find d_k.

$$\begin{aligned}\log\left[\frac{c_{k+1}}{c_k}\right] &= \frac{1}{2}\log\left[\frac{k+1}{k}\right] + \log\left[\frac{\exp\left(\int_B^{k+1}\frac{b(t)}{t}dt\right)}{\exp\left(\int_B^{k}\frac{b(t)}{t}dt\right)}\right] \\ &= \frac{1}{2}\left[\log(k+1) - \log k\right] + \int_k^{k+1}\frac{b(t)}{t}dt\,.\end{aligned}$$

(This shows also that $c_{k+1}/c_k = \mathrm{O}(1)$.) Since

$$\begin{aligned}\left|\int_k^{k+1}\frac{b(t)}{t}dt\right| &\leq \left[\max_{k\leq t\leq k+1}|b(t)|\right]\left[\log(k+1) - \log k\right] \\ &\leq \frac{1}{4}\left[\log(k+1) - \log k\right]\end{aligned}$$

if k is large enough,

$$\log\left[\frac{c_{k+1}}{c_k}\right] \geq \frac{1}{4}\left[\log(k+1) - \log k\right] \geq \frac{1}{8}\frac{1}{k} = d_k^{(0)}\,,$$

if k is large enough. This shows also that c_k is increasing if k is large enough. We have $D_n^{(0)} = \sum_{k=1}^n d_k^{(0)} \sim \frac{\log n}{8} \to \infty$.

To prove (3.1), consider

$$\begin{aligned}\sum_{k=2}^{n}\sum_{l=k}^{n} d_k^{(0)} d_l^{(0)} \alpha_{kl} &\leq c \sum_{k=2}^{n}\sum_{l=k}^{n} \frac{1}{k}\frac{1}{l}\frac{1}{\log k} \\ &\leq c(\log n)(\log\log n) \leq c(\log n)^2(\log\log n)^{-(1+\varepsilon)} \\ &\leq c(D_n^{(0)})^2 \left(\log D_n^{(0)}\right)^{-(1+\varepsilon)} .\end{aligned}$$

Thus the a.s. CLT is valid with $d_k^{(0)}$ and $D_n^{(0)}$ and consequently (3.3) is satisfied. $\square$

Ibragimov and Lifshits (1999) Theorem 2.5 contains a similar result with a more general condition $\sum_{n=2}^{\infty} \frac{\alpha(n)}{n \log n} < \infty$. However, we think that our condition $\alpha(n) \leq c/\log n$ is not a major restriction, because a typical sufficient condition for the CLT itself is $\sum_{n=1}^{\infty} \alpha(n)^{\frac{1+\delta}{2+\delta}} < \infty$, $\delta \geq 0$, see Hall and Heyde (1980), Corollary 5.3.

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