ANNALES
UNIVERSITATIS MARIAE CURIE – SKŁODOWSKA
LUBLIN – POLONIA

VOL. L V, 12 SECTIO A 2001

ALEXANDER R. PRUSS

Discrete harmonic measure, Green's functions and symmetrization: a unified probabilistic approach*

Abstract. We use probabilistic methods based on the work of Haliste (1965) to obtain various new versions of theorems of Baernstein (1974) on the effects of circular symmetrizations on harmonic measure and Green's functions. Our results are quite general and include a number of cases of symmetrization in discrete settings. For instance, in the setting of the discrete cylinder $\mathbb{Z}\times\mathbb{Z}_m$ we obtain complete generalized analogues of Baernstein's results on harmonic measure and Green's functions, and even get a discrete version of Beurling's (1933) shove theorem.

1. Introduction. Suppose $\mathbb{Z}_m$ be the integers modulo m, let $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ be the unit sphere of dimension n, use $\mathbb{H}^n$ to denote n-dimensional hyperbolic space, and put $\mathbb{T} = \mathbb{S}^1$.

1991 *Mathematics Subject Classification.* Primary 31C20, 60J05, 60J15, 60J65. Secondary 31A15, 31C12.

Key words and phrases. Random walk, discrete harmonic measure, discrete Green's function, Steiner symmetrization, circular symmetrization, non-increasing rearrangement, Baernstein $*$-functions, regular trees, line graphs of regular trees, hyperbolic plane, Brownian motion.

*The research was partially supported by Professor J. J. F. Fournier's NSERC Grant #4822. Most of the research was done while the author was at the University of British Columbia. The author is most grateful to the referee for a careful reading and a number of helpful suggestions, including the reference to [12].

Suppose R_n is a simple random walk on $\mathbb{Z}^2$, and let $P^z(\cdot)$ indicate probabilities where the walk is conditioned to start at z. Given a set $D \subset \mathbb{Z}^2$, let

$$\tau_D = \inf\{n \geq 0 : R_n \notin D\}$$

be the first time that the random walk exits D. Fix $N \in \mathbb{Z}$. Assume that the line $L_N = \{N\} \times \mathbb{Z}$ lies outside D. We are interested in the probability $w_N(D)$ that the random walk conditioned to start at $(0,0)$ hits L_N before hitting any other part of the complement of D. Evidently,

$$w_N(D) = P^{(0,0)}(R_{\tau_D} \in L_N).$$

The quantity $w_N(D)$ may also be described as the *discrete harmonic measure* of L_N at $(0,0)$ in D. Define

$$\Theta(x; D) = |\{y : (x,y) \in D\}|, \tag{1.1}$$

where $x \in \mathbb{Z}$ and $|\cdot|$ indicates cardinalities of sets. Then, let $D^\boxminus$ be the *discrete Steiner symmetrization* of D, defined by requiring that $\Theta(x; D) = \Theta(x; D^\boxminus)$ for all $x \in D$ and that for every $x \in D$ the set $\{y : (x,y) \in D^\boxminus\}$ be an interval of the form $\{-n, -n+1, \ldots, n\}$ or $\{-n, -n+1, \ldots, n, n+1\}$. These conditions uniquely define $D^\boxminus$. (See Figure 1.1.) We will then prove that

$$w_N(D) \leq w_N(D^\boxminus). \tag{1.2}$$

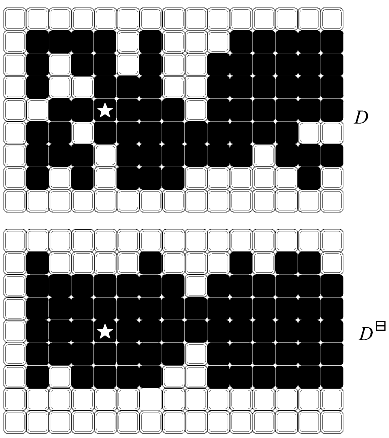

FIG. 1.1. The discrete Steiner symmetrization on $\mathbb{Z}^2$. Black squares indicate the points of the sets D and $D^\boxminus$ and white squares indicate the complements of the sets. The stars indicate the origin

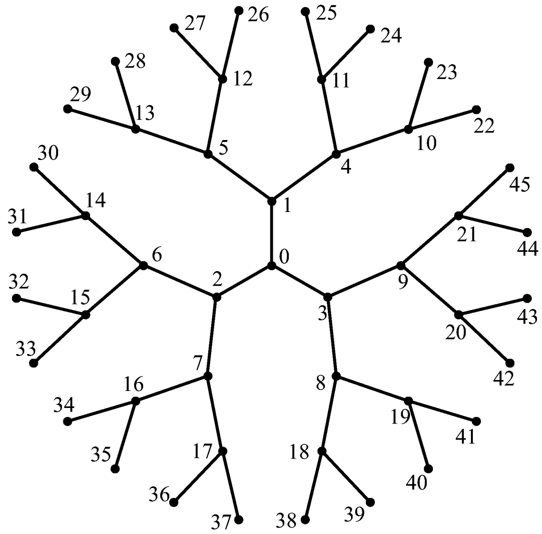

FIG. 1.2. The ordering $\lessdot$ on the tree T_3 is induced by the standard ordering $<$ on the labels in this diagram. The root of the tree is marked "0". (Figure reprinted with permission from the *Duke Mathematical Journal* [15])

For $w \in D \subseteq \mathbb{Z}^2$, let

$$g(z, w; D) = E^z[|\{n \in \mathbb{Z}_0^+ : n < \tau_D \text{ and } R_n = w\}|]$$

be the Green function in D. This is just the expected number of visits to w prior to leaving D when starting at z. It will be proved that:

$$g((x,y),(x',y');D) \le g((x,0),(x',0);D^{\boxminus}). \tag{1.3}$$

Inequality (1.2) follows from the methods of Quine [17] and is a discrete analogue of the continuous result in Theorem 8.1 of Haliste [9]. Inequalities (1.2) and (1.3) are special cases of Theorems (3.1) and (3.2) (respective) which will be proved later in this paper, and are representative of the kinds of results that we shall give in this paper.

In general, we will be able to handle not only simple random walks on $\mathbb{Z}^2$, but also simple random walks (as well as a more general class of walks than just simple random walks) on any graph $X \times Y$, where Y is $\mathbb{Z}$, the circle $\mathbb{Z}_m$, the p-regular tree T_p or the line graph $L(T_p)$, and X has constant degree, and an analogue of (1.2) will be valid in all these cases (see Corollary 3.3, below). The only difference between these cases is in how the symmetrization $D^{\boxminus}$ will be defined. The quantity $\Theta(x; D)$ will always be defined by (1.1). On $X \times \mathbb{Z}_m$, the symmetrization $D^{\boxminus}$ is defined almost in the same way as on

$\mathbb{Z}\times\mathbb{Z}$. On $X\times T_p$ it is defined by requiring that we should have $\Theta(x; D^{\boxminus}) = \Theta(x; D)$ for all x and that for each fixed x the set $\{y : (x, y) \in D^{\boxminus}\}$ be either all of T_p or of the form $\{y \in T_p : y \lessdot y_0\}$ for some $y_0 \in T_p$, where $\lessdot$ is the ordering on the tree displayed in Figure 1.2. Our proofs depend crucially on certain symmetrization-convolution inequalities of Pruss [15].

We not only obtain results for discrete harmonic measures and Green's functions, but also domination inequalities for certain generalized harmonic measures and Green's functions of unsymmetrized and symmetrized subsets of $X\times Y$ where Y is as in the previous paragraph (see Theorems 3.1 and 3.2, below). In fact, we handle some even more general cases, including problems involving Brownian motions on $\mathbb{R}^n$, $\mathbb{S}^n$ or $\mathbb{H}^n$.

An approximate circular symmetrization inequality on $\mathbb{Z}^2$ has been used by Kesten [12] to give an upper bound on a certain hitting probability for a random walk on $\mathbb{Z}^2$. It is quite possible that our exact inequalities could be used for similar purposes.

A companion paper [14] obtains similar results for some yet more general difference equations, but unfortunately requires additional approximation arguments to handle some of the cases directly handled by the methods of the present paper. Moreover, the methods of the present paper have the advantage of being probabilistic and more transparent; the methods of the companion paper are analytic and based on a modification of a method of Baernstein [2].

Terms such as "positive" and "decreasing" will be used in a non-strict sense (i.e., as meaning "non-negative" and "non-decreasing", respectively) unless otherwise qualified. We shall always use the same symbol to denote a graph and to denote its collection of vertices. Thus, if G is a graph, then "$x \in G$" means "x is a vertex of G" and $H \subseteq G$ denotes a collection H of vertices of G.

2. Notation and some preliminary results of a general nature.

2.1 Symmetrizations. Let $(X, \mathcal{F}, \mu)$ be a measure space. By a *symmetrization* on X we shall mean a map $\#: \mathcal{F} \to \mathcal{F}$ with the following properties:

(i) $X^\# = X$ and $\varnothing^\# = \varnothing$

(ii) if $A \subseteq B$ are measurable, then $A^\# \subseteq B^\#$

(iii) if $A_1 \subseteq A_2 \subseteq \cdots$ are measurable, then

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^\# = \bigcup_{i=1}^{\infty} A_i^\#.$$

(iv) $(A^\#)^\# = A^\#$ for all measurable sets A.

Actually, property (ii) is a special case of (iii), but we list it separately for ease of use. We call a set A *#-symmetric* (or just *symmetric* if no confusion is possible) if $A^{\#} = A$. If A coincides μ-almost everywhere with a #-symmetric set, then we call A *almost #-symmetric*. We say that the symmetrization # is *measure-preserving* if we have $\mu(A^{\#}) = \mu(A)$ for all measurable sets A.

We say that $\#: \mathcal{F} \to \mathcal{F}$ is a *Schwarz-type symmetrization* on X if # is a measure-preserving symmetrization satisfying the auxiliary condition that $A^{\#} \subseteq B^{\#}$ whenever $\mu(A) \leq \mu(B)$. A classical example of Schwarz-type symmetrization on $\mathbb{R}^n$ is *Schwarz symmetrization*, given by letting $A^{\#}$ be an open ball centred about the origin with the same volume as A.

Remark 2.1. The collection of the #-symmetric sets for a given Schwarz-type symmetrization is totally ordered under set inclusion and the mapping $A \mapsto \mu(A)$ gives an order isomorphism between this collection and a subset of $[0, \infty]$.

Given an extended real function f on some set X, define the level set

$$f_\lambda = \{x \in X : f(x) > \lambda\}$$

for $\lambda \in \mathbb{R}$. Given a symmetrization # on X and a measurable real function f, let

$$f^{\#}(x) = \sup\{\lambda : x \in (f_\lambda)^{\#}\},$$

for every $x \in X$. It can be easily proved that $(f^{\#})_\lambda = (f_\lambda)^{\#}$ for every $\lambda \in \mathbb{R}$ (this last fact actually uses property (iii) of symmetrization), and it follows that $f^{\#}$ is measurable. The function $f^{\#}$ is known as the *#-symmetrization* of f. Note that $(f^{\#})^{\#} = f^{\#}$ by property (v) of symmetrization. We say that f is *#-symmetric* if $f^{\#} = f$. An equivalent condition is to require that f_λ should be #-symmetric for all $\lambda \in \mathbb{R}$. We say that f is *almost #-symmetric* if it is equal μ-almost everywhere to a #-symmetric function. Observe that if $f = 1_A$ is the indicator function of a set A, then $f^{\#} = 1_{A^{\#}}$.

We say that functions f and g are *equimeasurable* if $\mu(f_\lambda) = \mu(g_\lambda)$ for all $\lambda \in \mathbb{R}$. If f and g are equimeasurable and # is a Schwarz-type symmetrization, then $f^{\#} = g^{\#}$. It follows that if f is almost #-symmetric, then f coincides μ-almost everywhere with a #-symmetric function g. Then, f and g are equimeasurable and so $f^{\#} = g^{\#} = g$. Hence f coincides μ-almost everywhere with $f^{\#}$. We shall often implicitly use this remark.

If # is a measure-preserving symmetrization, then for every measurable f on X we have f and $f^{\#}$ equimeasurable.

Example 2.1. Suppose that X is $\mathbb{Z}$ or $\mathbb{Z}_m$. Given a set $A \subseteq X$, if $|A| = \infty$, where $|\cdot|$ denotes cardinalities, then let $A^{\#} = \mathbb{Z}$. If $|A| = 0$, then let $A^{\#} = \varnothing$. If $|A| < \infty$, then let $A^{\#}$ be the unique set of the form $\{-n, -n+1, \ldots, n-$

$1, n\}$ or of the form $\{-n, -n+1, \ldots, n-1, n, n+1\}$ for some $n \geq 0$ such that $|A^{\#}| = |A|$. Then, $\#$ is a Schwarz-type symmetrization on the measure space X equipped with counting measure.

Example 2.2. Suppose that X is the p-regular tree T_p for some $p \geq 1$. This is an infinite tree each of whose vertices has degree p. We distinguish a vertex o known as the *root* of T_p and fix a "spiral-like" ordering $\lessdot$ on the vertices (see Figure 1.2 for an illustration that should make it clear how to inductively choose such an ordering, and see Pruss [15] for the explicit definition). Then, if $A \subseteq X$ has infinite cardinality, let $A^{\#} = X$. But if A has finite cardinality n, then let $A^{\#}$ be the collection of the first n elements of X, where by "first" we mean "first with respect to $\lessdot$". It is easy to see that $\#$ is a Schwarz-type symmetrization on X equipped with counting measure.

Example 2.3. Suppose that X is the line graph $L(T_p)$ of the p-regular tree T_p. This is a graph whose vertices are the edges of T_p, where vertices e_1 and e_2 of $L(T_p)$ are adjacent if and only if considered as edgeds of T_p they have precisely one vertex of T_p in common. The ordering $\lessdot$ on T_p from the preceding example induces a unique ordering $\lessdot$ on the vertices $L(T_p)$ in a natural way. This ordering on $L(T_p)$ is defined as follows. Suppose that $e_1 = \{v_1, w_1\}$ and $e_2 = \{v_2, w_2\}$ are distinct vertices of $L(T_p)$, i.e., distinct edges of T_p. Suppose also that the labels are such that $v_1 \lessdot v_2$ and $w_1 \lessdot w_2$. We then set $e_1 \lessdot e_2$ if and only if either we have $v_1 \lessdot w_1$ or we have both $v_1 = w_1$ and $v_2 \lessdot w_2$. (See Pruss [15] for details and a diagram.) As in the previous example, if $A \subseteq X$ has infinite cardinality, let $A^{\#} = X$, and otherwise let $A^{\#}$ be the collection of the $\lessdot$-first n elements of X. Once again, $\#$ will be a Schwarz-type symmetrization on X equipped with counting measure.

Example 2.4. Let X be either $\mathbb{R}^n$, or the sphere $\mathbb{S}^n$, or the hyperbolic space $\mathbb{H}^n$, equipped with the appropriate volume measure. Fix an origin o in X (if $X = \mathbb{R}^n$, then let $o = 0$; in the other two cases, any point is an acceptable origin). Given a set $A \subseteq X$, if A has infinite volume, let $A^{\#} = X$. Otherwise, let $A^{\#}$ be an open metric ball centred about o and having the same volume as A. If X is $\mathbb{R}^n$ or $\mathbb{H}^n$, then it is evident that $\#$ is a Schwarz-type symmetrization on X equipped with volume measure.

We now make a subtle technical remark about the case $X = \mathbb{S}^n$. In the case of $\mathbb{S}^n$, there is a small ambiguity if $A \subseteq \mathbb{S}^n$ has the same volume as $\mathbb{S}^n$. Let $-o$ be the point of $\mathbb{S}^n$ antipodal to o. It is customary to define $A^{\#} = \mathbb{S}^n \backslash \{-o\}$ if $A \neq \mathbb{S}^n$ and $A^{\#} = \mathbb{S}^n$ if $A = \mathbb{S}^n$. This means we do not have a Schwarz-type symmetrization in the strict sense of the definition, since $A^{\#}$ does not depend only on the measure of A. In fact, we do not quite have a symmetrization since if A_m is an increasing sequence of proper subsets of $\mathbb{S}^n$ whose union is $\mathbb{S}^n$, then $-o \notin A_m^\#$ while $-o \in (\mathbb{S}^n)^\#$, so that property (iii) fails (but this is the only way in which it can fail).

However, if we consider the symmetrization $\#$ as acting on $\mathbb{S}^n \backslash \{-o\}$ then we have a perfectly good Schwarz-type symmetrization. In the present paper, it makes no difference what we do here because the point $-o$ has null measure and we will be working with discrete time processes. If we were working with continuous time Brownian motion for $n = 1$ (which would be equivalent to the setting of Baernstein [1]), then it would matter whether we included $-o$ in the symmetrization in the case where $\mathbb{S}^1 \backslash A$ has null measure. In that case we would have to use the different definition in the previous paragraph according to which what $A^\#$ would be would depend on whether $\mathbb{S}^1 \backslash A$ is empty or not. However, because the null set $\{-o\}$ is not important to us (except in Remark 3.6), it will not matter whether we consider $\#$ a symmetrization on $\mathbb{S}^n \backslash \{-o\}$ or an "almost symmetrization" on $\mathbb{S}^n$. Except for one brief mention in Remark 3.6, we shall thus ignore this issue, and act as if $\#$ were a symmetrization on $\mathbb{S}^n$.

The symmetrizations $\#$ of the preceding examples will be called *the canonical Schwarz-type symmetrizations* on $\mathbb{Z}$, $\mathbb{Z}_m$, T_p, $L(T_p)$, $\mathbb{R}^n$, $\mathbb{S}^n$ and $\mathbb{H}^n$, respectively. Although there may be some freedom in choice of origin or, in the case of T_p (and consequently of $L(T_p)$), in the choice of ordering, nonetheless all the choices are equivalent up to isometry.

Now, given a set Y equipped with a symmetrization $\#$, and given an arbitrary set X, we may define a *product symmetrization* $\boxminus$ on $X \times Y$ given by

$$A^{\boxminus} = \bigcup_{x \in X} [\{x\} \times A_x^\#],$$

where $A_x = \{y : (x, y) \in A\}$. If $\#$ is a Schwarz-type symmetrization, then we can call $\boxminus$ a *Steiner-type symmetrization*. A classical example is when $X = Y = \mathbb{R}$ and $\#$ is Schwarz symmetrization; then, $\boxminus$ is *Steiner symmetrization about the real axis* on $\mathbb{R}^2$. Steiner symmetrization was introduced by Steiner [19] in his attack on the isoperimetric problem in $\mathbb{R}^2$. If $X = Y = \mathbb{Z}$, and $\#$ is the canonical Schwarz-type symmetrization on $\mathbb{Z}$, then $\boxminus$ is the *discrete Steiner symmetrization* on $\mathbb{Z}^2$, defined in the Introduction, and in fact if X is any constant degree graph while Y is $\mathbb{Z}$, $\mathbb{Z}_m$, T_p or $L(T_p)$, then $\boxminus$ agrees with the definition outlined in the Introduction.

If $X = \mathbb{Z}$ and $Y = \mathbb{Z}_m$, then $\mathbb{Z} \times \mathbb{Z}_m$ is a discrete tube. If $\#$ is the canonical Schwarz-type symmetrization on $\mathbb{Z}_m$, then $\boxminus$ can be considered a discrete version of circular symmetrization on $\mathbb{C}$ in light of the conformal equivalence of $\mathbb{C} \backslash \{0\}$ with the tube $\mathbb{R} \times \mathbb{T}$. Thus, our results about Steiner-type symmetrization on $\mathbb{Z} \times \mathbb{Z}_m$ will be discrete analogues of circular symmetrization results of Baernstein [1]. Steiner-type symmetrization on $\mathbb{Z} \times \mathbb{Z}_m$ was apparently first considered by Quine [17].

The following two elementary propositions are quite important. The first is essentially due to Hardy and Littlewood (cf. Theorem 368 of Hardy, Littlewood and Pólya [10] and Lemma 2.1 of Kawohl [11]).

Proposition 2.1. *Let $f_1, f_2, \ldots, f_n$ be positive measurable functions on a space X equipped with a measure μ. If $\#$ is a measure-preserving symmetrization on X, then*

$$\int_X f_1 f_2 \cdots f_n \, d\mu \le \int_X f_1^\# f_2^\# \cdots f_n^\# \, d\mu.$$

Proof. Writing $f_k(x) = \int_0^\infty 1_{(f_k)_\lambda}(x)\, d\lambda$, with an analogous expression for $f_k^\#$, together with Fubini's theorem allows us to reduce to case where $f_1, f_2, \ldots, f_n$ are indicator functions of sets $A_1, A_2, \ldots, A_n$, respectively ("layer-cake principle"). The inequality to be proved then becomes

$$\mu(A_1 \cap \cdots \cap A_n) \le \mu(A_1^\# \cap \cdots \cap A_n^\#).$$

Let $A = A_1 \cap \cdots \cap A_n$ and $A' = A_1^\# \cap \cdots \cap A_n^\#$. Then, $A^\# \subseteq A_k^\#$ for all k since $A \subseteq A_k$ and by property (ii) of symmetrizations. It follows that $A^\# \subseteq A'$. Hence, $\mu(A) = \mu(A^\#) \le \mu(A')$ as desired. $\square$

The next result is also useful. It is a generalization of Theorem 369 of Hardy, Littlewood and Pólya [10].

Proposition 2.2. *Let f be a positive integrable function on a set X equipped with a measure μ, and let $\#$ be a measure-preserving symmetrization on X. Suppose that*

$$\int_X fg \, d\mu \le \int_X fg^\# \, d\mu$$

for every positive function g on X. Then f is almost $\#$-symmetric.

Proof of Proposition 2.2. Fix $\lambda \in \mathbb{R}$. Let $A = f_\lambda$ for some $\lambda > 0$. Note that by the integrability of f, we have $\mu(A) < \infty$. Then, by assumption we have

$$\int_A f \, d\mu \le \int_{A^\#} f \, d\mu.$$

Let $A_1 = A \cap A^\#$ and let $A_2 = A^\# \backslash A$. Let $B_2 = A \backslash A_1$. Evidently $A^\#$ is the disjoint union of A_1 and A_2, while A is the disjoint union of A_1 with B_2. It follows that

$$\int_{A_1} f \, d\mu + \int_{B_2} f \, d\mu \le \int_{A_1} f \, d\mu + \int_{A_2} f \, d\mu.$$

Hence,

$$\int_{B_2} f\,d\mu \le \int_{A_2} f\,d\mu. \tag{2.1}$$

But $f > \lambda$ on $B_2 \subseteq A$, while $f \le \lambda$ on $A_2 \subseteq X\backslash A$. Moreover, B_2 and A_2 have the same measure, since $\mu(A_1) + \mu(B_2) = \mu(A) = \mu(A^\#) = \mu(A_1) + \mu(A_2)$. It follows that the only way (2.1) can hold is if $\mu(A_2) = \mu(B_2) = 0$. Thus, A and $A^\#$ coincide up to a set of measure zero. Hence, f_λ and $(f_\lambda)^\#$ are equal modulo sets of measure zero for each fixed $\lambda > 0$. Writing

$$f(x) = \max(0, \sup\{q \in \mathbb{Q} : q > 0 \text{ and } x \in f_q\})$$

and

$$f^\#(x) = \max(0, \sup\{q \in \mathbb{Q} : q > 0 \text{ and } x \in (f_q)^\#\}),$$

it follows that f and $f^\#$ coincide μ-almost everywhere. $\square$

Remark 2.2. Later on in the paper we shall have to use approximation arguments. To this end, note that if f_n is a sequence of measurable functions on a measure space X with $f_n(x) \uparrow f(x)$ as $n \to \infty$ for all fixed x and if $\#$ is a symmetrization, then $(f_n)^\#(x) \uparrow f^\#(x)$ as $n \to \infty$ for all fixed x by property (iii) of symmetrization. We also note that if f_n is an increasing sequence of #-symmetric functions, necessarily converging pointwise to a limit f, then $(f_n)^\# \uparrow f^\#$ and since $(f_n)^\# = f_n$ it follows that $f = f^\#$ and hence that f is symmetric.

If we are working on a σ-finite measure space X and $\#$ is a Schwarz-type symmetrization, then we may always approximate a positive #-symmetric function f from below by bounded functions supported on sets of finite measure as follows. Letting $\gamma_n(t) = \min(t, n)$, we easily see that if f is #-symmetric, then so is $\gamma_n \circ f$. Let $A_1 \subseteq A_2 \subseteq \cdots$ be a sequence of measurable sets with finite measure whose union is X. Replacing A_i by $A_i^\#$ and using properties (i) and (iii) of symmetrization, we may assume the A_i are symmetric. Since the intersection of a finite number of #-symmetric sets is #-symmetric as the #-symmetric sets are totally ordered under inclusion (Remark 2.1), it follows the function $f \cdot 1_{A_n}$ is #-symmetric whenever f is #-symmetric. Then $(\gamma_n \circ f) \cdot 1_{A_n}$ is a sequence of #-symmetric functions increasing pointwise to f.

2.2. Symmetrization-convolution inequalities. Our main tool is that of symmetrization-convolution inequalities. Suppose that $\#$ is a symmetrization on a measure space $(X, \mathcal{F}, \mu)$. Let κ be a measure defined on the product σ-algebra $\mathcal{F} \times \mathcal{F}$ on $X \times X$. We say that κ *satisfies a #-symmetrization-convolution inequality* if

$$\int_{X\times X} f(x)g(y)\,d\kappa(x,y) \le \int_{X\times X} f^\#(x)g^\#(y)\,d\kappa(x,y)$$

whenever f and g are positive measurable functions on X. If K is a positive function on $X \times X$, then we say that K *satisfies a #-symmetrization-convolution inequality* if $d\kappa = K(x,y)d\mu(x)d\mu(y)$ does. The Riesz-Sobolev inequality (see Riesz [18] in the case of $n = 1$, and Brascamp, Lieb and Luttinger [7] for some much more general results) implies that if $X = \mathbb{R}^n$ while $K(x,y) = k(|x-y|)$ where k is a positive decreasing function, then K satisfies a Schwarz-symmetrization-convolution inequality. Baernstein and Taylor [3] have shown that if $X = \mathbb{S}^n$ and $K(x,y) = k(|x-y|)$ where k is a positive decreasing function, then K satisfies a Schwarz-symmetrization-convolution inequality. Beckner [4] has obtained the same result if $X = \mathbb{H}^n$ and $K(x,y) = k(d(x,y))$, where d is the hyperbolic metric.

Hardy and Littlewood (see Theorem 371 of Hardy, Littlewood and Pólya [10]) have shown that if $X = \mathbb{Z}$ and # is the canonical Schwarz-type symmetrization on $\mathbb{Z}$, then $K(x,y) = k(|x-y|)$ for decreasing functions k satisfies a #-symmetrization-convolution inequality. Pruss [15] has shown that if X is $\mathbb{Z}_m$, T_p or $L(T_p)$, and if # is the canonical Schwarz-type symmetrization on X, then the function $K(x,y) = k(d(x,y))$ for positive decreasing functions k satisfies a #-symmetrization-convolution inequality, where d is the graph distance (length of shortest path) on the circular graph $\mathbb{Z}_m$, on the tree T_p or on the line graph $L(T_p)$. Pruss [15] has also shown the same thing if X is the edge graph of the octahedron and # is defined appropriately.

Finally, Proposition 2.1 shows that for any measure-preserving symmetrization # on any measure space X, the diagonal measure δ_μ defined by

$$\int_{X\times X} f(x,y)\,d\delta_\mu(x,y) = \int_X f(x,x)\,d\mu(x)$$

satisfies a #-symmetrization-convolution inequality.

The main hypothesis that we shall make for our results is that we have an appropriate symmetrization-convolution inequality. Thus, if new symmetrization-convolution inequalities were to be discovered, then we would get new results. Note that in discrete cases it is difficult to get symmetrization-convolution inequalities. For instance, it is known [15] that on the cube $\mathbb{Z}_2^3$ and on the ternary plane $\mathbb{Z}_3^2$ there are no Schwarz-type symmetrizations which would yield a symmetrization-convolution inequality of the type that we have on $\mathbb{Z}$, $\mathbb{Z}_m$, T_p, $L(T_p)$ or on the octahedron edge-graph.

2.3. Hardy-Littlewood-Pólya domination. Let X be a space equipped with a measure μ and let f and g be positive measurable functions on X. We say that f *is dominated by* g (in the sense of Hardy, Littlewood and Pólya) and write $f \preceq g$ providing

$$\int_X \Phi(f(x))\,d\mu(x) \le \int_X \Phi(g(x))\,d\mu(x),$$

for all increasing convex functions Φ. It is well known that $f \preceq g$ if and only if for every $\alpha \in (0, \infty]$ we have

$$\sup_A \int_A f\,d\mu \le \sup_A \int_A g\,d\mu, \tag{2.2}$$

where the suprema are taken over all measurable sets $A \subseteq X$ with $\mu(A) \le \alpha$.

Prpopsition 2.3. *Suppose that # is a Schwarz-type symmetrization on X and that the positive function g is #-symmetric. Then the following are equivalent for a positive measurable function f on X:*

(a) $f \preceq g$

(b) $\int_E f\,d\mu \le \int_{E^\#} g\,d\mu$ *for all measurable* $E \subseteq X$

(c) $\int_X fh\,d\mu \le \int_X gh^\#\,d\mu$ *for all positive measurable functions h on X.*

The proof is given in §5.1.

Definition 2.1. Let f and g be two positive functions on X. We write $f \trianglelefteq g$ if $\int_X fh\,d\mu \le \int_X gh^\#\,d\mu$ for all positive measurable functions h.

Remark 2.3. Assume that # is a Schwarz-type symmetrization. If $f \trianglelefteq g$, then (2.2) follows and so $f \preceq g$. Conversely, by Proposition 2.3, if $f \preceq g$ and g is #-symmetric, then $f \trianglelefteq g$. Proposition 2.2 shows that $f \trianglelefteq f$ implies that f is almost #-symmetric, assuming f is integrable.

Definition 2.2. We say that a measure μ is *homogeneously divisible* if for any measurable sets A and B with $\mu(A) \le \mu(B)$, there exists a measurable subset B' of B with $\mu(A) = \mu(B')$.

Lebesgue measure (whether on $\mathbb{R}^n$, $\mathbb{S}^n$ or $\mathbb{H}^n$) is homogeneously divisible. So is counting measure on any set. In fact, a measure is homogeneously divisible if and only if either it is purely atomic with all atoms having equal weight or it is purely non-atomic. The author is grateful to Professor Herman Rubin for this last observation.

Lemma 2.1. *Assume that our measure μ is σ-finite and homogeneously divisible, and that # is a Schwarz-type symmetrization. Let f and g be positive and measurable. If $f \trianglelefteq g$, then also $f^\# \trianglelefteq g$.*

The proof is given in §5.1.

3. Statement of results.

3.1. The processes. First we describe a class of processes for which our results are valid.

Let Y be a space with a homogeneously divisible σ-finite measure μ. Let # be a Schwarz-type rearrangement on Y. Let X be any measure space with a measure ν, and let $\boxminus$ be the product symmetrization on $Z \stackrel{\text{def}}{=} X \times Y$ induced by $\#$. Let $\pi_1 \colon Z \to X$ and $\pi_2 \colon Z \to Y$ be defined by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$.

Suppose that $\{R_n\}_{n=0}^{\infty}$ is a process on Z and let $\mathcal{F}$ be the σ-field generated by $\{\pi_1(R_n)\}_{n=0}^{\infty}$. Assume that almost surely given $\mathcal{F}$, the process $\{\pi_2(R_n)\}_{n=0}^{\infty}$ is a Markov process. Let κ_n be the $\mathcal{F}$-conditional transition-probability measure for $\pi_2(R_n)$ on $Y \times Y$ defined by

(3.1)
$$\int_{Y \times Y} f(x, y)\, d\kappa_n(x, y) = \int E[f(x, \pi_2(R_{n+1})) \mid \mathcal{F}, \pi_2(R_n) = x]\, d\mu(x).$$

Then, κ_n is a randomly-chosen measure on the set $Y \times Y$ such that $\int_{Y \times Y} f(x)g(y) d\kappa_n(x, y)$ is an $\mathcal{F}$-measurable random variable for any non-random μ-measurable f and g. Note that if $k_{n,y}$ is the randomly-chosen measure defined by $k_{n,x}(U) = P(\pi_2(R_{n+1}) \in U \mid \mathcal{F}, R_n = x)$, then

$$dk_{n,x}(y) d\mu(x) = d\kappa_n(x, y). \tag{3.2}$$

For a measure κ on $Y \times Y$ and a function f on Y, we write $f * \kappa$ for the measure on Y defined by

$$\int_Y g\, d(f * \kappa) = \int_Y f(x)g(y)\, d\kappa(x, y), \tag{3.3}$$

and $\kappa * f$ for the measure on Y defined by

$$\int_Y g\, d(\kappa * f) = \int_Y g(x)f(y)\, d\kappa(x, y). \tag{3.3}$$

We now assume that almost surely given $\mathcal{F}$, for any μ-measurable positive function f on Y, the measure $f * \kappa_n$ has μ-density f^{κ_n} while the measure $\kappa * g$ has μ-density f_{κ_n}. In all concrete cases of interest, the validity of this assumption will be obvious.

Recall the diagonal measure δ_μ defined in Section 2.2. The following is our central assumption.

Definition 3.1. We say that $\{R_n\}$ *is $\boxminus$-symmetrizable with constant c* providing it is a process on $X \times Y$ with the properties given above and satisfies the additional condition that for every $n \geq 0$ the random measure $\kappa_n + c\delta_\mu$ almost surely satisfies a $\#$-symmetrization-convolution inequality, where $c \geq 0$.

The nicest cases are when $c = 0$, but some very natural discrete cases do have $c > 0$.

We now give a few interesting examples of symmetrizable processes.

Example 3.1. Let Y be $\mathbb{R}^m$, $\mathbb{S}^m$, $\mathbb{H}^m$, $\mathbb{Z}_m$, $\mathbb{Z}$, T_p or $L(T_p)$, and let μ be our canonical measure on Y. Let $\#$ be the canonical Schwarz-type symmetrization on Y. (Another case that we can include here is if Y is the octahedron edge graph and the canonical Schwarz-type symmetrization $\#$ on it is defined as in Pruss [15].) Let r_n be a process on Y such that

$$P(r_{n+1} \in U \mid r_n = x) = \int_U k_n(d(x,y))\, d\mu(y),$$

for any measurable $U \subseteq Y$, where k_n is a decreasing function and d is the canonical metric on Y. Let X be any set and let $\{A_n\}$ be any process on X independent of $\{r_n\}$. Then, $R_n = (A_n, r_n)$ is $\boxminus$-symmetrizable with constant 0. To see this note that $d\kappa_n(x,y) = k_n(d(x,y))\, d\mu(x) d\mu(y)$, and as mentioned in Section 2.2 such a κ_n in our cases satisfies $\#$-symmetrization-convolution, which implies that $\{R_n\}$ is $\boxminus$-symmetrizable with constant 0.

Example 3.2. Let Y be $\mathbb{R}^n$, $\mathbb{S}^n$ or $\mathbb{H}^n$. Let B_t be Brownian motion on Y. Let $0 \le t_0 \le t_1 \le t_2 \le \cdots$ be any real increasing process independent of $\{B_t\}_{t\in[0,\infty)}$, and let $\{A_n\}_{n=0}^{\infty}$ be any process on a measure space X such that $\{A_n\}_{n=0}^{\infty}$ is also independent of $\{B_t\}_{t\in[0,\infty)}$ (but possibly dependent on the t_n). Then $R_n = (A_n, B_{t_n})$ is $\boxminus$-symmetrizable with constant 0. To see this, note that the transition probability density between B_{t_n} and $B_{t_{n+1}}$ given $\mathcal{G} \overset{\text{def}}{=} \sigma(\{t_n\}_{n=0}^{\infty})$ is just $K_{t_{n+1}-t_n}(x,y)$ where K_t is the heat kernel on X, and, given $\mathcal{G}$, this must be of the form $k_{t_{n+1}-t_n}(d(x,y))$ on $\mathbb{R}^m$, $\mathbb{S}^m$ or $\mathbb{H}^m$ for a decreasing function $k_{t_{n+1}-t_n}$, so that $\{R_n\}$ given $\mathcal{G}$ is $\boxminus$-symmetrizable with constant 0, and it follows immediately that $\{R_n\}$ is unconditionally $\boxminus$-symmetrizable with constant 0. One of the most interesting cases is when we just have $t_n = n$, in which case $R_n = (A_n, B_n)$.

Example 3.3. Let Y be the linear graph $\mathbb{Z}$, the circle graph $\mathbb{Z}_m$ for $m \ge 2$, the p-regular tree T_p for $p \ge 2$, the line graph $L(T_q)$ of T_q for $q \ge 2$ or the octahedron edge graph, and let $\#$ be the canonical Schwarz-type symmetrization on Y. Let p be the degree of Y; if $Y = T_p$ then p is automatically equal to the degree of Y; if $Y = \mathbb{Z}$ or $Y = \mathbb{Z}_m$, then let $p = 2$; if $Y = L(T_q)$ then put $p = 2(q-1)$. Let X be an arbitrary graph with a constant degree r (i.e., any graph such that all vertices have r edges emanating from them). Let $Z = X \times Y$ be the graph whose vertices are all points (x,y) with $x \in X$ and $y \in Y$ and whose edges are pairs $((x,y),(x_1,y_1))$ where either we have $x = x_1$ and (y,y_1) is an edge of Y, or we have $y = y_1$ and (x,x_1) is an edge of X. Then, Z is a constant degree graph of degree $s = r + p$. Let R_n be the standard nearest-neighbour random walk on Z, so that

$$P(R_{n+1} = z \mid R_n = w) = \begin{cases} \frac{1}{s} & \text{if } w \text{ and } z \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

(For instance, if $X = Y = \mathbb{Z}$ then R_n is just the standard random walk on $\mathbb{Z}^2$.)

I claim that $\{R_n\}$ is $\boxminus$-symmetrizable with some constant $c \geq 0$. It suffices to check that $\kappa_0 + c\delta_\mu$ satisfies a #-symmetrization-convolution inequality for some $c \geq 0$. Let $c = p^{-1}$. Let $x_0 = \pi_1(R_0)$ and $x_1 = \pi_1(R_1)$. Given $\mathcal{F}$ and R_0, if $x_0 = x_1$, then $\pi_2(R_1)$ is uniformly distributed over the p neighbours of the point $\pi_2(R_0)$ in Y. Hence, if $x_0 = x_1$, then

$$\int_{Y \times Y} f(a)f(b)\, d\kappa_0(a,b) = \frac{1}{p} \sum_{a \in Y} \sum_{b \in N(a)} f(a)f(b)$$

given $\mathcal{F}$, where $N(a)$ is the set of neighbours of a in Y. Therefore, under these circumstances,

$$\begin{aligned}\int_{Y \times Y} f(a)f(b)\, d(\kappa_0(a,b) + p^{-1}\delta_\mu(a,b)) &= \frac{1}{p} \sum_{a \in Y} \sum_{b \in N(a) \cup \{a\}} f(a)f(b) \\ &= \frac{1}{p} \sum_{a,b \in Y} f(a)k(d(a,b))f(b),\end{aligned}$$

where $k(t) = 1$ if $t \leq 1$ and $k(t) = 0$ otherwise, while μ is the counting measure on Y. Since k is a decreasing function, it follows from the symmetrization-convolution inequalities of Pruss [15] (discussed in Section 2.2 of the present paper) that

$$\sum_{a,b \in Y} f(a)k(d(a,b))g(b) \leq \sum_{a,b \in Y} f^{\#}(a)k(d(a,b))g^{\#}(b).$$

Hence, $\kappa_0 + p^{-1}\delta_\mu$ satisfies a #-symmetrization-convolution inequality given $\mathcal{F}$ if $x_0 = x_1$. On the other hand, if $x_0 \neq x_1$, then $\pi_2(R_0) = \pi_2(R_1)$, and so given $\mathcal{F}$ if $x_0 \neq x_1$ we have $\kappa_0 = \delta_\mu$, and as mentioned in Section 2.2, the measure δ_μ does satisfy a #-symmetrization-convolution inequality. Hence, given $\mathcal{F}$, if $x_0 \neq x_1$, then we have $\kappa_0 + p^{-1}\delta_\mu$ satisfying a #-symmetrization-convolution inequality, as desired.

Example 3.4. Other constructions like the one in the preceding example are possible. For instance, while in the previous example we had an equal probability $1/(r+p)$ of moving from $(x,y) \in X \times Y$ to any of its $r+p$ neighbours in $X \times Y$, we could instead have a probability q/p of moving to any of the neighbours of the form (x,y') where y' is a neighbour of y in Y, and a probability $(1-q)/r$ of moving to any of the neighbours of the form (x',y) where x' is a neighbour of x in X. We could even handle some cases where X has non-constant degree. The thing to note is that our results do not depend very much on what X is.

Example 3.5. A particularly interesting case in Example 3.3 is when $X = \mathbb{Z}$ and $Y = \mathbb{Z}_m$, so that $Z = \mathbb{Z} \times \mathbb{Z}_m$ is the discrete cylinder and R_n is the simple random walk on Z. Another interesting case is when $X = \mathbb{Z}^{m-1}$ and $Y = \mathbb{Z}$ so that $Z = \mathbb{Z}^m$ and R_n is the simple random walk on Z.

3.2. Generalized harmonic measures and Green's functions. Given a $\boxminus$-symmetrizable process R_n on $Z = X \times X$, we now define a certain stopping time. Let $s\colon Z \to [0,1]$ be measurable. The value of s at a point z will represent the probability that R_n survives for one step while standing at z. More precisely, let $\{X_n\}_{n=0}^{\infty}$ be a sequence of independent identically distributed random variables uniformly distributed on $[0,1]$ and independent of the process $\{R_n\}$. Let

$$\tau_s = \inf\{n \geq 0 : X_n > s(R_n)\}$$

be the first time that R_n fails to survive a step.

Let $A \subseteq Z$ be measurable. Then, the *generalized harmonic measure* of A at z with respect to the survival function s and the process $\{R_n\}$ is defined to be

$$\omega(z, A; s) = P^z(\tau_s < \infty \text{ and } R_{\tau_s} \in A),$$

where we use $P^z(\cdot)$ and $E^z[\cdot]$ to indicate probabilities and expectations, respectively, for the process R_n conditioned by $\{R_0 = z\}$.

The generalized harmonic measure $\omega(z, A; s)$ is the probability that when the process R_n started at z terminates by failing to survive a step, it terminates inside the set A. The particularly interesting case is when s is the indicator function 1_D of some measurable set D and A is outside D (typically in the boundary of D). Then, we shall write $\omega(z, A; D) = \omega(z, A; 1_D)$. This is the standard harmonic measure of A at z in D, i.e., the probability that R_n when started at z first exists D at a point of A.

Another interesting quantity is the *generalized Green's function* $g(z, A; s)$ which is the expected number of times that the process R_n started at z visits the set A before terminating. More precisely,

$$\begin{aligned} g(z, A; s) &= E^z[|\{n \in \mathbb{Z}_0^+ : n < \tau_s \text{ and } R_n \in A\}|] \\ &= \sum_{n=0}^{\infty} P^z(n < \tau_s \text{ and } R_n \in A). \end{aligned}$$

For fixed z, the function $g(z, \cdot; s)$ is a measure on Z. If this measure is absolutely continuous with respect to the product measure $\nu \times \mu$ on Z, then we shall write $g(z, w; s)$ for the value of its $(\nu \times \mu)$-density at $w \in Z$. If ν and μ are counting measures, then $g(z, w; s) = g(z, \{w\}; s)$.

If $s = 1_D$, then again we write $g(z, A; D) = g(z, A; 1_D)$ for $D, A \subseteq Z$ and $g(z, w; D) = g(z, w; 1_D)$ for $D \subseteq Z$ and $w \in Z$. Note that the function $g(z, w; D)$ is then the standard (probabilistic) Green's function on D. If Z is a discrete set and the measure $\nu \times \mu$ on it is counting measure, then $g(z, w; D)$ is the expected number of times that R_n visits the point w before exiting D, assuming $R_0 = z$.

3.3. Symmetrization for generalized harmonic measures and Green's functions. Let $\{R_n\}$ be a process as in Section 3.1. Throughout, we let

$$\psi_c(t) = \frac{t}{1 - p + pt}, \tag{3.5}$$

where $p = c/(1+c)$. Note that ψ_c is strictly increasing and maps $[0, 1]$ onto itself. Moreover, ψ_0 is the identity function.

Theorem 3.1. *Suppose that $\{R_n\}$ is $\boxminus$-symmetrizable with some constant $c \geq 0$. Let $s, s' \colon Z \to [0, 1]$, $U \subseteq X$ and $A \subseteq U \times Y$ be measurable. Assume that s and s' vanish identically on $U \times Y$ and that $\psi_c(s(x, \cdot)) \preceq \psi_c(s'(x, \cdot))$ with $s'(x, \cdot)$ being almost #-symmetric for all fixed $x \in X$. Then,*

$$\omega((x, \cdot), A; s) \preceq \omega((x, \cdot), A^{\boxminus}; s'),$$

for each fixed $x \in X$, with the function on the right hand side being almost #-symmetric.

Remark 3.1. This result should not particularly surprise experts, given the work of Haliste [9, Section 8] and Borell [6].

Remark 3.2. Note that $s'(x, \cdot)$ is almost #-symmetric if and only if $\psi_c(s'(x, \cdot))$ is almost #-symmetric since $\psi_c \colon [0, 1] \to [0, 1]$ is strictly increasing and continuous so that $s'(x, \cdot)$ and $\psi_c(s'(x, \cdot))$ have the same collection of level sets.

Since we trivially have $\psi_c(s(x, \cdot)) \preceq \psi_c(s^{\boxminus}(x, \cdot))$ as the two functions are equimeasurable, we immediately obtain the following corollary.

Corollary 3.1. *Suppose that $\{R_n\}$ is $\boxminus$-symmetrizable with some constant $c \geq 0$. Let $s \colon Z \to [0, 1]$, $U \subseteq X$ and $A \subseteq U \times Y$ be measurable. Assume that s vanishes identically on $U \times Y$. Then, for each fixed $x \in X$ we have*

$$\omega((x, \cdot), A; s) \preceq \omega((x, \cdot), A^{\boxminus}; s^{\boxminus}),$$

and the function on the right hand side is almost #-symmetric.

Theorem 3.2. *Suppose that* $\{R_n\}$ *is a process which is* $\boxminus$*-symmetrizable with constant* c*. Let* $s, s' : Z \to [0,1]$ *and* $A \subseteq Z$ *be measurable. Suppose that* $\psi_c(s(x,\cdot)) \preceq \psi_c(s'(x,\cdot))$ *with* $s'(x,\cdot)$ *being almost* $\#$*-symmetric for all fixed* $x \in X$*. Then,*

$$g((x,\cdot), A; s) \preceq g((x,\cdot), A^{\boxminus}; (s')^{\boxminus}),$$

for each fixed $x \in X$*, with the function on the right hand side being almost* $\#$*-symmetric.*

Remark 3.3. If s and s' are indicator functions of sets, while $X = \mathbb{Z}$ and either $Y = \mathbb{Z}$ or $Y = \mathbb{Z}_m$, and if R_n is a simple random walk on $Z = X \times Y$, then Theorem 3.1 follows directly from the methods of Quine [17], and Theorem 3.2 can probably also be proved by Quine's methods (in the same special case).

Corollary 3.2. *Suppose that* $\{R_n\}$ *is a process which is* $\boxminus$*-symmetrizable with some constant* $c \geq 0$*. Let* $s: Z \to [0,1]$ *and* $A \subseteq Z$ *be measurable. Then, for each fixed* $x \in X$ *we have*

$$g((x,\cdot), A; s) \preceq g((x,\cdot), A^{\boxminus}; s^{\boxminus}),$$

and the function on the right hand side is almost $\#$*-symmetric.*

Remark 3.4. In many instances one can prove that $\omega((x,\cdot), A^{\boxminus}; s^{\boxminus})$ and $g((x,\cdot), A^{\boxminus}; s^{\boxminus})$ are continuous and that continuous functions which are almost $\#$-symmetric are in fact $\#$-symmetric.

Given a nonempty discrete space Y (i.e., a space Y equipped with counting measure) and a Schwarz-type symmetrization of Y, we say that $o \in Y$ is the *origin* of Y if $\{y\}^{\#} = \{o\}$ for any singleton $\{y\} \subseteq Y$. Note that since we are talking about a Schwarz-type symmetrization, the origin exists and is of course unique. If Y is $\mathbb{Z}$ or $\mathbb{Z}_m$, then $o = 0$. If $Y = T_p$, then o is the root of the tree. If $Y = L(T_p)$, then o is the edge $\{o_{T_p}, 1_{T_p}\}$ of T_p, where o_{T_p} is the root of T_p and 1_{T_p} is the $\lessdot$-smallest vertex of T_p among those greater than o_{T_p}.

Corollary 3.3. *Suppose that* Y *is a discrete space equipped with a Schwarz-type symmetrization* $\#$ *and that* o *is the origin of* Y*. Then under the conditions of Theorem 3.1 we have*

$$\omega((x,y), A; s) \leq \omega((x,o), A^{\boxminus}; s'),$$

for all $x \in X$ *and* $y \in Y$*, and under the conditions of Theorem 3.2 we have*

$$g((x,y), A; s) \leq g((x,o), A^{\boxminus}; s'),$$

also for all $x \in X$ and $y \in Y$.

Remark 3.5. Letting $X = Y = \mathbb{Z}$, $s = 1_D$, $U = \{N\}$ and letting R_n be the simple random walk on $\mathbb{Z}^2$ (Example 3.3) we obtain inequality (1.2) of our Introduction.

Proof of Corollary 3.3. By Theorem 3.1,

$$\omega((x,\cdot),A;s) \preceq \omega((x,\cdot),A^{\boxminus};s')$$

and the quantity on the right hand side is #-symmetric. By Proposition 2.3(b) it follows that

$$\sum_{a\in\{y\}} \omega((x,a),A;s) \le \sum_{a\in\{y\}^{\#}} \omega((x,a),A^{\boxminus};s').$$

Since $\{y\}^{\#} = \{o\}$ we are done. The case of Green's functions is handled analogously. $\square$

Theorem 3.3. *Suppose that $\{R_n\}$ is a process which is $\boxminus$-symmetrizable with constant 0. Let $s\colon Z \to [0,1]$ be measurable. Then, for every $N \in \mathbb{Z}_0^+$ and each fixed $x \in X$ we have*

$$(y \mapsto P^{(x,y)}(\tau_s > N)) \preceq (y \mapsto P^{(x,y)}(\tau_{s^{\#}} > N)),$$

and $y \mapsto P^{(x,y)}(\tau_{s^{\#}} > N)$ is almost #-symmetric.

Theorem 3.3 will not in general hold if we merely assume that $\{R_n\}$ is $\boxminus$-symmetrizable for a constant $c > 0$.

Remark 3.6. While all our results are for discrete time processes, an approximation argument can often be used to get results for continuous time. Thus, for instance, if we are interested in Brownian motion B_t on $X \times Y$ where Y is one $\mathbb{R}^n$, $\mathbb{S}^n$ and $\mathbb{H}^n$, we can discretize the time coordinate by considering the discrete time processes $R_n^\varepsilon = B_{\varepsilon n}$ and taking a limit as $\varepsilon \to 0+$. By Example 3.2, our results will apply to the process R_n^ε. Hence, we can obtain symmetrization inequalities for ordinary harmonic measures and Green's functions (i.e., for the case where $s = 1_D$ for an open set D) on $X \times Y$ where Y is one of $\mathbb{R}^n$, $\mathbb{S}^n$ and $\mathbb{H}^n$ and X is any manifold. To prove these results, one could first use an approximation argument whereby one first proves them by using the limit $\varepsilon \to 0+$ with R_n^ε in the case where the complement of D is the closure of an open set. Then, one could approximate a general D (working now in the continuous Brownian motion case and no longer with any reference to R_n^ε) by domains whose complements are closures of open sets.

In particular, we can obtain probabilistic proofs of Baernstein and Taylor's results [1][3] on symmetrization, Green's functions and harmonic measures. To produce versions where s takes values in $[0, 1]$ would require a more subtle approximation argument, but probably can also be done.

A small subtlety occurs if $Y = \mathbb{S}^1$ and X is one dimensional. In that case, because the "line" $X \times \{-o\}$ may have strictly positive capacity with respect to Brownian motion (even though it has null capacity with respect to $\{R_n^\varepsilon\}$ for every fixed $\varepsilon > 0$), it matters very much to us that we define the "symmetrization" $\#$ on $\mathbb{S}^1$ as in Example 2.4 with $A^\# = \mathbb{S}^1$ in the case where $A = \mathbb{S}^1$ and $A^\# = \mathbb{S}^1 \setminus \{-o\}$ in the case where $\mathbb{S}^1 \setminus A$ is a non-empty set of measure zero.

Remark 3.7. The methods used in our paper can be adapted to the case of a Schwarz-type *rearrangement* $\#$ mapping subsets of Y to subsets of another space $\hat{Y}$. Functions on Y can then be easily rearranged to form functions on $\hat{Y}$, just as in this paper, and a function on $\hat{Y}$ can be called *$\#$-symmetric* if it is of the form $f^\#$ for some f on Y. Then, we must have a pair of processes, R_n on $X \times Y$ and $\hat{R}_n$ on $X \times \hat{Y}$, and we must require that we have a convolution-rearrangement inequality between the measures induced by the conditioned transition kernels κ_n and $\hat{\kappa}_n$ associated with R_n and $\hat{R}_n$. We also need to make an additional assumption that $f^{\hat{\kappa}_n}$ and $f_{\hat{\kappa}_n}$ are almost $\#$-symmetric with probability one whenever f is $\#$-symmetric.

Actually, this approach can yield new results even in continuous cases. For instance, Y could be a manifold and $\hat{Y}$ a manifold of revolution with an appropriate isoperimetric inequality between them (cf. Gallot [8]). By the methods of [8, Proof of Theorem 5.4(iii)], such isoperimetric inequalities can be shown to imply a corresponding convolution-rearrangement inequality for heat kernels. Assume that if f is $\#$-symmetric, then $f^{\hat{\kappa}_n}$ and $f_{\hat{\kappa}_n}$ are almost $\#$-symmetric (under appropriate conditions on $\hat{Y}$ this will follow from a maximum principle since $f^{\hat{\kappa}_n}$ and $f_{\hat{\kappa}_n}$ are solutions at some time $t > 0$ of the heat equation on $\hat{Y}$ with initial data f; moreover, we will have $f^{\hat{\kappa}_n} = f_{\hat{\kappa}_n}$ by the symmetry of the heat kernel). These methods combined with the time-discretization described in the preceding remark, should yield rearrangement results for harmonic measures and Green's functions defined in terms of Brownian motion on $X \times Y$ and on $X \times \hat{Y}$. We do not give the details as they do not contain any new insights but only require a notation complicated by the fact that Y and $\hat{Y}$ do not coincide.

4. A discrete Beurling shove theorem. In this section we are working with a random walk R_n on $Z = \mathbb{Z} \times G$, where G is one of the graphs $\mathbb{Z}$, $\mathbb{Z}_m$, T_p, $L(T_p)$ or the edge-graph of the octahedron, either in the setting of Example 3.3 or of Example 3.4. Let o be the origin of G.

Let $D = \mathbb{Z}^- \times G$, where $\mathbb{Z}^- = \{-1, -2, \dots\}$. Let $T = \{0\} \times G$. Then the following result is a discrete analogue of Beurling's shove theorem (see pp. 58–62 of his thesis [5] or §IV.5.4 of Nevanlinna [13]).

Theorem 4.1. *Let H be a finite non-empty subset of $\mathbb{Z}^- \times \{o\}$, and set $U = D\backslash H$. Let $U^\diamond = D\backslash H'$, where $H' = \{-|H|, -|H|+1, \dots, -1\} \times \{o\}$. Then,*

$$\omega((t,o),T;U) \leq \omega((t,o),T;U^\diamond), \tag{4.1}$$

whenever $t < \inf\{t' : (t',o) \in H\}$.

The reason for the name "shove theorem" is evident in this formulation insofar as H' is formed from H by shoving all of its elements to the right in $\mathbb{Z}^- \times \{o\}$ and thus eliminating any gaps in H.

Remark 4.1. The original Beurling shove theorem may be described by letting D be the half-cylinder $\mathbb{R}^- \times \mathbb{T}$, where $\mathbb{R}^- = (-\infty, 0)$, setting $T = \{0\} \times \mathbb{T}$ and letting H be a finite union of closed intervals in $(-\infty, 0] \times \{1\}$. Then, the theorem asserts that $\omega((-\infty,1),T;D\backslash H) \leq \omega((-\infty,1),T;D\backslash H')$, where ω is a continuous harmonic measure, and H' is a single interval of the form $[-|H|, 0] \times \mathbb{T}$, where $|H|$ is the linear measure of H. (Actually, the way Beurling [5] originally formulated his result was on the unit disc and not on the half-cylinder, but our description is conformally equivalent to his.) Since $\mathbb{Z}_m$ is a discrete analogue of $\mathbb{T}$ and $\mathbb{Z}^-$ is a discrete analogue of $\mathbb{R}^-$, we are justified in calling Theorem 4.1 a discrete Beurling shove theorem, at least in the case $G = \mathbb{Z}_m$.

Remark 4.2. One may conjecture a number of generalizations of Theorem 4.1. One such would be to consider a survival function s instead of U, such that s and $s^\diamond$ vanish identically on $Z\backslash D$ and are identically 1 everywhere on D except possibly on $\mathbb{Z}^- \times \{o\}$, while $s^\diamond(\cdot, o)$ is the decreasing rearrangement of $s(\cdot, o)$. Then the conjecture of course is that (4.1) continues to hold. This conjecture appears to be nontrivial even in the case of $m = 1$ (in this case the process is a one-dimensional random walk); in this case the conjecture *is* known to be true, but the proof is not very easy [16].

The proof of Theorem 4.1 will be done almost exactly as in the classical continuous case as soon as we establish two lemmas.

Let $U \subseteq Z = \mathbb{Z} \times G$. Write ∂U for the collection of points of $Z\backslash U$ which lie precisely one simple random walk step away from U, i.e.,

$$\partial U = \{z \in Z\backslash U : \exists w \in U \text{ such that } P^w(R_1 = z) > 0\}.$$

Write $\bar{U} = U \cup \partial U$. Let f be any function on $\bar{U}$. Define

$$\Delta f(z) = E^z[f(R_1)] - f(z),$$

for $z \in U$. We say that a function f is a **harmonic function on** U (with respect to $\{R_n\}$) if it is defined on $\bar{U}$ and satisfies $\Delta f(z) = 0$ for all $z \in U$.

Example 4.1. The function $z \mapsto g(z, w; U)$ is a harmonic function on $U \backslash \{w\}$ for any fixed $w \in Z$. This is easiest seen directly from the definition of $g(z, w; U) = g(z, \{w\}; 1_U)$.

Example 4.2. The function $z \mapsto \omega(z, A; U)$ is a harmonic function on U for A outside U. This is also easy to see from the definition of $\omega(z, A; U) = \omega(z, A; 1_U)$.

We recall the following maximum principle, which is far from optimal, but suffices for our needs.

Proposition 4.1. *Let f be a bounded harmonic function on a set $U \subseteq D$. Assume that $f(z) \le C$ for all $z \in \partial U$. Then, $f(z) \le C$ for all $z \in U$.*

The proof follows immediately from the facts that $f(R_{\min(n,\tau_U)})$ is a martingale, where $\tau_U = \inf\{n \ge 0 : R_n \notin U\}$, and that $P^z(\tau_U < \infty) = 1$ for all $z \in Z$ since $U \subseteq D = \mathbb{Z}^- \times G$ while the first component of R_n is in effect a simple random walk on $\mathbb{Z}$ (with some delays during which the walk runs around on G) and hence cannot remain in $\mathbb{Z}^-$ forever.

We now state our two lemmas which provide the keys to the proof of Theorem 4.1.

Lemma 4.1. *In the setting of Theorem 4.1, let $h(t_1,t_2) = g((t_1,o),(t_2,o); D)$. Then for fixed $t_2 \in \mathbb{Z}^-$ the function $h(\cdot, t_2)$ is increasing on $(-\infty, t_2] \cap \mathbb{Z}^-$, and decreasing on $[t_2, -1] \cap \mathbb{Z}^-$. Similarly, for fixed $t_1 \in \mathbb{Z}^-$ the function $h(t_1, \cdot)$ is increasing on $(-\infty, t_1] \cap \mathbb{Z}^-$, and decreasing on $[t_2, -1] \cap \mathbb{Z}^-$.*

Proof. Fix $t_2 \in \mathbb{Z}^-$. First suppose $t_1 > t_2$. We shall show that $h(t_1 - 1, t_2) \ge h(t_1, t_2)$. Let $D_1 = \{t_1, t_1 + 1, \dots, -1\} \times G$. Then, it is easy to see that

$$h(t_1, t_2) = \sum_{\alpha \in G} \omega((t_1, o), \{(t_1 - 1, \alpha)\}; D_1) g((t_1 - 1, \alpha), (t_2, o); D).$$

But by Corollary 3.3 (just let $(x, y) = (t_1 - 1, \alpha)$, $s = s' = 1_D$, and $A = \{(t_2, o)\}$) we have $g((t_1 - 1, \alpha), (t_2, o); D) \le g((t_1 - 1, o), (t_2, o); D) = h(t_1 - 1, t_2)$. Thus,

$$\begin{aligned} h(t_1, t_2) &\le g((t_1 - 1, o), (t_2, o); D) \sum_{\alpha \in G} \omega((t_1, o), \{(t_1 - 1, \alpha)\}; D_1) \\ &= \omega((t_1, o), \{t_1 - 1\} \times G\}; D_1) h(t_1 - 1, t_2) \\ &\le h(t_1 - 1, t_2). \end{aligned}$$

The inequality $h(t_1+1,t_2) \geq h(t_1,t_2)$ in the case $t_1 < t_2$ is proved very similarly. The case of t_1 fixed can be handled just as above (or else it can be noted that it follows from the fact that $h(t_1,t_2) = h(t_2,t_1)$ for the random walks of Examples 3.3 and 3.4.) $\square$

Now, for a subset U of $\mathbb{Z} \times G$, recall that we have let

$$\tau_U = \inf\{n \geq 0 : R_n \notin U\}.$$

Define

$$\tilde{\tau}_U = \inf\{n > 0 : R_n \notin U\}.$$

The following lemma then is valid for any random walk, not just the random walks of Examples 3.3 and 3.4. It extends in an easy way to a number of situations.

Lemma 4.2. *In the setting of Theorem 4.1, let $\phi(z) = 1-\omega(z,T;U)$. Then*

$$\phi(z) = \sum_{w \in H} g(z,w;D)\psi(w),$$

for a positive function ψ on H. More precisely, we may take

$$\psi(w) = P^w(\tilde{\tau}_U = \tilde{\tau}_D).$$

Proof. For $\psi(w) = P^w(\tilde{\tau}_U = \tilde{\tau}_D)$, by the definition of the Green's function and by Fubini's theorem we have

$$\begin{aligned}
&\sum_{w \in H} g(z,w;D)\psi(w) \\
&\quad = \sum_{w \in H} E^z\left[\sum_{n=0}^{\tau_D-1} 1_{\{R_n=w\}}\right] P^w(\tilde{\tau}_U = \tilde{\tau}_D) \\
&\quad = \sum_{n=0}^{\infty}\sum_{w \in H} P^z(R_n = w \text{ and } \tau_D > n) P^w(\tilde{\tau}_U = \tilde{\tau}_D) \\
&\quad = \sum_{n=0}^{\infty} P^z\big(R_n \in H,\, \tau_D > n \text{ and } (R_k \in U, \forall k \in \{n+1,\dots,\tau_D-1\})\big).
\end{aligned}$$

But it is easy to see that the events within the $P^z(\cdot)$ are disjoint for distinct values of n since $H \subseteq U^c$. Moreover, it is easy to see that the union over $n \in \mathbb{Z}_0^+$ of these events is the event $\{\exists n \in \mathbb{Z}_0^+.\,(R_n \in H \text{ and } n < \tau_D)\}$. Clearly, if the random walk starts at z, the probability of this event is $\omega(z,H;D) = \phi(z)$. $\square$

The rest of the proof follows by the original methods of Beurling [5] (see also §IV.5.4 of Nevanlinna [13]), but we give it to make it completely clear how to apply the methods in the discrete case.

Proof of Theorem 4.1. Let $\mathbb{Z}_0^- = \{0\} \cup \mathbb{Z}^-$. For conciseness, given a subset L of $\mathbb{Z}_0^- \times \{o\}$, write

$$\inf L \stackrel{\text{def}}{=} \inf\{l : (l,o) \in L\}.$$

Let $t_0 = \inf H$. We proceed by induction on $N = |t_0| - |H|$. First, if $N = 0$, then $H = H'$ and we are done. Suppose now that the result has been proved whenever $|t_0| - |H| < N$ and that $N \geq 1$. Let $t_1 = \inf\{t \in \mathbb{Z}^- : t \geq t_0, t \notin H\}$. Since $N \geq 1$, we have $t_1 \in \{t_0 + 1, \ldots, -1\}$, and moreover $\{t_0, \ldots, t_1 - 1\} \subseteq H$.

Define

$$H_1 = (H \cap [t_1, -1]) \cup \{t_0 + 1, \ldots, t_1\}.$$

It is easy to see that $|H_1| = |H|$ and that H_1 is in fact just H with the hole at t_1 deleted by shifting everything to the left of the hole by one point to the right. Moreover, $|\inf H_1| = t_0 + 1$ so that $|\inf H_1| - |H| < N$ as $t_0 < 0$. Thus, if we form $(H_1)'$ from H_1 in the same way that H' is formed from H, by our induction hypothesis we will have

$$\omega((t,o), T; D\backslash H_1) \leq \omega((t,o), T; D\backslash (H_1)'),$$

whenever $t < \inf H_1$, and in particular whenever $t < \inf H$. But $|H_1| = |H|$ so that $(H_1)' = H'$. Thus, the desired inequality (4.1) will follow as soon as we prove that

$$\omega((t,o), T; D\backslash H) \leq \omega((t,o), T; D\backslash H_1) \tag{4.2}$$

whenever $t < \inf H$. Write $H = A_1 \cup A_2$ where $A_1 = \{t_0, \ldots, t_1 - 1\}$ and $A_2 = [t_1 + 1, -1] \cap H$. Let $\phi(z) = 1 - \omega(z, T; D\backslash H)$. Then, by Lemma 4.2 we have

$$\phi(z) = \phi_1(z) + \phi_2(z),$$

where

$$\phi_i(z) = \sum_{w \in A_i} g(z, w; D)\psi(w),$$

for $i = 1, 2$. For $(x, y) \in \mathbb{Z}_0^- \times G$, let $\hat{\phi}(x, y) = \phi_1(x-1, y) + \phi_2(x, y)$. I claim that

$$\hat{\phi}(z) \geq 1 - \omega(z, T; D\backslash H_1) \tag{4.3}$$

for all $z \in \mathbb{Z}_0^- \times G$. Suppose for now that this claim holds. Then, for $t < \inf H$ we have

$$\omega((t,o),T;D\backslash H_1) \geq 1 - \phi_1(t-1,o) - \phi_2(t,o).$$

But $\phi_1(t-1,o) \leq \phi_1(t,o)$ since $g((t,o),(u,o);D)$ is increasing in t for $t < u$ (Lemma 4.1) and since ψ is positive while $A_1 \subseteq [-t_0,-1] \times \{o\}$. Thus,

$$\omega((t,o),T;D\backslash H_1) \geq 1 - \phi_1(t,o) - \phi_2(t,o) = 1 - \phi(t,o) = \omega((t,o),T;D\backslash H),$$

which is precisely what we were supposed to prove.

Thus, we need only verify (4.3). But $\hat{\phi}$ is a bounded discrete harmonic function on $D\backslash H_1$ since ϕ_1 and ϕ_2 are harmonic on $D\backslash A_1$ and $D\backslash A_2$, respectively, (the ϕ_i are sums of Green's functions to which we can apply Example 4.1), and $\omega(\cdot,T;D\backslash H_1)$ is harmonic in $D\backslash H_1$ (Example 4.2), while

$$H_1 = \{(x+1,y) : (x,y) \in A_1\} \cup A_2.$$

Thus, the maximum principle (Proposition 4.1) implies that to show (4.3) it suffices to verify that (4.3) holds on $\partial(D\backslash H_1) \subseteq H_1 \cup T$.

But on T, the inequality (4.3) holds trivially as its right hand side vanishes while the left is positive. Suppose now that $z \in H_1 = \{(x+1,y) : (x,y) \in A_1\} \cup A_2$. Then the right hand side of (4.3) equals 1. There are two cases to consider. First suppose that $z \in \{(x+1,y) : (x,y) \in A_1\}$. Then, $z = (x+1,o)$, where $(x,o) \in A_1$. We have

$$\hat{\phi}(z) = \phi_1(x,o) + \phi_2(x+1,o).$$

But whenever $w \in A_2$ and $(x,o) \in A_1$, we have $g((x+1,o),w;D) \geq g((x,o),w;D)$ by Lemma 4.1 so that $\phi_2(x+1,o) \geq \phi_2(x,o)$, and so

$$\hat{\phi}(z) \geq \phi_1(x,o) + \phi_2(x,o) = \phi(x,o).$$

Now, $(x,o) \in A_1 \subseteq H$ so that $\phi(x,o) = 1 - \omega((x,o),T;D\backslash H) = 1$, and so (4.3) is verified. Suppose now that $z = (x,o) \in A_2$. Then,

$$\hat{\phi}(z) = \phi_1(x-1,o) + \phi_2(x,o).$$

But whenever $w \in A_1$ and $(x,o) \in A_2$ we have $g((x-1,o),w;D) \geq g((x,o),w;D)$ by Lemma 4.1, so that $\phi_1(x-1,o) \geq \phi_1(x,o)$, and thus

$$\hat{\phi}(z) \geq \phi_1(x,o) + \phi_2(x,o) = \phi(x,o) = 1,$$

since $(x,o) \in H$ as before, so that again (4.3) is verified. $\square$

Remark 4.3. The perceptive reader may notice that the proof of Theorem 4.1 may be slightly modified to relax the assumption that D is $\mathbb{Z}^- \times G$; indeed, any $\boxminus$-symmetric subset D of $\mathbb{Z}^- \times G$ will work just as well provided that for every $(x, y) \in D$ the set $\{(x', y) : x' \leq x\}$ is contained in D.

5. Proofs.

5.1. Proofs of two auxiliary results. Proof of Proposition 2.3. First note that (b) and (c) are equivalent. For, if we let $h = 1_A$ then we obtain (b) from (c). On the other hand, we may write

$$h(x) = \int_0^\infty 1_{h_\lambda}(x)\, d\lambda,$$

and

$$h^\#(x) = \int_0^\infty 1_{h_\lambda^\#}(x)\, d\lambda.$$

Together with Fubini's theorem, this proves that (b) implies (c).

Suppose now that (b) is true. Then, since E and $E^\#$ always have the same measure, it is easy to see that (2.2) must always hold, and so (a) follows.

Hence it remains to prove that (a) implies (b). Fix a measurable set E. Let $\alpha = \mu(E)$. Then, by (a), (2.2), Proposition 2.1 and the symmetry of g we have

$$\begin{aligned}\int_E f\, d\mu &\leq \sup_A \int_A g\, d\mu = \sup_A \int_X 1_A g\, d\mu \\ &\leq \sup_A \int_X 1_{A^\#} g^\#\, d\mu = \sup_A \int_{A^\#} g\, d\mu.\end{aligned}$$

The supremum here is taken over all measurable A with $\mu(A) \leq \alpha$. But $\mu(E) = \alpha$, and since $\#$ is of Schwarz type it follows that $A^\# \subseteq E^\#$ if $\mu(A) = \alpha$, so that (a) implies (b) as desired. $\square$

Proof of Lemma 2.1. The reasoning in the proof of Proposition 2.3 shows that we only need to prove

$$\int_A f^\#\, d\mu \leq \int_{A^\#} g\, d\mu$$

for all measurable sets A.

In fact, by Proposition 2.1 it suffices to prove that

$$\int_{A^\#} f^\#\, d\mu \leq \int_{A^\#} g\, d\mu,$$

for all A, or, equivalently, to prove that

$$\int_A f^\# \, d\mu \le \int_A g \, d\mu$$

for all $\#$-symmetric A. Approximating f from below shows that we may assume that f has support in a set of at most finite measure (see Remark 2.2). Fix a non-empty $\#$-symmetric set A. Let $\lambda = \inf_A f^\#$. First I claim that $(f^\#)_\lambda \subseteq A$. Clearly by definition of the level set $(f^\#)_\lambda$ it suffices to prove that $(f^\#)_t \subseteq A$ for every $t > \lambda$. Fix $t > \lambda$. By definition of λ, there is an $x \in A$ with $f^\#(x) < t$. Hence, $x \notin (f^\#)_t$. Therefore, $A \not\subseteq (f^\#)_t$. By Remark 2.1 and since both A and $(f^\#)_t$ are $\#$-symmetric, it follows that $(f^\#)_t \subseteq A$.

Let $A_1 = (f^\#)_\lambda$ and let $A_2 = A \backslash A_1$. By construction, $f^\# = \lambda$ everywhere on A_2. Now, let $B_3 = \{x : f(x) = \lambda\}$. If f_t has finite measure for some $t < \lambda$, which will necessarily be the case if either $\lambda > 0$ (since f is supported in a set of finite measure) or if our measure space is finite, then equimeasurability easily shows that $\mu(A_2) \le \mu\{x : f^\#(x) = \lambda\} = \mu(B_3)$. But if $\lambda = 0$ and our measure space is infinite, then B_3 has infinite measure (since f is supported in a set of finite measure) and we trivially have $\mu(A_2) \le \mu(B_3)$. Hence in either case $\mu(A_2) \le \mu(B_3)$. By the homegeneous divisibility property of μ, there exists a subset B_2 of B_3 with $\mu(B_2) = \mu(A_2)$. Let $B_1 = f_\lambda$. By equimeasurability we have $\mu(B_1) = \mu(A_1)$. Let $B = B_1 \cup B_2$. Since B_1 and B_2 are disjoint, and likewise A_1 and A_2 are disjoint, we have $\mu(A) = \mu(B)$. Thus, $B^\# = A^\# = A$. Since $A_1 = (f^\#)_\lambda$ and $B_1 = f_\lambda$, we have

$$\int_{A_1} f^\# \, d\mu = \int_{B_1} f \, d\mu$$

by equimeasurability. We also have

$$\int_{A_2} f^\# \, d\mu = \lambda\mu(A_2) = \lambda\mu(B_2) = \int_{B_2} f \, d\mu.$$

Thus, since $f \trianglelefteq g$, we have

$$\int_A f^\# \, d\mu = \int_B f \, d\mu \le \int_{B^\#} g \, d\mu = \int_A g \, d\mu,$$

as desired. $\square$

5.2. Reduction to the case of symmetrizable processes with $c = 0$. Suppose that our process $\{R_n\}$ is $\boxminus$-symmetrizable with constant $c \ge 0$ and that κ_n is defined as in (3.1).

We shall show that if we can prove our Theorems 3.1 and 3.2 for processes which are $\boxminus$-symmetrizable with constant 0, then we obtain them for constant $c > 0$ as well.

Since $\{R_n\}$ is $\boxminus$-symmetrizable with constant $c \ge 0$, the measure $\kappa_n + c\delta_\mu$ satisfies a #-symmetrization-convolution inequality. Now, define the process $\{N_n\}_{n=0}^\infty$ to be a Markov process with values in $\mathbb{Z}_0^+$ such that $N_0 = 0$ with probability one and

$$P(N_n = N_{n-1} + 1 \mid N_{n-1}) = \frac{1}{1+c}$$

while

$$P(N_n = N_{n-1} \mid N_{n-1}) = \frac{c}{1+c}.$$

Assume that $\{N_n\}_{n=0}^\infty$ is independent of $\{R_n\}_{n=0}^\infty$. Note that if $c = 0$ then $N_n = n$ for all n with probability one. Now, define

$$\hat{R}_n = R_{N_n}.$$

I claim that $\{\hat{R}_n\}$ is $\boxminus$-symmetrizable with constant 0. Let $\mathcal{F} = \sigma\big(\{\pi_1(R_n)\}_{n=0}^\infty\big)$. Let $\mathcal{G} = \mathcal{F} \vee \sigma\big(\{N_n\}_{n=0}^\infty\big)$. It suffices to show that for all positive measurable f and g on Y we have

$$\begin{aligned} &\int_Y E[g(\pi_2(\hat{R}_{n+1})) \mid \mathcal{G}, \pi_2(\hat{R}_n) = x] f(x)\, d\mu(x) \\ &\qquad \le E[g^\#(\pi_2(\hat{R}_{n+1})) \mid \mathcal{G}, \pi_2(\hat{R}_n) = x] f^\#(x)\, d\mu(x). \end{aligned} \tag{5.1}$$

Let $\mathcal{F} = \sigma\{\pi_1(R_n)\}_{n=0}^\infty$. Fix $n \ge 0$. Set $\alpha = N_n$ and $\alpha' = N_{n+1}$. Then,

$$\begin{aligned} E[g(\pi_2(\hat{R}_{n+1})) \mid \mathcal{G}, \pi_2(\hat{R}_n) = x] &= E[g(\pi_2(R_{\alpha'})) \mid \mathcal{F}, \pi_2(R_\alpha) = x, \alpha] \\ &= \frac{1}{1+c} E[g(\pi_2(R_{\alpha+1})) \mid \mathcal{F}, \pi_2(R_\alpha) = x, \alpha, \alpha' = \alpha + 1] \\ &\quad + \frac{c}{1+c} E[g(\pi_2(R_\alpha)) \mid \mathcal{F}, \pi_2(R_\alpha) = x, \alpha, \alpha' = \alpha] \\ &= (1+c)^{-1}(E[g(\pi_2(R_{\alpha+1})) \mid \mathcal{F}, \pi_2(R_\alpha) = x, \alpha] + cg(x)). \end{aligned}$$

It follows that

$$\begin{aligned} (1+c) &\int_Y E[g(\pi_2(\hat{R}_{n+1})) \mid \mathcal{G}, \pi_2(\hat{R}_n) = x] f(y)\, d\mu(x) \\ &= \int_Y f(x) g(y) d\kappa_\alpha(x,y) + c \int_Y f(x) g(x)\, d\mu(x) \\ &= \int_Y f(x) g(y) d(\kappa_\alpha(x,y) + c\delta_\mu(x,y)). \end{aligned}$$

By choice of c and using the independence of α from $\{R_m\}_{m=0}^{\infty}$ we have $\kappa_\alpha + c\delta_\mu$ almost surely satisfying a #-symmetrization-convolution inequality given $\mathcal{G}$ and (5.1) follows.

Hence $\{\hat{R}_n\}$ is $\boxminus$-symmetrizable with constant 0 as desired. We now show that Theorems 3.1 and 3.2 for $\{\hat{R}_n\}$ imply corresponding theorems for $\{R_n\}$. Let $\hat{\omega}$ and $\hat{g}$ respectively denote generalized harmonic measures and Green's functions with respect to $\{\hat{R}_n\}$, and let ω and g denote the analogous functions for $\{R_n\}$.

We may describe the process $\{\hat{R}_n\}$ as follows. A step of this random walk consists of first flipping a coin and with probability $p = c/(1+c)$ staying put, while with probability $1-p = 1/(1+c)$ taking a step with transition probabilities associated with $\{R_m\}$. Let S_n be the event $\{N_{n+1} = N_n + 1\}$, i.e., the event that the flip of the coin was such that we took the step with transition probabilities associated with $\{R_m\}$.

Now, note that the distribution of $\hat{R}_{n+1}$ conditioned on the event S_n given N_n is the same as the distribution of R_{N_n+1} given N_n. Moreover, the probability that the random walk $\{\hat{R}_n\}$ starting at time k at the point z will survive until one of the events S_n happens is equal to

$$(1-p)s(z)+(1-p)p(s(z))^2+(1-p)p^2(s(z))^3+\cdots = \frac{(1-p)s(z)}{1-ps(z)} = \phi(s(z)),$$

where $\phi(t) = \frac{(1-p)t}{1-pt}$. Note that ϕ is a strictly increasing function mapping $[0,1]$ onto itself. The above shows that $\{\hat{R}_n\}$ with survival probabilities s behaves much as $\{R_n\}$ with survival probabilities $\phi \circ s$ would. This observation shows that

$$\omega(z, A; \phi \circ s) = \hat{\omega}(z, A; s),$$

or, equivalently, since $\psi_c = \phi^{-1}$ (where ψ_c is defined by (3.5)),

$$\omega(z, A; s) = \hat{\omega}(z, A; \psi_c \circ s).$$

This observation together with Remark 3.2 shows that Theorem 3.1 for the process $\{\hat{R}_n\}$ which has symmetrizability constant 0 (recall at this point that $\psi_0(t) = t$ for all t) implies the same theorem for the process $\{R_n\}$ which has an arbitrary symmetrizability constant c.

Now, if the random walk $\{\hat{R}_n\}$ is at a point w at a given time, then the probability that it survives at least k contiguous steps before one of the events S_n happening (i.e., before taking a step in accordance with the transition probabilities of $\{R_n\}$) is

$$(s(w))^k p^{k-1}.$$

The expected number of contiguous steps that it makes without one of the S_n happening is then equal to

$$\sum_{k=1}^{\infty}(s(w))^k p^{k-1} = \frac{s(w)}{1-s(w)p} = (1-p)^{-1}\phi(s(w)).$$

On the other hand, the expected number of contiguous steps that the process $\{R_n\}$ conditioned to start at w and equipped with survival probabilities $\phi\circ s$ (where ϕ is as before) will survive and stay at w before taking a step with the transition probabilities of $\{R_n\}$ is $\phi(s(w))$. For the process R_n will necessarily take the next step according to its own transition probabilities unless it fails to survive, and the survival probability is $\phi(s(w))$. Combining these observations with the results of the previous paragraph, we can easily convince ourselves that the identity

$$\hat{g}(z, A; \phi \circ s) = (1-p)^{-1} g(z, w; s),$$

must hold. As before, this shows that Theorem 3.2 for $\{\hat{R}_n\}$ implies the same theorem for $\{R_n\}$.

5.3. Proofs for the case of symmetrizability with constant zero. Our proofs are based on the methods of Haliste [9].

Suppose that $\{r_n\}$ is a process on Y such that the process $\{(n, r_n)\}_{n=0}^{\infty}$ on $\mathbb{Z}_0^+ \times Y$ is $\boxminus$-symmetrizable with constant 0. Let $T_S = \inf\{n \geq 0 : X_n \geq S(n, r_n)\}$ for a function S on Y.

Proposition 5.1. *Suppose we are in the above-described setting and that S and S' are measurable functions from $\mathbb{Z}_0^+ \times Y$ into $[0,1]$ with $S(n,\cdot) \preceq S'(n,\cdot)$ and $S'(n,\cdot)$ being #-symmetric for all fixed n. Then,*

$$\begin{aligned}&\big(y \mapsto P(T_S > N \text{ and } r_N \in A \mid r_0 = y)\big)\\ &\qquad\qquad \trianglelefteq \big(y \mapsto P(T_{S'} > N \text{ and } r_N \in A^{\#} \mid r_0 = y)\big),\end{aligned}$$

for each fixed $n \geq 0$, $x \in \mathbb{Z}_0^+$ and $A \subseteq Y$, with the function on the right hand side being almost #-symmetric.

The proof will be given later.

Lemma 5.1. *Let f and g be positive almost #-symmetric functions on a σ-finte measure space Y, where # is a Schwarz-type symmetrization. Then fg and $f+g$ are almost #-symmetric. Moreover, if $f_1, f_2, f_3, \ldots$ are almost #-symmetric and positive, then so is $\sum_{i=1}^{\infty} f_i$.*

Proof of Lemma 5.1. Since our measure space is σ-finite, via Remark 2.2 we may approximate f and g from below by symmetric functions from L^2.

Hence, it suffices to consider the case where fg is integrable. But then for any measurable positive h we have

$$\int_Y fgh\,d\mu \le \int_Y f^{\#}g^{\#}h^{\#}\,d\mu = \int_Y fgh^{\#}\,d\mu$$

by Proposition 2.1 and by the almost #-symmetry of f and g. By Proposition 2.2, it follows that fg is almost #-symmetric. Likewise, to prove that $f+g$ are almost #-symmetric, it suffices to consider the case of integrable f and g. But then,

$$\int_Y (f+g)h\,d\mu \le \int_Y f^{\#}h^{\#}\,d\mu + \int_Y g^{\#}h^{\#}\,d\mu = \int_Y (f+g)h^{\#}\,d\mu,$$

and as before it follows from Proposition 2.2 that $f+g$ is almost #-symmetric. By induction, it follows that if $f_1, f_2, \ldots$ are almost #-symmetric, then $\sum_{i=1}^{n} f_i$ is almost #-symmetric for all i and since this sum increases to $\sum_{i=1}^{\infty} f_i$ as the f_i are positive, it follows from Remark 2.2 that our infinite sum is also almost #-symmetric. □

Lemma 5.2. *Let $f(y,\omega)$ be a positive measurable function on $Y\times\Omega$, where Ω is a probability space and Y is a σ-finite measure space equipped with a Schwarz-type symmetrization #. Assume that for almost every $\omega\in\Omega$ the function $f(\cdot,\omega)$ is almost #-symmetric. Then, $y\mapsto E[f(y,\omega)]$, where the expectation is taken with respect to ω, is almost #-symmetric.*

Proof of Lemma 5.2. Let μ be our measure on Y. If ψ_n and A_n are as in Remark 2.2 (except that now we have Y in place of X), then $\psi_n(f(y,\omega))\cdot 1_{A_n}(y)$ is an almost #-symmetric function of y for almost all ω (this observation uses Lemma 5.1). Let $f_n(y,\omega)=\psi_n(f(y,\omega))\cdot 1_{A_n}(y)$. By Proposition 2.1 if g is positive and measurable, then we have

$$\int_Y f_n(y,\omega)g(y)\,d\mu(y) \le \int_Y f_n(y,\omega)g^{\#}(y)\,d\mu(y)$$

for almost all ω since $f_n(y,\omega)$ is an almost #-symmetric function of y for almost all ω. By Fubini's theorem,

$$\int_Y E[f_n(y,\omega)]g(y)\,d\mu(y) \le \int_Y E[f_n(y,\omega)]g^{\#}(y)\,d\mu(y).$$

Hence, by Proposition 2.2, since $y\mapsto E[f_n(y,\omega)]$ is integrable (as it is bounded and supported on a set A_n of finite measure), it must be almost #-symmetric. But by monotone convergence as $n\to\infty$ this function increases to $E[f(y,\omega)]$ and by Remark 2.2 the latter function must also be almost #-symmetric. □

First, assuming Proposition 5.1 and Lemma 5.2, we give the proofs of Theorems 3.1 and 3.2 in the case of symmetrizability constant 0, since the case of symmetrizability constant $c > 0$ follows by the work of the previous section.

Proof of Theorem 3.1 in the case $\boldsymbol{c = 0}$. Let $N = \inf\{n \geq 0 : \pi_1(R_n) \in U\}$. Note that $N \in \mathcal{F}$, where $\mathcal{F} = \sigma\big(\{\pi_1(R_n)\}_{n=0}^{\infty}\big)$. Let $A_N = \{y : (\pi_1(R_N), y) \in A\}$. Then,

$$\omega(z, A; s) = P^z(N \leq \tau_s \text{ and } \pi_2(R_N) \in A_N \text{ and } N < \infty).$$

This observation uses the assumption that s vanishes on $U \times Y$. Moreover, if $N < \infty$, then we have $(A_N)^{\#} = \{y : (\pi_1(R_N), y) \in A^{\#}\}$. Thus, repeating the above argument, we see that

$$\omega(z, A^{\#}; s') = P^z(N \leq \tau_{s'} \text{ and } \pi_2(R_N) \in (A_N)^{\#} \text{ and } N < \infty).$$

Let $r_n = \pi_2(R_n)$ for all n. Then, if $\{R_n\}$ is symmetrizable with constant 0, the process $\{(n, r_n)\}$ is also symmetrizable with constant 0 conditionally on $\mathcal{F}$. Letting $S(n, y) = s(\pi_1(R_n), y)$ and $S'(n, y) = s'(\pi_1(R_n), y)$ for $n < N$ and putting $S(n, y) = S'(n, y) = 1$ for all $n \geq N$, we have $S(n, \cdot) \trianglelefteq S'(n, \cdot)$ for each fixed n almost surely given $\mathcal{F}$. It is easy to see that the inequalities $T_S > N$ and $\tau_s \geq N$ are almost surely equivalent, as are the inequalities $T_{S'} > N$ and $\tau_{s'} \geq N$. An application of Proposition 5.1 to the process $\{(n, r_n)\}$ conditioned on $\mathcal{F}$ thus implies that almost surely

$$\begin{aligned}\big(y \mapsto 1_{\{N<\infty\}} P^{(x,y)}(\tau_s \geq N \text{ and } \pi_2(R_N) \in A_N \mid \mathcal{F})\big)& \\ \trianglelefteq \big(y \mapsto 1_{\{N<\infty\}} P^{(x,y)}(\tau_{s'} \geq N \text{ and } \pi_2(R_N) \in (A_N)^{\#} \mid \mathcal{F})\big).&\end{aligned}$$

Fix $x \in X$. If $y \mapsto L(y)$ is the left hand side of the above expression and $y \mapsto R(y)$ is the right hand side, then what the above expression says is that almost surely we have

$$\int_Y L(y) g(y)\, d\mu(y) \leq \int_Y R(y) g^{\#}(y)\, d\mu(y)$$

for all positive measurable g. Taking expectations of both sides of this inequality and using Fubini's theorem we see that

$$\int_Y E^{(x,y)}[L(y)] g(y)\, d\mu(y) \leq \int_Y E^{(x,y)}[R(y)] g^{\#}(y)\, d\mu(y).$$

Hence,

$$(y \mapsto E^{(x,y)}[L(y)]) \trianglelefteq (y \mapsto E^{(x,y)}[R(y)]).$$

But

$$E^{(x,y)}[L(y)] = P^{(x,y)}(N < \infty \text{ and } \tau_s \geq N \text{ and } \pi_2(R_N) \in A_N)$$

which equals $\omega((x,y),A;s)$ as can be easily verified. Likewise, $E^{(x,y)}[R(y)] = \omega((x,y),A^{\boxminus};s')$. Hence, we have $\omega((x,\cdot),A;s) \trianglelefteq \omega((x,\cdot),A^{\boxminus};s')$ and by Proposition 2.3 we can replace "$\trianglelefteq$" by "$\preceq$" here.

Now, by Proposition 5.1, the random function $R(y)$ is almost surely an almost $\#$-symmetric function of y. It follows by Lemma 5.2 that $\omega((x,\cdot),A^{\#};s')$ is also almost $\#$-symmetric. $\square$

Proof of Theorem 3.2 in the case $c = 0$. By definition of the Green function and by Fubini's theorem, we have

$$\begin{aligned} g(z,A;s) &= E^z\left[\sum_{N=0}^{\tau_s-1} 1_{\{R_N\in A\}}\right] \\ &= E^z\left[\sum_{N=0}^{\infty} 1_{\{R_N\in A \text{ and } N<\tau_s\}}\right] \\ &= \sum_{N=0}^{\infty} P^z(R_N \in A \text{ and } N < \tau_s). \end{aligned} \tag{5.2}$$

We will obtain the desired inequality between $g((x,\cdot),A;s)$ and $g((x,\cdot),A^{\boxminus};s')$ as soon as we show that for all fixed N and x we have

$$P^{(x,\cdot)}(R_N \in A \text{ and } N < \tau_s) \trianglelefteq P^{(x,\cdot)}(R_N \in A^{\boxminus} \text{ and } N < \tau_s). \tag{5.3}$$

But much as in Theorem 3.1 (although even more easily), (5.3) follows immediately by conditioning on $\mathcal{F} = \sigma\big(\{\pi_1(R_n)\}_{n=0}^{\infty}\big)$ and applying Proposition 5.1 to the process $\{(n,r_n)\}$ given $\mathcal{F}$, where $r_n = \pi_2(R_n)$.

The almost $\#$-symmetry of the right hand side of (5.3) then follows by the same conditioning argument together with Lemma 5.2, just as in the proof of Theorem 3.1. By Lemma 5.1 and equation (5.2), the almost $\#$-symmetry of the right hand side of (5.3) for all N implies the almost $\#$-symmetry of $g((x,\cdot),A^{\boxminus};s')$. $\square$

The proof of Theorem 3.3 is even easier.

Proof of Theorem 3.3. Let $A = Y$. Put $S(n,y) = s(\pi_1(R_n),y)$ and $S'(n,y) = s'(\pi_1(R_n),y)$. Applying Proposition 5.1 to the process $\{(n,r_n)\}$ conditioned on $\mathcal{F} = \sigma\big(\{\pi_1(R_n)\}_{n=0}^{\infty}\big)$ we obtain the result as in the preceding two proofs. $\square$

Hence it remains to prove Proposition 5.1.

The following inequality is crucial here. Let κ_n be one of the measures defined by (3.1) for the process in Proposition 5.1. Note that κ_n is then an ordinary (non-random) measure since the σ-field defined by the first component of the process $\{(n, r_n)\}$ is trivial. Let f^{κ_n} and f_{κ_n} be the μ-densities of $f * \kappa_n$ and $\kappa_n * f$, respectively, as before.

Lemma 5.3. *If f is a positive #-symmetric measurable function, then f^{κ_n} and f_{κ_n} are #-symmetric.*

Proof. It is clear that if f is bounded and supported in a set of finite measure, then f^{κ_n} and f_{κ_n} are integrable if κ_n is defined by (3.1). Recall Remark 2.2. Since a general #-symmetric positive function can be approximated by bounded #-symmetric functions with support in a set of finite measure, we may assume that f is bounded and supported in a set of finite measure, and hence that f^{κ_n} and f_{κ_n} are integrable. Fix any positive measurable g. Then,

$$\begin{aligned}\int_Y f^{\kappa_n} g\, d\mu &= \int_{Y\times Y} f(x)g(y)\, d\kappa_n(x,y)\\ &\le \int_{Y\times Y} f^{\#}(x)g^{\#}(y)\, d\kappa_n(x,y)\\ &= \int_{Y\times Y} f(x)g^{\#}(y)\, d\kappa_n(x,y) = \int_Y f^{\kappa_n} g^{\#}\, d\mu,\end{aligned}$$

since our process is symmetrizable with constant 0 and as $f = f^{\#}$. By Proposition 2.2 it follows that f^{κ_n} is almost #-symmetric since it is integrable. The almost #-symmetry of f_{κ_n} is proved analogously. □

Lemma 5.4. *Suppose that f and F are positive μ-measurable functions on Y with $f \preceq F$ and F almost #-symmetric. Then, $f_{\kappa_n} \trianglelefteq F_{\kappa_n}$.*

Proof. Fix any positive measurable function g on Y. Then,

$$\begin{aligned}\int_Y f_{\kappa_n} g\, d\mu &= \int_{Y\times Y} g(x)f(y)\, d\kappa_n(x,y)\\ &\le \int_{Y\times Y} g^{\#}(x)f^{\#}(y)\, d\kappa_n(x,y)\\ &= \int_Y (g^{\#})^{\kappa_n} f^{\#}\, d\mu\\ &\le \int_Y (g^{\#})^{\kappa_n} F\, d\mu = \int_Y F_{\kappa_n} g^{\#}\, d\mu.\end{aligned}$$

Here, the first inequality followed from the symmetrizability with constant 0 of our process. The second followed from the fact that by Lemma 5.4

the function $(g^{\#})^{\kappa_n}$ is #-symmetric while by Lemma 2.1 we have $f^{\#} \trianglelefteq F$ if $f \trianglelefteq F$. From our chain of inequalities it follows that $f_{\kappa_n} \trianglelefteq F_{\kappa_n}$, as desired. $\square$

Lemma 5.5. *Suppose that $f \trianglelefteq f'$ and that $g \trianglelefteq g'$ with g' almost #-symmetric. Then, $fg \trianglelefteq f'g'$.*

Proof. Fix a positive measurable function h on Y. We have

$$\begin{aligned}\int_Y fgh\,d\mu &\le \int_Y f^{\#}g^{\#}h^{\#}\,d\mu \le \int_Y f^{\#}g'h^{\#}\,d\mu \\ &\le \int_Y f'g'h^{\#}\,d\mu.\end{aligned}$$

The first inequality here came from Proposition 2.1. The second came from the fact that $f^{\#}h^{\#}$ is almost #-symmetric by Lemma 5.1 while $g^{\#} \trianglelefteq g'$ by Lemma 2.1. The final inequality came from the facts that $f^{\#} \trianglelefteq f'$ by Lemma 2.1 and that $g'h^{\#}$ is almost #-symmetric by Lemma 5.1 as g' is almost #-symmetric. $\square$

Proof of Proposition 5.1. For convenience, we write S_n for $S(n,\cdot)$ and S'_n for $S'(n,\cdot)$. Let κ_n be our measure on $Y \times Y$ defined by

$$\int_{Y\times Y} f(x,y)\,d\kappa_n(x,y) = \int E[f(x,r_{n+1}) \mid r_n = x]\,d\mu(x).$$

Let

$$\alpha(y) = P(T_S > N \text{ and } r_N \in A \mid r_0 = y)$$

and

$$\alpha'(y) = P(T_{S'} > N \text{ and } r_N \in A \mid r_0 = y).$$

Let $g_N = S_N 1_A$ and $g'_N = S'_N 1_{A^{\#}}$. By Lemma 5.5, $g_N \trianglelefteq g'_N$ and by Lemma 5.1, g'_N is almost #-symmetric. For $0 \le n \le N-1$, if g_{n+1} has been defined, put

$$g_n = S_n(g_{n+1})_{\kappa_n}.$$

Likewise put

$$g'_n = (S'_n g'_{n+1})_{\kappa_n}.$$

By an application of Lemmas 5.1, 5.3, 5.4 and 5.5, we see that if $g_{n+1} \trianglelefteq g'_{n+1}$ and g'_{n+1} is almost #-symmetric, then $g_n \trianglelefteq g'_n$ and g'_n is almost #-symmetric. By iteration, $g_0 \trianglelefteq g'_0$ and g'_0 is almost #-symmetric.

I now claim that $\alpha = g_0$ and $\alpha' = g_0'$. If this claim is correct then we are done. It suffices to show that $\alpha = g_0$ since the other equality is analogous. Let $k_{n,y}$ be the measure $P(r_{n+1} \in \cdot \mid r_n = y)$. Then,

$$\alpha(y_0) = S_0(y_0) \int_Y dk_{0,y_0}(y_1)\, S_1(y_1) \int_Y dk_{1,y_1}(y_2) \\ \cdots S_{N-1}(y_{n-1}) \int_Y dk_{N-1,y_{N-1}}(y_N) S_N(y_N) 1_A(y_N).$$

The innermost integrand then equals $g_N(y_N)$. By (3.2), the innermost integral is $(g_N)_{\kappa_{N-1}}(y_{N-1})$. Thus, $g_{N-1}(y_{N-1})$ is equal to the integrand of the second-innermost integral (i.e., of the integral with respect to y_{N-1}). Iterating these observations we may conclude that $\alpha(y_0) = g_0(y_0)$, as desired. $\square$

References

[1] Baernstein II, A., *Integral means, univalent functions and circular symmetrization*, Acta Math. **133** (1974), 139–169.

[2] ———, *A unified approach to symmetrization*, In Alvino, A., et al. (eds.) Partial Differential Equations of Elliptic Type, Symposia Mathematica, vol. 35, Cambridge University Press, Cambridge, 1994, pp. 47–91.

[3] Baernstein II, A., B.A. Taylor, *Spherical rearrangements, subharmonic functions and $*$-functions in n-space*, Duke Math. J. **43** (1976), 245–268.

[4] Beckner, W., *Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality*, Ann. Math. **138** (1993), 213–242.

[5] Beurling, A., *Études sur un Problème de Majoration*, Thèse pour le doctorat, Almqvist & Wiksell, Uppsala, 1933.

[6] Borell, C., *An inequality for a class of harmonic functions in n-space.*, In Essén, M., The $\cos \pi\lambda$ Theorem, Lecture Notes in Mathematics, vol. 467, Springer-Verlag, New York, 1975, pp. 99–112.

[7] Brascamp, H.J., E.H. Lieb and J.M. Luttinger, *A general rearrangement inequality for multiple integrals*, J. Funct. Anal. **17** (1974), 227–237.

[8] Gallot, S., *Inégalités isopérimétriques et analytiques sur les variétés Riemanniannes*, Astérisque **163–164** (1988), 31–91.

[9] Haliste, K., *Estimates of harmonic measures*, Ark. Mat. **6** (1965), 1–31.

[10] Hardy, G.H., J.E. Littlewood and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1964.

[11] Kawohl, B., *Rearrangements and Convexity of Level Sets in PDE*, Lecture Notes in Mathematics, vol. 1150, Springer Verlag, Berlin/Heidelberg, 1985.

[12] Kesten, H., *Hitting probabilities of random walks on $\mathbb{Z}^d$*, Stochastic Processes and their Applications **25** (1987), 165–184.

[13] Nevanlinna, R., *Analytic Functions*, Springer Verlag, New York, 1970.

[14] Pruss, A.R., *Symmetrization inequalities for difference equations on graphs*, Adv. Appl. Math. **22** (1999), 338–370.

[15] ———, *Discrete convolution-symmetrization inequalities and the Faber-Krahn inequality on regular trees*, Duke Math. J. **91** (1998), 463–514.

[16] Pruss, A.R., *One-dimensional random walks, decreasing rearrangements and discrete Steiner symmetrization*, Ann. Inst. H. Poincaré Probab. Statist. **33** (1997), 83–112.

[17] Quine, J.R., *Symmetrization inequalities for discrete harmonic functions*, Preprint (undated).

[18] Riesz, F., *Sur une inégalité intégrale*, J. London Math. Soc. **5** (1930), 162–168.

[19] Steiner, J., *Einfache Beweise der isoperimetrischen Hauptsätze*, J. Reine Angew. Math. **18** (1838), 281–296; Reprinted in: Steiner, J., *Gesammelte Werke*, vol. 2, Reimer, Berlin, 1882, pp. 75–91.

Department of Philosophy
University of Pittsburgh
Pittsburgh, PA 15260, U.S.A.
e-mail: pruss+@pitt.edu

received August 16, 2000