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Torsions of connections on higher order tangent bundles

Dedicated to Professor Ivan Kolář
on the occasion of his 65-th birthday

Abstract. The torsion of a connection on a natural bundle is defined as the Frölicher-Nijenhuis bracket of some natural affinor and the connection itself. Using the complete description of all natural affinors on r-th order tangent bundles, we determine all torsions of connections on such bundles.

1. Introduction. Let F be a natural bundle on the category $\mathcal{M}f_m$ of all m-dimensional manifolds and their local diffeomorphisms and $\Gamma : FM \to J^1FM$ be a general connection on the fibered manifold $FM \to M$. Denoting by $Q : TFM \to TFM$ an arbitrary natural tangent valued one-form (in other words an affinor) on FM, the Frölicher-Nijenhuis bracket $\tau := [\Gamma, Q]$ of Γ and Q is called a (general) torsion of Γ. Using such a point of view,

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the complete list of all natural affinors on FM induces the list of all general torsions of connections on FM.

In this paper we describe all natural affinors and all general torsions on the r-th order tangent bundle $T^{(r)}M$. For the definition of $T^{(r)}M$ see Chapter 3. Roughly speaking, higher order tangent bundles are used in generalizations of geometric constructions which depend on partial derivatives of higher order. An example of the second order construction is the osculating plane to a space curve. Then Pohl defined general osculating spaces of order r, which are also called r-th order tangent bundles, [20]. Obviously, the first order tangent bundle $T^{(1)}M$ is the well known tangent bundle TM of a manifold M. Such an approach has been used also in [6]. We point out that higher order tangent bundles have many applications also in higher order mechanics. For example, in the theory of gross higher order bodies, it is often necessary to consider higher order tensor fields with values in tensor products of higher order tangent and cotangent spaces, [19]. We remark that connections on higher order tangent bundles were studied e.g. by Pohl [21], Klein [5] and Cenkl [1].

All manifolds and maps are assumed to be infinitely differentiable. In what follows we will use the terminology and notations from [8].

2. Torsions of a general connection. The concept of a torsion was primarily introduced for a classical linear connection Γ on a manifold M. Such a connection can be equivalently considered as a linear connection on the tangent bundle TM of M and also as a principal connection on the frame bundle PM of M. Using the first interpretation of Γ as a linear connection on TM, we can define the torsion of Γ as the covariant exterior differential in the sense of Koszul of the identity tensor of M,

$$\tau(X,Y) := \bigtriangledown_\Gamma \mathrm{Id}_{TM}(X,Y)$$

which yields the classical formula

$$\tau(X,Y) = \frac{1}{2}\left(\bigtriangledown_X Y - \bigtriangledown_Y X - [X,Y]\right)$$

for every vector fields X, Y on M. Taking into account the second interpretation of Γ, we can take the canonical $\mathbb{R}^m$-valued form $\theta : TPM \to \mathbb{R}^m$ of PM, $m = \dim M$ and introduce the torsion of Γ as the standard covariant differential of θ,

$$\tau(U,V) := d\theta(hU, hV)$$

where h means the horizontal projection of Γ and d denotes the exterior differential operator.

Now let $p : Y \to M$ be an arbitrary fibered manifold and $\Gamma : Y \to J^1Y$ a (general) connection on Y. Recall that an *affinor* on Y is defined as a linear morphism $TY \to TY$ over the identity of Y. By the Frölicher-Nijenhuis theory, affinors on Y are exactly tangent valued one-forms on Y, i.e. sections from $C^\infty(TY \otimes T^*Y)$. Denoting by (x^i, y^p) the canonical coordinates on Y, the coordinate form of an affinor φ on Y is

$$(dx^i, dy^p) \mapsto (\varphi^i_j dx^j + \varphi^i_p dy^p, \varphi^p_i dx^i + \varphi^p_q dy^q).$$

An affinor φ on Y is called *vertical*, if φ has values in the vertical tangent bundle, i.e. $\varphi \in C^\infty(VY \otimes T^*Y)$, in coordinates

$$(dx^i, dy^p) \mapsto (0, \varphi^p_i dx^i + \varphi^p_q dy^q).$$

Finally, using the canonical inclusion $T^*M \subset T^*Y$ of cotangent bundles, we can define vertical affinors of the form $\varphi \in C^\infty(VY \otimes T^*M)$, which are sometimes called *soldering forms*.

Taking into account the identification of a connection Γ on Y with its horizontal projection $TY \to TY$ (denoted also by Γ), we obtain another affinor on Y. In this situation we can compute the Frölicher-Nijenhuis bracket $[\Gamma, \varphi] \in C^\infty(TY \otimes \wedge^2 T^*Y)$. Clearly, if φ is a vertical affinor, then $[\Gamma, \varphi] \in C^\infty(VY \otimes \wedge^2 T^*Y)$ and if φ is a soldering form, then $[\Gamma, \varphi] \in C^\infty(VY \otimes \wedge^2 T^*M)$.

Definition. A *natural affinor* on a natural bundle F is a system of affinors $Q_M : TFM \to TFM$ for every m-manifold M satisfying $TFf \circ Q_M = Q_N \circ TFf$ for every local diffeomorphism $f : M \to N$.

Kolář and Modugno [7] introduced the following general definition of the torsion, which generalizes the classical torsion of a linear connection.

Definition. Let Q be a non identity natural affinor on a natural bundle F and let $\Gamma : FM \to J^1(FM \to M)$ be a connection on FM. The Frölicher-Nijenhuis bracket $[\Gamma, Q]$ is called the (general) torsion of Γ.

Since the torsion is defined by means of some natural affinor, the complete list of natural affinors on a natural bundle F enables us to describe all torsions of connections on F. Such an approach has been used for the first time by Kolář and Modugno [7], who determined all natural affinors on an arbitrary product preserving bundle and have described all torsions on the tangent bundle TM, on the bundle of k-dimensional 1-velocities $T^1_k M$, on the bundle of 1-dimensional 2-velocities $T^2_1 M$ and on the frame bundle PM. Further, Kureš has studied general torsions on iterated tangent bundles, on the bundles $T^r_1 M$ of 1-dimensional velocities of order r and on non–holonomic bundles of higher order velocities, see [11], [9] and [10]. Kolář and Modugno [7] have also described torsions of linear connections on the cotangent bundle T^*M and in [3] we have studied torsions on higher order cotangent bundles $T^{r*}M$. We remark that Mikulski, [15], has recently described all natural affinors on the bundle J^rTM of the r-jet prolongation of the tangent bundle and all natural affinors on the dual bundle $(J^rT^*)^*$ of the r-jet prolongation of the cotangent bundle, [16]. We point out that the above results of Mikulski can be used in the theory of general torsions.

In what follows we will need only the coordinate form of the Frölicher-Nijenhuis bracket of a vertical affinor $\varphi \in C^\infty(VY \otimes T^*M)$ (i.e. of a soldering form) and a connection $\Gamma : Y \to J^1Y$. If

$$dy^p = F_i^p(x,y)dy^p$$

are equations of Γ, then the horizontal projection $TY \to TY$ of Γ is of the form

$$(dx^i, dy^p) \mapsto (dx^i, F_i^p dx^i),$$

so that the corresponding affinor can be written in the form

$$\delta_j^i \frac{\partial}{\partial x^i} \otimes dx^j + F_i^p \frac{\partial}{\partial y^p} \otimes dx^i.$$

If

$$\varphi_i^p(x,y)\frac{\partial}{\partial y^p} \otimes dx^i$$

is the coordinate expression of φ, then by [7] the Frölicher-Nijenhuis bracket $[\Gamma, \varphi] \in (VY \otimes \wedge^2 T^*M)$ is of the form

$$(1) \qquad \left(\frac{\partial \varphi_j^p}{\partial x^i} + F_i^q \frac{\partial \varphi_j^p}{\partial y^q} - \varphi_j^q \frac{\partial F_i^p}{\partial y^q} \right) \frac{\partial}{\partial y^p} \otimes (dx^i \wedge dx^j).$$

We remark that the coordinate form of the Frölicher-Nijenhuis bracket of Γ with a general affinor $\varphi \in C^\infty(TY \otimes T^*Y)$ has been computed by Kureš in [12].

3. Natural affinors on higher order tangent bundles. The space

$$T^{r*}M = J^r(M, \mathbb{R})_0$$

of all r-jets from a manifold M into reals with target zero is a vector bundle over M, which is called the *r-th order cotangent bundle*. The dual vector bundle

$$T^{(r)}M = (T^{r*}M)^*$$

is called the *r-th order tangent bundle*, [8]. For every map $f : M \to N$ the jet composition $A \mapsto A \circ (j_x^r f)$, $x \in M$, $A \in (T^{r*}N)_{f(x)}$ defines a linear map

$(T^{r*}N)_{f(x)} \to (T^{r*}M)_x$. Denoting by $(T^{(r)}f)_x : (T^{(r)}M)_x \to (T^{(r)}N)_{f(x)}$ the dual map, we have constructed a vector bundle functor $T^{(r)}$, which is defined on the whole category $\mathcal{M}f$ of all smooth manifolds and all smooth maps. Clearly, for $r > 1$ the functor $T^{(r)}$ does not preserve products, while for $r = 1$ we obtain the classical tangent bundle TM. Denote by (x^i) the canonical coordinates on M and by $(u_i, u_{ij}, \dots, u_{i_1 \dots i_r})$ the induced coordinates on $T^{r*}M$. Then the linear functional on $(T^{r*}M)_x$ of the form

$$v^i u_i + v^{ij} u_{ij} + \cdots + v^{i_1 \dots i_r} u_{i_1 \dots i_r}$$

defines additional fiber coordinates $(v^i, v^{ij}, \dots, v^{i_1 \dots i_r})$ on $T^{(r)}M$ (symmetric in all indices). Finally, on the tangent bundle $TT^{(r)}M$ we have local coordinates

$$(x^i, v^i, v^{ij}, \dots, v^{i_1 \dots i_r}, X^i = dx^i, V^i = dv^i, V^{ij} = dv^{ij}, \dots, V^{i_1 \dots i_r} = dv^{i_1 \dots i_r}).$$

Now we construct an affinor $Q : TT^{(r)}M \to VT^{(r)}M$ as follows. The jet projection $J^r(M, \mathbb{R})_0 \to J^1(M, \mathbb{R})_0$ yields the inclusion $i_M : T^{(1)}M = TM \to T^{(r)}M$. Since $\pi_M : T^{(r)}M \to M$ is a vector bundle, the vertical bundle $VT^{(r)}M \subset TT^{(r)}M$ is of the form

$$VT^{(r)}M = T^{(r)}M \oplus T^{(r)}M.$$

Denoting by $p_M : TM \to M$ the tangent bundle projection, we have $p_{T^{(r)}M} : TT^{(r)}M \to T^{(r)}M$. Define an affinor

$$Q : TT^{(r)}M \to VT^{(r)}M \tag{2}$$

by

$$Q(A) = (p_{T^{(r)}M}(A), i_M(T\pi_M(A)).$$

In coordinates,

$$(dx^i, dv^i, dv^{ij}, \dots, dv^{i_1 \dots i_r}) \mapsto (0, dx^i, 0, \dots, 0),$$

i.e.

$$Q = \delta^i_j \frac{\partial}{\partial v^i} \otimes dx^j. \tag{3}$$

Clearly, $Q \in C^\infty(VT^{(r)}M \otimes T^*M)$ is a soldering form. Let

$$t : TT^{(r)}M \to TT^{(r)}M \tag{4}$$

be the natural transformation over the identity of $T^{(r)}M$ constructed by

$$t(A) = (p_{T^{(r)}M}(A), p_{T^{(r)}M}(A)) \in T^{(r)}M \oplus T^{(r)}M = VT^{(r)}M \subset TT^{(r)}M.$$

In coordinates,

$$\overline{X}^i = 0, \overline{V}^i = v^i, \overline{V}^{ij} = v^{ij}, \dots, \overline{V}^{i_1 \dots i_r} = v^{i_1 \dots i_r}.$$

Since $TT^{(r)}M \to T^{(r)}M$ is a vector bundle, natural transformations $TT^{(r)}M \to TT^{(r)}M$ over the identity of $T^{(r)}M$ form a vector space. Denote by

$$\mathrm{Id} : TT^{(r)}M \to TT^{(r)}M \tag{5}$$

the identity natural transformation. The first author deduced that all natural transformations $TT^{(2)}M \to TT^{(2)}M$ over the identity of $T^{(2)}M$ are linearly generated by three above transformations Id, Q and t, [2]. Generalizing the proof of Proposition 2 from [2] from $r = 2$ to an arbitrary integer r we show

Lemma 1. *All natural transformations $TT^{(r)} \to TT^{(r)}$ over the identity of $T^{(r)}$ are of the form*

$$\alpha \mathrm{Id} + \beta Q + \gamma t \tag{6}$$

with any $\alpha, \beta, \gamma \in \mathbb{R}$.

From Lemma 1 it follows directly

Lemma 2. *All natural affinors on $T^{(r)}M$ are linearly generated by the identity affinor* Id *and by Q.*

We remark that natural affinors on $T^{(r)}M$ were also determined by Gancarzewicz and Kolář, [4]. The geometrical properties of the r-th order tangent bundle were recently studied also by Mikulski. For example, Mikulski [17] has classified all natural operators transforming vector fields on M into vector fields on $T^{(r)}M$ and all linear natural liftings of one-forms to $T^{(r)}M$, [18].

4. Torsions of connections on $\boldsymbol{T^{(r)}M}$. A general connection Γ on $T^{(r)}M$ has the coordinate form

$$dv^i = \Gamma^i_j(x,v)dx^j, \dots, dv^{i_1 \dots i_r} = \Gamma^{i_1 \dots i_r}_j(x,v)dx^j, \tag{7}$$

i.e.

$$\Gamma = \delta^i_j \frac{\partial}{\partial x^i} \otimes dx^j + \Gamma^i_j(x,v) \frac{\partial}{\partial v^i} \otimes dx^j + \dots + \Gamma^{i_1 \dots i_r}_j(x,v) \frac{\partial}{\partial v^{i_1 \dots i_r}} \otimes dx^j.$$

Since Q is the only non trivial affinor on $T^{(r)}$, we have one general torsion

$$\tau := [\Gamma, Q] \in C^\infty(VT^{(r)}M \otimes \wedge^2 T^*M).$$

From (1) we obtain by a direct evaluation

Proposition 1. *The coordinate form of the torsion τ on $T^{(r)}M$ is*

$$\tau = \frac{\partial \Gamma^k_j}{\partial v^i}\frac{\partial}{\partial v^k} \otimes (dx^i \wedge dx^j) + \cdots + \frac{\partial \Gamma^{k_1\dots k_r}_j}{\partial v^i}\frac{\partial}{\partial v^{k_1\dots k_r}} \otimes (dx^i \wedge dx^j). \tag{8}$$

For $r = 1$ this torsion coincides with the torsion on the tangent bundle, [7].

Denote by $p_r : T^{(r)}M \to TM$ the projection. We say that a connection Γ on $T^{(r)}M$ is *projectable* over a connection on TM if there is a connection $\Delta : TM \to J^1TM$ such that $J^1p_r \circ \Gamma = \Delta \circ p_r$. Clearly, Δ has the coordinate form

$$dv^i = \Gamma^i_j dx^j.$$

Proposition 1 directly implies

Proposition 2. *If $\tau = 0$, then the connection Γ is projectable over a connection Δ on TM without torsion.*

A linear connection Γ on $T^{(r)}M$ has equations

$$\begin{aligned} dv^i &= (\Gamma^i_{j\ell_1} v^{\ell_1} + \Gamma^i_{j\ell_1\ell_2} v^{\ell_1\ell_2} + \cdots + \Gamma^i_{j\ell_1\dots\ell_r} v^{\ell_1\dots\ell_r}) dx^j \\ &\vdots \\ dv^{i_1\dots i_r} &= (\Gamma^{i_1\dots i_r}_{j\ell_1} v^{\ell_1} + \Gamma^{i_1\dots i_r}_{j\ell_1\ell_2} v^{\ell_1\ell_2} + \cdots + \Gamma^{i_1\dots i_r}_{j\ell_1\dots\ell_r} v^{\ell_1\dots\ell_r}) dx^j \end{aligned}$$

so that its torsion is of the form

$$\tau = (\Gamma^k_{ji}\frac{\partial}{\partial v^k} + \Gamma^{k_1k_2}_{ji}\frac{\partial}{\partial v^{k_1k_2}} + \cdots + \Gamma^{k_1\dots k_r}_{ji}\frac{\partial}{\partial v^{k_1\dots k_r}}) \otimes (dx^i \wedge dx^j). \tag{9}$$

We can see that if Γ is linear, then its torsion τ does not depend on fiber coordinates $(v^i, v^{ij}, \dots, v^{i_1\dots i_r})$.

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