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WŁODZIMIERZ M. MIKULSKI

Liftings of 1-forms to the bundle of affinors

Dedicated to Professor Ivan Kolář
on the occasion of his 65-th birthday

Abstract. All natural operators $T^* \rightsquigarrow T^*(T \otimes T^*)$ over n-manifolds are described. Non-existence of canonical volume forms on some natural bundles is deduced.

We study how a 1-form ω on a n-manifold M induces a 1-form $B(\omega)$ on $TM \otimes T^*M$. This problem is reflected in natural operators $T^* \rightsquigarrow T^*(T \otimes T^*)$ over n-manifolds, [5]. Using the results from [1] and [2], we prove that the set of natural operators $T^* \rightsquigarrow T^*(T \otimes T^*)$ over n-manifolds is a free $C^\infty(\mathbf{R}^n)$-module. We construct a basis of this module. See [6] (or [3]), for the similar problem with T (or T^*) instead of $T \otimes T^*$.

We deduce non-existence of canonical volume forms (simplectic, cosimplectic, contact structures) on some natural bundles, e.g. $T \otimes T^*$.

From now on $\pi : TM \otimes T^*M \to M$ is the bundle projection for every n-manifold M, $x^1, ..., x^n$ are the usual coordinates on $\mathbf{R}^n$ and $\partial_i = \frac{\partial}{\partial x^i}$ for $i = 1, ..., n$.

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All manifolds and maps are assumed to be of class C^∞.

I. The natural operators $T^* \rightsquigarrow T^*(T \otimes T^*)$. If $L : V \to V$ is an endomorphism of an n-dimensional vector space V then $a_1(L), ..., a_n(L)$ denote the coefficients of the characteristic polynomial

$$W_L(\lambda) = \det(\lambda\, \mathrm{id}_V - L) = \lambda^n + a_1(L)\lambda^{n-1} + ... + a_{n-1}(L)\lambda + a_n(L)\ .$$

1. Example. For every n-manifold M we have $a_1, ..., a_n : TM \otimes T^*M \to \mathbf{R}$ (as $T_xM \otimes T^*_xM = \mathrm{End}(T_xM)$) and $da_1, ..., da_n \in \Omega^1(TM \otimes T^*M)$. $da_1, ..., da_n : T^* \rightsquigarrow T^*(T \otimes T^*)$ are constant natural operators over n-manifolds.

2. Example. For every n-manifold M, $\omega \in \Omega^1(M)$ and $i = 1, ..., n$ let $B^{(i)}(\omega) \in \Omega^1(TM \otimes T^*M)$, $B^{(i)}(\omega)_\tau = \omega_x \circ \tau^{n-i} \circ T_\tau \pi$, $\tau \in T_xM \otimes T^*_xM$, $x \in M$, $\tau^{n-i} = \tau \circ ... \circ \tau$ $(n-i)$-times. $B^{(1)}, ..., B^{(n)} : T^* \rightsquigarrow T^*(T \otimes T^*)$ are nat. operators over n-manifolds.

The set of natural operators $T^* \rightsquigarrow T^*(T \otimes T^*)$ over n-manifolds is a $C^\infty(\mathbf{R}^n)$-module: $(fB)(\omega) = f((a_i)_{i=1}^n)B(\omega)$, $B : T^* \rightsquigarrow T^*(T \otimes T^*)$, $\omega \in \Omega^1(M)$, $f \in C^\infty(\mathbf{R}^n)$.

3. Theorem. *The operators da_i and $B^{(i)}$ for $i = 1, ..., n$ form a basis of the $C^\infty(\mathbf{R}^n)$-module of natural operators $T^* \rightsquigarrow T^*(T \otimes T^*)$ over n-manifolds.*

Proof. Let $B : T^* \rightsquigarrow T^*(T \otimes T^*)$ be a natural operator over n-manifolds. We have to show that B is the linear combination of the operators da_i and $B^{(i)}$ for $i = 1, ..., n$ with uniquely determined coefficients from $C^\infty(\mathbf{R}^n)$. □

4. Lemma. *$B(\omega) = B(0)$ on $V(TM \otimes T^*M)$ for $\omega \in \Omega^1(M)$.*

Proof. Let $v \in V_\tau(T\mathbf{R}^n \otimes T^*\mathbf{R}^n)$, $\tau \in T_0\mathbf{R}^n \otimes T^*_0\mathbf{R}^n$, $\omega \in \Omega^1(\mathbf{R}^n)$. Since $t\,\mathrm{id}_{\mathbf{R}^n}$ for $t \neq 0$ preserve v, $B((t\,\mathrm{id}_{\mathbf{R}^n})^*\omega)_\tau(v) = B(\omega)_\tau(v)$. If $t \to 0$, $B(\omega)_\tau(v) = B(0)_\tau(v)$. □

5. Lemma. *There exist the maps $f_i \in C^\infty(\mathbf{R}^n)$ such that $B(\omega)_\tau((v^o)^C_\tau) = \sum_{i=1}^n f_i(a_1(\tau), ..., a_n(\tau))\omega_0(\tau^{n-i}(v))$ for $\tau \in T_0\mathbf{R}^n \otimes T^*_0\mathbf{R}^n$, $\omega \in \Omega^1(\mathbf{R}^n)$, $v \in T_0\mathbf{R}^n$, where v^o is the constant vector field on $\mathbf{R}^n$ with $v^o_0 = v$ and $(v^o)^C$ its complete lifting.*

Proof. Consider $\tau \in T_0\mathbf{R}^n \otimes T^*_0\mathbf{R}^n$. Given $\omega \in \Omega(\mathbf{R}^n)$, by [5], $B(\omega)_\tau$ depends only on $j^r_0\omega$ for some finite $r = r(\tau)$. Define $\overline{B}_\tau : (J^r_0T^*\mathbf{R}^n) \times T_0\mathbf{R}^n \to \mathbf{R}$, $\overline{B}_\tau(j^r_0\omega, v) = B(\omega)_\tau((v^o)^C_\tau)$, $\omega \in \Omega^1(\mathbf{R}^n)$, $v \in T_0\mathbf{R}^n$. By the invariance of B with respect to $t\,\mathrm{id}_{\mathbf{R}^n}$ for $t \in \mathbf{R}_+$ and the homogeneous function theorem, [5], $\overline{B}_\tau$ depends linearly on (ω_0, v).

So, we can define $\tilde{B} : T_0\mathbf{R}^n \otimes T_0^*\mathbf{R}^n \to T_0\mathbf{R}^n \otimes T_0^*\mathbf{R}^n$, $\tilde{B}(\tau)(\omega_0, v) = B(\omega)_\tau((v^o)^C_\tau)$, $\omega \in \Omega^1(\mathbf{R}^n)$, $v \in T_0\mathbf{R}^n$. By the $GL(\mathbf{R}^n)$-invariance of B, $\tilde{B}$ is $GL(\mathbf{R}^n)$-equivariant. Then, by Proposition 2.2 in [1], there exist the maps $f_i \in C^\infty(\mathbf{R}^n)$ with $\tilde{B}(\tau) = \sum_{i=1}^n f_i(a_1(\tau), ..., a_n(\tau))\tau^{n-i}$ for $\tau \in T_0\mathbf{R}^n \otimes T_0^*\mathbf{R}^n$. □

Replacing B by $B - \sum_{i=1}^n f_i B^{(i)}$, we can assume that B is constant.

6. Lemma. *There exist the maps $g_i \in C^\infty(\mathbf{R}^n)$ with $B = \sum_{i=1}^n g_i da_i$.*

Proof. Since B is constant, we can define new natural operator $B^o : T \otimes T^* \rightsquigarrow T^*$ such that $B^o(\tau) = \tau^* B$ for any tensor field τ on an n-manifold M.

By Theorem 2.2 in [2], there exist the maps $g_{ij} \in C^\infty(\mathbf{R}^n)$ for $i, j = 1, ..., n$ such that $B^o(\tau) = \sum_{i,j=1}^n g_{ij}(a_1(\tau), ..., a_n(\tau))d(a_i(\tau)) \circ \tau^{n-j}$ for any τ as above.

Since the correspondence $B \to B^o$ is injective, it remains to verify that $g_{ij} = 0$ for $i = 1, ..., n$ and $j = 1, ..., n-1$. We can assume $n \geq 2$.

Consider $i_o = 1, ..., n$ and $j_o = 1, ..., n-1$ and $b = (b_1, ..., b_n) \in \mathbf{R}^n$. Let $A \in gl(n)$ be such that $Ae_i = e_{i+1}$ for $i = 1, ..., n-1$ and $Ae_n = -b_n e_1 - ... - b_1 e_n$.

Let τ be the tensor field on $\mathbf{R}^n$ of type $(1,1)$ such that $\tau_x \in \mathrm{End}(T_x\mathbf{R}^n)$ has matrix A with respect to $\partial_{1|x}, ..., \partial_{n|x}$ for $x \in \mathbf{R}^n$. Then $a_i(\tau) = b_i$, $i = 1, ..., n$.

Let η be the tensor field on $\mathbf{R}^n$ of type $(1,1)$ such that η_x has matrix $A - x^{n-j_o+1}E_{n-i_o+1,n}$ for any $x = (x^1, ..., x^n) \in \mathbf{R}^n$, where $E_{k,l} \in gl(n)$ has 1 in the (k,l) position and 0 in other positions. Then $a_i(\eta) = b_i + \delta^i_{i_o} x^{n-j_o+1}$ for $i = 1, ..., n$.

Clearly $T\eta(\partial_{1|0}) = T\tau(\partial_{1|0})$ and $d(a_i(\tau)) = db_i = 0$ for $i = 1, ..., n$. Then $B^o(\eta)_0(\partial_{1|0}) = B^o(\tau)_0(\partial_{1|0}) = 0$. Then, since $(\eta_0)^{n-j}(\partial_{1|0}) = \partial_{n-j+1|0}$ for $j = 1, ..., n$, $\sum_{i,j=1}^n g_{ij}(b)d(a_i(\eta))(\partial_{n-j+1|0}) = 0$. Since $\frac{\partial}{\partial x^{n-j+1}}(a_i(\eta)) = \delta^i_{i_o}\delta^{n-j+1}_{n-j_o+1}$, $g_{i_o j_o}(b) = 0$.

The proof of Theorem 3 is complete. □

7. Corollary. *The operators $B^{(i)}$ (or da_i) for $i = 1, ..., n$ form a basis of the $C^\infty(\mathbf{R}^n)$-module of linear natural operators $T^* \rightsquigarrow T^*(T \otimes T^*)$ (or canonical 1-forms on $T \otimes T^*$) over n-manifolds.*

8. Remark. If $n \geq 2$, there are non-zero canonical 2-forms on $T \otimes T^*$ (for example, $da_i \wedge da_j$ for $1 \leq i < j \leq n$) but there are no canonical simplectic structures because there are no canonical volume forms, see Proposition 10 (c).

II. Non-existence of canonical volume forms on some natural bundles.

9. Lemma. *Let F be a natural bundle over n-manifolds. Suppose there is $v_o \in F_0\mathbf{R}^n$ with $F(t\,\mathrm{id}_{\mathbf{R}^n})(v_o) = v_o$ for all $t \in \mathbf{R}$ and there are a basis $u_1, ..., u_k$ of $V_{v_o}F\mathbf{R}^n$ and $a_1, ..., a_k \in \mathbf{R}$ with $TF(t\,\mathrm{id}_{\mathbf{R}^n})(u_j) = t^{a_j}u_j$ for $j = 1, ..., k$ and $t \in \mathbf{R}_+$. Suppose also that there is a canonical volume form Ω on F. Then $n + \sum_{j=1}^{k} a_j = 0$.*

Proof. Using basis $\partial^C_{1\,|v_o}, ..., \partial^C_{n\,|v_o}, u_1, ..., u_k$ of $T_{v_o}F\mathbf{R}^n$ we see that $t\,\mathrm{id}_{\mathbf{R}^n}$ maps Ω_{v_o} into $t^{-(n+\sum_{j=1}^{k} a_j)}\Omega_{v_o}$. On the other hand, $t\,\mathrm{id}_{\mathbf{R}^n}$ preserves Ω_{v_o} and $\Omega_{v_o} \neq 0$. □

10. Proposition. *There are no canonical volume forms on:*

(a) $(F_{|\mathcal{M}f_n})^*$ *for every bundle functor* $F : \mathcal{M}f \to \mathcal{VB}$ *with* $\dim(F_0\mathbf{R}^n) - \dim(F\mathbf{R}^0) \geq n + 1$ *(in particular, on* $T^{r*}_p = J^r(.,\mathbf{R}^p)_0$ *for* $p, r \in \mathbf{N}$ *with* $(p, r) \neq (1, 1)$*, on* J^rT^* *for* $r \in \mathbf{N}$ *and on* Λ^pT^* *for* $n \geq 4$ *and* $p = 2, ..., n - 2$*);*

(b) $\Lambda^{n-1}T^*$ *for* $n \geq 3$*;*

(c) $\otimes^pT \otimes \otimes^qT^*$ *for* $p, q \in \mathbf{N} \cup \{0\}$ *with* $(p, q) \neq (0, 1)$ *if* $n \geq 2$ *and* $(p, q) \neq (p, p + 1)$ *if* $n = 1$*;*

(d) $\otimes^p(\Lambda^nT^*)$ *for* $p \geq 2$*.*

Proof. ad (a) Put $v_o = 0 \in (F_0\mathbf{R}^n)^*$. There are a basis $u_1, ..., u_k$ of $V_{v_o}(F\mathbf{R}^n)^*$ and $a_1, ..., a_k \in \{0, -1, ...\}$ with $T((F)^*(t\,\mathrm{id}_{\mathbf{R}^n}))(u_j) = t^{a_j}u_j$ for $j = 1, ..., k$ and $t > 0$, [4]. We see $\mathrm{card}\{j \mid a_j = 0\} = \dim(F\mathbf{R}^0)$. So, $n + \sum_{j=1}^{k} a_j < 0$. Apply Lemma 9.

ad (b)—(d) Consider $v_o = 0$ over $0 \in \mathbf{R}^n$ and the obvious bases $u_1, ..., u_k$ of $F_0\mathbf{R}^n \tilde{=} V_{v_o}F\mathbf{R}^n$ for $F = \otimes^pT\otimes\otimes^qT^*$, $\Lambda^{n-1}T^*$, $\otimes^p(\Lambda^nT^*)$. Find $a_1, ..., a_k$ with $F(t\,\mathrm{id}_{\mathbf{R}^n})(u_j) = t^{a_j}u_j$ for $j = 1, ..., k$ and $t \in \mathbf{R}_+$ and apply Lemma 9. □

11. Remark. (a) There is the well-known simplectic structure (and the volume form) on T^*. If $(p, r) = (1, 1)$, $T^{r*}_p \tilde{=} T^*$. If $n = 1$, $\otimes^pT\otimes\otimes^{p+1}T^* \tilde{=} T^*$ by a contraction.

(b) We have a volume form $d\Theta$ on Λ^nT^*, $\Theta_\omega(v_1, ..., v_n) = \omega(T\pi(v_1), ..., T\pi(v_n))$, $v_1, ..., v_n \in T_\omega\Lambda^nT^*M$, $\omega \in \Lambda^nT^*_xM$, $x \in M$.

(c) If $F : \mathcal{M}f \to \mathcal{VB}$ is a bundle functor with $\dim(F_0\mathbf{R}^n) - \dim(F\mathbf{R}^0) = n$ ($< n$), then a volume form on $(F_{|\mathcal{M}f_n})^*$ can exist or not, see (a) and 10(b) ((b) and 10(d)).

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Institute of Mathematics
Jagiellonian University
Reymonta 4, 30-059 Kraków, Poland
e-mail: mikulski@im.uj.edu.pl

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