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An estimate of the growth of spirallike mappings relative to a diagonal matrix

Abstract. In this paper we give an upper estimate of the growth of a component of normalized spirallike mappings relative to a diagonal matrix on the Euclidean unit ball.

1. Introduction. Let f be a univalent mapping in the unit disc Δ with $f(0) = 0$ and $f'(0) = 1$. Then the classical growth theorem is as follows:

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}.$$

It is well known that the above growth theorem cannot be generalized to normalized biholomorphic mappings on the Euclidean unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ $(n \geq 2)$. Barnard, FitzGerald and Gong [1] and Chuaqui [2] extended the above growth theorem to normalized starlike mappings on $\mathbb{B}^n$. Dong and Zhang [3] generalized the above result to normalized starlike mappings on the unit ball in complex Banach spaces.

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It is interesting to consider the possibility of extending the above growth theorem to a larger family of biholomorphic mappings. Hamada and Kohr [5] generalized the above growth theorem to spirallike mappings of type α on the unit ball $\mathbb{B}$ in an arbitrary complex Banach space and gave an example of a normalized spirallike mapping in the sense of Suffridge [8] such that the same growth theorem does not hold. This example also shows that the growth of normalized spirallike mappings cannot be estimated from above.

In this paper we give an upper estimate of the growth of a component of spirallike mappings relative to a diagonal matrix.

2. Preliminaries. For complex Banach spaces X, Y, let $\mathcal{L}(X,Y)$ be the space of all continuous linear operators from X into Y with the standard operator norm. By I we denote the identity in $\mathcal{L}(X,X)$. Let G be a domain in X and let $f : G \to Y$. The mapping f is said to be holomorphic on G, if for any $z \in G$, there exists $Df(z) \in \mathcal{L}(X,Y)$ such that

$$\lim_{h\to 0} \frac{\|f(z+h) - f(z) - Df(z)h\|}{\|h\|} = 0.$$

Let $\mathcal{H}(G)$ be the set of holomorphic mappings from a domain $G \subset X$ into X.

A mapping $f \in \mathcal{H}(G)$ is said to be locally biholomorphic on G if its Fréchet derivative $Df(z)$ as an element of $\mathcal{L}(X,X)$ is nonsingular at each $z \in G$. A mapping $f \in \mathcal{H}(G)$ is said to be biholomorphic on G if $f(G)$ is open in X, the inverse f^{-1} exists and is holomorphic on $f(G)$. Let $\mathbb{B}$ denote the unit ball with respect to the norm $\|\cdot\|$ on X. A mapping $f \in \mathcal{H}(\mathbb{B})$ is said to be normalized if $f(0) = 0$ and $Df(0) = I$. For each $z \in X \setminus \{0\}$, we define

$$T(z) = \{z^* \in \mathcal{L}(X,\mathbb{C}) : \|z^*\| = 1, z^*(z) = \|z\|\}.$$

By the Hahn-Banach theorem, $T(z)$ is nonempty. Let

$$\mathcal{N} = \{g \in \mathcal{H}(\mathbb{B}) \,:\, g(0) = 0,\ \Re z^*(g(z)) > 0 \text{ for all } z \in \mathbb{B} \setminus \{0\}, z^* \in T(z)\}.$$

For $h \in \mathcal{N}$, let

$$k = k(Dh(0)) = \inf\{\Re z^*(Dh(0)z) : \|z\| = 1, z^* \in T(z)\}$$

and

$$m = m(Dh(0)) = \sup\{\Re z^*(Dh(0)z) : \|z\| = 1, z^* \in T(z)\}.$$

By Lemma 4 of Gurganus [4],

$$\frac{1-\|z\|}{1+\|z\|}\Re z^*(Dh(0)z) \le \Re z^*(h(z)) \le \frac{1+\|z\|}{1-\|z\|}\Re z^*(Dh(0)z) \tag{2.1}$$

for all $z \in \mathbb{B} \setminus \{0\}$, $z^* \in T(z)$. Therefore, $0 \le k \le m < \infty$.

From (2.1), we obtain the following lemma (cf. [7, Lemma 2.2]).

Lemma 2.1. *Let $h \in \mathcal{N}$ and let $z \in \mathbb{B} \setminus \{0\}$. If $v(t) = v(z,t)$ is a solution to the initial value problem*

$$\frac{\partial v}{\partial t} = -h(v), \quad v(0) = z$$

defined for all $t \geq 0$, then

$$\frac{\|v(z,t)\|}{(1-\|v(z,t)\|)^2} \leq e^{-kt} \frac{\|z\|}{(1-\|z\|)^2} \tag{2.2}$$

and

$$e^{-mt} \frac{\|z\|}{(1+\|z\|)^2} \leq \frac{\|v(z,t)\|}{(1+\|v(z,t)\|)^2} \tag{2.3}$$

holds for all $t \geq 0$.

Proof. From (2.1), we have

$$k\frac{1-\|z\|}{1+\|z\|}\|z\| \leq \Re z^*(h(z)) \leq m\frac{1+\|z\|}{1-\|z\|}\|z\| \tag{2.4}$$

for all $z \in \mathbb{B} \setminus \{0\}$, $z^* \in T(z)$. By the uniqueness of the solution, $v(t) \neq 0$ for all $t \geq 0$. For any t, t' with $0 \leq t < t'$,

$$\begin{aligned} \big| \|v(t)\| - \|v(t')\| \big| &\leq \|v(t) - v(t')\| \\ &= \left\| \int_t^{t'} \frac{dv(\tau)}{d\tau} d\tau \right\| \\ &\leq \int_t^{t'} \left\| \frac{dv(\tau)}{d\tau} \right\| d\tau \\ &= \int_t^{t'} \|-h(v(\tau))\| \, d\tau. \end{aligned}$$

Since $\|h(v(t))\|$ is continuous for $t \geq 0$, it follows that $\|v(t)\|$ is absolutely continuous. Therefore, $\|v(t)\|$ is differentiable a.e. on $[0,\infty)$ and

$$\frac{\partial \|v\|}{\partial t} = \Re v^* \left(\frac{\partial v}{\partial t} \right)$$

for $v^* \in T(v(t))$ a.e. on $[0,\infty)$ by Lemma 1.3 of Kato [6]. Then

$$\frac{\partial \|v\|}{\partial t} = -\Re v^* \left(h(v) \right)$$

for $v^* \in T(v(t))$. Therefore, from (2.4), we have

$$-m\frac{1+\|v(t)\|}{1-\|v(t)\|}\|v(t)\| \leq \frac{\partial\|v\|}{\partial t} \leq -k\frac{1-\|v(t)\|}{1+\|v(t)\|}\|v(t)\|$$

for all $t \geq 0$. Then

$$\int_0^t \frac{1+\|v(t)\|}{\|v(t)\|(1-\|v(t)\|)}\frac{\partial\|v\|}{\partial t}dt \leq \int_0^t -kdt$$

and

$$\int_0^t -mdt \leq \int_0^t \frac{1-\|v(t)\|}{\|v(t)\|(1+\|v(t)\|)}\frac{\partial\|v\|}{\partial t}dt.$$

Since

$$\begin{aligned}\int_0^t \frac{1+\|v(t)\|}{\|v(t)\|(1-\|v(t)\|)}\frac{\partial\|v\|}{\partial t}dt &= \int_{\|z\|}^{\|v(t)\|} \frac{1+x}{x(1-x)}dx \\ &= \log\|v(t)\| - 2\log(1-\|v(t)\|) \\ &- \{\log\|z\| - 2\log(1-\|z\|)\}\end{aligned}$$

and

$$\begin{aligned}\int_0^t \frac{1-\|v(t)\|}{\|v(t)\|(1+\|v(t)\|)}\frac{\partial\|v\|}{\partial t}dt &= \int_{\|z\|}^{\|v(t)\|} \frac{1-x}{x(1+x)}dx \\ &= \log\|v(t)\| - 2\log(1+\|v(t)\|) \\ &- \{\log\|z\| - 2\log(1+\|z\|)\},\end{aligned}$$

we obtain the inequalities (2.2) and (2.3). This completes the proof. □

The following definition is due to Suffridge [8].

Definition 2.1. Let $f : \mathbb{B} \to X$ be a normalized biholomorphic mapping. Let $A \in \mathcal{L}(X, X)$ be such that

$$\inf\{\Re z^*(A(z)) : \|z\| = 1, z^* \in T(z)\} > 0. \tag{2.5}$$

We say that f is spirallike relative to A if $e^{-tA}f(\mathbb{B}) \subset f(\mathbb{B})$ for all $t \geq 0$, where

$$e^{-tA} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}t^k A^k.$$

Lemma 2.2. *Suppose $A \in \mathcal{L}(X,X)$ satisfies the condition* (2.5)*. Let $f : \mathbb{B} \to X$ be a normalized spirallike mapping relative to A and let $v(z,t) = f^{-1}(e^{-tA}f(z))$. Then, for any $z \in \mathbb{B} \setminus \{0\}$ and $\varepsilon > 0$, there exists $t_0 > 0$ such that*

$$\|f(v(z,t)) - v(z,t)\| \le e^{-kt}\varepsilon$$

for all $t \ge t_0$.

Proof. Let $z \in \mathbb{B} \setminus \{0\}$ and $\varepsilon > 0$ be fixed. Since f is a normalized holomorphic mapping, there exists a neighborhood $U = U(\varepsilon)$ of 0 such that

$$\|f(y) - y\| \le \varepsilon \|y\| \frac{(1-\|z\|)^2}{\|z\|}$$

for $y \in U$. Let

$$h(z) = [Df(z)]^{-1}Af(z).$$

Since f is spirallike relative to A, $h \in \mathcal{N}$ by Theorem 11 of Suffridge [8]. Also,

$$\frac{\partial v}{\partial t} = -h(v), \quad v(z,0) = z.$$

Since $Dh(0) = A$, $k > 0$ by (2.5). Then, by (2.2), there exists a $t_0 > 0$ such that $v(z,t) \in U$ for all $t \ge t_0$. Therefore,

$$\|f(v(z,t)) - v(z,t)\| \le \varepsilon \|v(z,t)\| \frac{(1-\|z\|)^2}{\|z\|} \le e^{-kt}\varepsilon$$

for all $t \ge t_0$ from (2.2). This completes the proof. $\square$

3. The growth theorem. Let $\|\cdot\|$ be the Euclidean norm on $\mathbb{C}^n$ and let $\mathbb{B}^n$ be the Euclidean unit ball in $\mathbb{C}^n$. Hamada-Kohr [5] gave an example of a normalized spirallike mapping relative to a diagonal matrix such that its growth cannot be estimated from above. Consider the following

Example. Let

$$f(z_1, z_2) = (z_1, z_2 + az_1^2)$$

on the Euclidean unit ball in $\mathbb{C}^2$. Let A be a linear mapping such that

$$A(z_1, z_2) = (z_1, 2z_2).$$

Then $[Df(z)]^{-1}Af(z_1, z_2) = (z_1, 2z_2)$. Therefore, f is a normalized spirallike mapping relative to A for any $a \in \mathbb{C}$ by Theorem 11 of Suffridge [8]. Let $z^0 = (1/2, 0)$. Then $f(z^0) = (1/2, a/4)$ and $\|f(z^0)\| \to \infty$ as $a \to \infty$. Therefore, the growth of normalized spirallike mappings cannot be estimated from above.

Using Lemmas 2.1 and 2.2, we obtain the following

Theorem 3.1. *Let $A = (a_i^j)$ be a diagonal matrix. Assume that*

$$0 < \Re a_1^1 = \cdots = \Re a_l^l < \Re a_{l+1}^{l+1} \leq \ldots \leq \Re a_n^n.$$

Let $f : \mathbb{B}^n \to \mathbb{C}^n$ be a normalized spirallike mapping relative to A. Then

$$\|(f_1, \ldots, f_l, 0, \ldots, 0)\| \leq \frac{\|z\|}{(1 - \|z\|)^2}.$$

Proof. Let $M = (m_i^j)$ be the diagonal matrix such that $m_i^i = a_i^i$ for $1 \leq i \leq l$ and $m_i^i = a_1^1$ for $l + 1 \leq i \leq n$. Let $z \in \mathbb{B} \setminus \{0\}$ be fixed. Then $e^{tM} f(v(z, t)) = e^{t(M-A)} f(z)$ tends to $(f_1(z), \ldots, f_l(z), 0, \ldots, 0)$ as $t \to \infty$. Let $\varepsilon > 0$ be fixed. Then there exists a $t_0 > 0$ such that

$$\|f(v(z, t)) - v(z, t)\| \leq e^{-kt}\varepsilon$$

for all $t \geq t_0$ by Lemma 2.2. We have

$$\begin{aligned}
\|e^{tM}(f(v(z, t)) - v(z, t))\| &\leq \|e^{tM}\| \|f(v(z, t)) - v(z, t)\| \\
&= \|e^{\Re tM}\| \|f(v(z, t)) - v(z, t)\| \\
&\leq e^{kt} e^{-kt}\varepsilon \\
&= \varepsilon.
\end{aligned}$$

Consequently, $e^{tM} v(z, t)$ tends to $(f_1(z), \ldots, f_l(z), 0, \ldots, 0)$ as $t \to \infty$. Since

$$\|e^{tM} v(z, t)\| \leq e^{kt} \|v(z, t)\| \leq \frac{\|z\|}{(1 - \|z\|)^2}$$

by Lemma 2.1, the theorem follows. $\square$

By the same argument as in the proof of the above theorem, we obtain the following corollaries.

Corollary 3.1. *Let A be a normal matrix and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A. Assume that $\Re\lambda_1 = \cdots = \Re\lambda_n > 0$. Let $f : \mathbb{B}^n \to \mathbb{C}^n$ be a normalized spirallike mapping relative to A. Then*

$$\frac{\|z\|}{(1 + \|z\|)^2} \leq \|f(z)\| \leq \frac{\|z\|}{(1 - \|z\|)^2}.$$

Corollary 3.2. *Let $\mathbb{B}$ be the unit ball in an arbitrary complex Banach space X. Let $A = aI$ with $\Re a > 0$. Let $f : \mathbb{B} \to X$ be a normalized spirallike mapping relative to A. Then*

$$\frac{\|z\|}{(1 + \|z\|)^2} \leq \|f(z)\| \leq \frac{\|z\|}{(1 - \|z\|)^2}.$$

Corollary 3.2 was obtained by Hamada-Kohr [5]. Also, if we put $a = 1$ in Corollary 3.2, we obtain the growth theorem of normalized starlike mappings ([1], [2], [3]).

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