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On the growth of polynomials not vanishing in the unit disc

Dedicated to Professor Zdzisław Lewandowski on his 70-th birthday

Abstract. Let $\mathcal{P}_n^*$ denote the class of all polynomials of degree at most n not vanishing in the open unit disc. Furthermore, let $0 \le r < R \le 1$. We obtain some sharp lower and upper bounds for $|f(r)|/|f(R)|$ when f belongs to $\mathcal{P}_n^*$. In our investigations we make essential use of certain properties of functions analytic and bounded in the unit disc.

1. Introduction and statement of results. For any entire function f let

$$M(f;\rho) := \max_{|z|=\rho} |f(z)| \qquad (0 \le \rho < \infty)\,,$$

and denote by $\mathcal{P}_n$ the class of all polynomials of degree at most n. If f belongs to $\mathcal{P}_n$ then so does the polynomial $f^*(z) := z^n\overline{f(1/\overline{z})}$. Hence, by the maximum modulus principle $M(f^*;r^{-1}) \ge M(f^*;1)$ for $0 < r < 1$. However, $M(f^*;r^{-1}) = r^{-n}M(f;r)$, and so

$$M(f;r) \ge r^n M(f;1) \qquad (0 < r < 1)\,. \tag{1}$$

In (1) equality holds if and only if $f(z)$ is a constant multiple of z^n.

The following result of Rivlin [7] contains the sharp version of (1) for polynomials not vanishing in the open unit disc.

Theorem A. *Let $\mathcal{P}_n^*$ consist of all those polynomials in $\mathcal{P}_n$ which do not vanish in the open unit disc. Then for any f belonging to $\mathcal{P}_n^*$, we have*

$$M(f;r) \geq \left(\frac{1+r}{2}\right)^n M(f;1) \qquad (0 \leq r < 1)\,, \tag{2}$$

where equality holds if and only if $f(z) := c\,\left(z - \mathrm{e}^{\mathrm{i}\gamma}\right)^n, c \in \mathbb{C}, c \neq 0, \gamma \in \mathbb{R}$.

Here, we may also mention Mamedhanov [4] who observed that under the conditions of Theorem A, we have

$$\left|f\left(r\mathrm{e}^{\mathrm{i}\gamma}\right)\right| \geq \left(\frac{1+r}{2}\right)^n \left|f\left(\mathrm{e}^{\mathrm{i}\gamma}\right)\right| \qquad (0 \leq r < 1\,; \gamma \in \mathbb{R})\,.$$

Govil [1] noted that (2) can be replaced by the more general inequality

$$M(f;r) \geq \left(\frac{1+r}{1+R}\right)^n M(f;R) \qquad (0 \leq r < R \leq 1)\,. \tag{3}$$

He also proved the following result.

Theorem B. *Let $f(z) := \sum_{\nu=0}^n c_\nu z^\nu$ be a polynomial of degree at most n not vanishing in the open unit disc. If $f'(0) = 0$, then for $0 \leq r < R \leq 1$, we have*

$$M(f;r) \geq \left(\frac{1{+}r}{1{+}R}\right)^n \frac{M(f;R)}{1 - (n/4)(1{-}R)(R{-}r)\left((1{+}r)/(1{+}R)\right)^{n-1}}\,. \tag{4}$$

In [5] it was shown that under the conditions of Theorem B, we have

$$M(f;r) \geq \left(\frac{1+r^2}{1+R^2}\right)^{n/2} M(f;R) \qquad (0 \leq r < R \leq 1)\,, \tag{5}$$

which is sharp for even n.

A reader wondering about the value of the condition "$f'(0) = 0$" appearing in the statement of Theorem B might find some of the sections in [8, Section 6 in particular; 9; 6] persuasive. It may be added that if $f(z) := \sum_{\nu=0}^n c_\nu z^\nu$ satisfies the conditions of Theorem A, then the polynomial $f(z^2)$ is of degree at most $2n$ and satisfies the other two conditions of Theorem B.

Inequality (5) is only a special case of the following more general result [5, Corollary 1] which applies to all polynomials of degree at most n not vanishing in the open unit disc.

Theorem C. *Let* $f(z) := \sum_{\nu=0}^{n} c_\nu z^\nu \neq 0$ *for* $|z| < 1$. *Then*

$$(6)\quad \frac{M(f;r)}{M(f;R)} \geq \left(\frac{1+2\lambda r+r^2}{1+2\lambda R+R^2}\right)^{n/2} \qquad \left(0 \leq r < R \leq 1\,,\ \lambda := \left|\frac{c_1}{nc_0}\right|\right).$$

Note that if $f(z) := \sum_{\nu=0}^{n} c_\nu z^\nu \neq 0$ for $|z| < 1$, then $c_0 \neq 0$, and $|c_1/c_0| \leq n$. Hence, $\lambda := |c_1/nc_0| \leq 1$. For any $\lambda \in [0\,,\,1]$ and $\gamma \in \mathbb{R}$, the two zeros of the quadratic $1 + 2\lambda z \mathrm{e}^{-\mathrm{i}\gamma} + z^2 \mathrm{e}^{-2\mathrm{i}\gamma}$ lie on the unit circle, and so if n is even then $f_\gamma(z) := (1 + 2\lambda z\mathrm{e}^{-\mathrm{i}\gamma} + z^2\mathrm{e}^{-2\mathrm{e}\gamma})^{n/2}$ is a polynomial of degree n satisfying the conditions of Theorem C. It is clear that

$$M(f_\gamma;\rho) = (1 + 2\lambda\rho + \rho^2)^{n/2} \qquad (0 \leq \rho \leq 1)\,,$$

and so (6) becomes an equality for the polynomial f_γ, $\gamma \in \mathbb{R}$. The inequality is not sharp in the case where n is odd.

Here we prove the following result which tells us more than what Theorem C does.

Theorem 1. *Let* $f(z) := \sum_{\nu=0}^{n} c_\nu z^\nu \neq 0$ *for* $|z| < 1$. *Then, for any* $\gamma \in \mathbb{R}$, *we have*

$$(7)\quad \frac{|f(r e^{\mathrm{i}\gamma})|}{|f(R e^{\mathrm{i}\gamma})|} \geq \left(\frac{1+2\lambda r+r^2}{1+2\lambda R+R^2}\right)^{n/2} \qquad \left(0 \leq r < R \leq 1\,,\ \lambda := \left|\frac{c_1}{nc_0}\right|\right).$$

Obviously (7) implies (6).

We shall apply Theorem 1 to obtain the following result about polynomials having all their zeros on the unit interval.

Corollary 1. *Let* $P(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$ *have all its zeros on the unit interval* $[-1, 1]$, *and let* ζ *be any point of the complex plane, not belonging to* $[-1, 1]$. *Furthermore, let* A *be the semi-major axis of the ellipse passing through* ζ *and having* -1, 1 *as foci. Then*

$$|P(\zeta)| \geq \left(\frac{A+\Lambda}{1+\Lambda}\right)^n \left|P\left(\frac{\xi}{A}\right)\right| \qquad \left(\xi := \Re\zeta\,,\, \Lambda := \left|\frac{a_{n-1}}{na_n}\right|\right).$$

Upper bound for $\boldsymbol{|f(r\mathrm{e}^{\mathrm{i}\gamma})|/|f(R\mathrm{e}^{\mathrm{i}\gamma})|\,,\ 0 \leq r < R \leq 1\,.}$ For any entire function f let

$$m(f;\rho) := \min_{|z|=\rho} |f(z)| \qquad (0 \leq \rho < \infty)\,.$$

If $f(z) \neq 0$ in the open unit disc, then by the minimum modulus principle $m(f;r) \geq m(f;R)$ for $0 \leq r < R \leq 1$. How large can $m(f;r)/m(f;R)$ be if f satisfies the conditions of Theorem C? The following result contains an answer to this question.

Theorem 2. *Let* $f(z) := \sum_{\nu=0}^{n} c_\nu z^\nu \neq 0$ *for* $|z| < 1$, *and let* $\lambda := |c_1/nc_0|$. *Then, for any* $\gamma \in \mathbb{R}$, *we have*

$$\text{(8)} \quad \frac{\left|f\left(re^{\mathrm{i}\gamma}\right)\right|}{\left|f\left(Re^{\mathrm{i}\gamma}\right)\right|} \leq \left(\frac{1+r}{1+R}\right)^{(1-\lambda)n/2}\left(\frac{1-r}{1-R}\right)^{(1+\lambda)n/2} \qquad (0 \leq r < R < 1)\ .$$

In (8), equality holds for the polynomial

$$f_{1,\gamma}(z) := \left(1 + z\mathrm{e}^{-\mathrm{i}\gamma}\right)^{(1-\lambda)n/2}\left(1 - z\mathrm{e}^{-\mathrm{i}\gamma}\right)^{(1+\lambda)n/2}\ ,$$

where it is presumed that $(1-\lambda)n/2$ is an integer.

The following corollary is a simple consequence of Theorem 2.

Corollary 2. *Let* $f(z) := \sum_{\nu=0}^{n} c_\nu z^\nu \neq 0$ *for* $|z| < 1$, *and let* $\lambda := |c_1/nc_0|$. *Then*

$$\text{(9)} \quad m(f;r) \leq \left(\frac{1+r}{1+R}\right)^{(1-\lambda)n/2}\left(\frac{1-r}{1-R}\right)^{(1+\lambda)n/2} m(f;R) \qquad (0 \leq r < R < 1)\,.$$

Sharpness of the estimate for $\boldsymbol{m(f;r)/m(f;R)}$**.** We claim that (9) becomes an equality for $f_{1,\gamma}$ which is a polynomial of degree n provided that $(1-\lambda)n/2$ is an integer. It is enough to check this for $f_{1,0}$. Since for all real θ and all $\rho \in [0\,,\,1)$:

$$\left|f_{1,0}\left(\rho\mathrm{e}^{\mathrm{i}\theta}\right)\right| = (1 + 2\rho\cos\theta + \rho^2)^{(1-\lambda)n/4}(1 - 2\rho\cos\theta + \rho^2)^{(1+\lambda)n/4}\,,$$

we need to determine $\min_{-1\leq t\leq 1} A_\lambda(t)$, where

$$A_\lambda(t) := (1 + 2\rho t + \rho^2)^{1-\lambda}(1 - 2\rho t + \rho^2)^{1+\lambda} \qquad (0 \leq \lambda \leq 1)\,.$$

It is clear that

$$\min_{-1\leq t\leq 1} A_0(t) = A_0(\pm 1) = (1-\rho^2)^2\,,$$

and that

$$\min_{-1\leq t\leq 1} A_1(t) = A_1(1) = (1-\rho)^4\,.$$

Hence, equality holds in (9) for $f_{1,0}$ when $\lambda = 0$, and also when $\lambda = 1$.

Now let $0 < \lambda < 1$. An elemetary calculation gives

$$A'_\lambda(t) = -4\rho\{2\rho t + \lambda(1+\rho^2)\}\left(\frac{1 - 2\rho t + \rho^2}{1 + 2\rho t + \rho^2}\right)^{\lambda}\,.$$

For any $\rho \in (0,1)$, the only possible root of $A'_\lambda(t) = 0$ in $[-1,1]$ is $t = t_0 := -\lambda(1+\rho^2)/2\rho$. If $t_0 \notin [-1\,,\,1]$ then $A'_\lambda(t) < 0$ for all $t \in [-1\,,\,1]$ since $A'_\lambda(1) < 0$, and so

$$\min_{-1\le t\le 1} A_\lambda(t) = A_\lambda(1)\,.$$

In the case where t_0 belongs to $[-1\,,\,1]$ it is a point of local maximum since

$$A''_\lambda(t_0) = -8\rho^2\left(\frac{1-2\rho t_0+\rho^2}{1+2\rho t_0+\rho^2}\right)^\lambda < 0\,.$$

We conclude that

$$\min_{-1\le t\le 1} A_\lambda(t) = \min\{A_\lambda(-1)\,,\,A_\lambda(1)\} = A_\lambda(1)\,.$$

Consequently,

$$\min_{|z|=\rho} |f_{1,0}(z)| = (1+\rho)^{(1-\lambda)n/2}(1-\rho)^{(1+\lambda)n/2} \qquad (0 \le \rho < 1)\,,$$

and so (9) becomes an equality for $f_{1,0}$ which is a polynomial provided that $(1-\lambda)n/2$ is an integer.

2. A lemma. For the proofs of Theorems 1 and 2 we need the following auxiliary result.

Lemma 1. *Let* $f(z) := c_n\prod_{\nu=1}^n (z-z_\nu) = \sum_{\nu=0}^n c_\nu z^\nu \ne 0$ *for* $|z| < 1$. *Then* $zf'(z) - nf(z) \ne 0$ *for* $|z| < 1$, *and* $|f'(z)| \le |zf'(z) - nf(z)|$ *for* $|z| = 1$, *so that*

$$\varphi(z) := \frac{f'(z)}{zf'(z) - nf(z)} \tag{10}$$

is analytic on the closed unit disc. Furthermore, $|\varphi(z)| \le 1$ *for* $|z| \le 1$.

Proof of Lemma 1. The polynomial $f^*(z) := z^n\overline{f(1/\overline{z})}$ has all its zeros in the closed unit disc. Furthermore, any zero of f lying on the unit circle is also a zero of f^* of the same multiplicity. This allows us to conclude that $\psi(z) := f^*(z)/f(z)$ is analytic on the closed unit disc, and $\psi(z) = 1$ on the unit circle. Hence, by the maximum modulus principle $|\psi(z)| \le 1$ for $|z| \le 1$. It follows that

$$\left|\frac{f(z)}{f^*(z)}\right| = \left|\overline{\psi\left(\frac{1}{\overline{z}}\right)}\right| \le 1 \qquad (|z| \ge 1)\,.$$

Consequently, $f(z)-\omega f^*(z)\neq 0$ for $|z|>1$ and $|\omega|>1$. In other words, the polynomial $f(z)-\omega f^*(z)$ has all its zeros in the closed unit disc for all ω such that $|\omega|>1$. By the Gauss–Lucas theorem [2, Theorem 4.4.1] we can say the same about its derivative $f'(z)-\omega {f^*}'(z)$. This implies that $|f'(z)|\le |{f^*}'(z)|$ for $|z|>1$. By continuity, $|f'(z)|\le |{f^*}'(z)|$ for $|z|=1$ also. Since

$$\left|{f^*}'(z)\right| = \left|z^{n-1}\overline{{f^*}'(z)}\right| = \left|z^{n-1}\overline{{f^*}'\left(\frac{1}{\overline{z}}\right)}\right| \qquad (|z|=1)$$

we see that

$$|f'(z)| \le \left|z^{n-1}\overline{{f^*}'\left(\frac{1}{\overline{z}}\right)}\right| \qquad (|z|=1)\,. \tag{11}$$

Finally, we observe that for all z on the unit circle

$$z^{n-1}\overline{{f^*}'\left(\frac{1}{\overline{z}}\right)} = c_{n-1}z^{n-1}+\ldots+(n-1)c_1 z + nc_0 = nf(z)-zf'(z)\,. \tag{12}$$

Since ${f^*}'$ has all its zeros in $|z|\le 1$, the polynomial $z^{n-1}\overline{{f^*}'(1/\overline{z})}$ has no zeros in the open unit disc, and so from (11) and (12) it follows that $\big|f'(z)/\big(zf'(z)-nf(z)\big)\big|\le 1$ for $|z|\le 1$. $\square$

3. Proofs of the theorems and of Corollary 1.

Proof of Theorem 1. Clearly,

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\log|f(\rho)| = \Re\frac{\mathrm{d}}{\mathrm{d}\rho}\log f(\rho) = \Re\frac{f'(\rho)}{f(\rho)} \qquad (0\le\rho<1)\,.$$

In terms of the function φ introduced in (10), we have

$$\rho\,\frac{f'(\rho)}{f(\rho)} = -\frac{n\rho\varphi(\rho)}{1-\rho\varphi(\rho)} = n-\frac{n}{1-\rho\varphi(\rho)}\,, \tag{13}$$

so that

$$\rho\,\Re\frac{f'(\rho)}{f(\rho)} = n-\Re\frac{n}{1-\rho\varphi(\rho)} \le n-\frac{n}{1+\rho|\varphi(\rho)|} \qquad (0\le\rho<1)\,. \tag{14}$$

Since $\varphi(0)=-c_1/nc_0$, and $|\varphi(z)|\le 1$ for $|z|\le 1$, it follows from the generalized Schwarz's lemma [3, Section 6.2] that

$$|\varphi(\rho)| \le \frac{\rho+|\varphi(0)|}{|\varphi(0)|\rho+1} = \frac{\rho+\lambda}{\lambda\rho+1} \qquad \left(0\le\rho<1\,, : \lambda := \left|\frac{c_1}{nc_0}\right|\right)\,. \tag{15}$$

From (14) and (15) it follows that

$$\rho\,\Re\,\frac{f'(\rho)}{f(\rho)} \le n - \frac{n}{1+(\rho^2+\lambda\rho)/(\lambda\rho+1)} = n\,\frac{\rho^2+\lambda\rho}{1+2\lambda\rho+\rho^2}\,,$$

and so

$$\Re\,\frac{f'(\rho)}{f(\rho)} \le n\,\frac{\rho+\lambda}{1+2\lambda\rho+\rho^2}\,.$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\log|f(\rho)| = \Re\,\frac{f'(\rho)}{f(\rho)} \le n\,\frac{\rho+\lambda}{1+2\lambda\rho+\rho^2} \qquad (0 \le \rho < 1)\,.$$

Hence, for $0 \le r < R \le 1$, we have

$$\log\frac{|f(R)|}{|f(r)|} = \int_r^R \frac{\mathrm{d}}{\mathrm{d}\rho}\log|f(\rho)|\,\mathrm{d}\rho \le \int_r^R n\,\frac{\rho+\lambda}{1+2\lambda\rho+\rho^2}\,\mathrm{d}\rho = \frac{n}{2}\,\log\frac{1+2\lambda R+R^2}{1+2\lambda r+r^2}\,.$$

This proves (7) in the case where γ is zero. The same argument applied to the polynomial $f(z\mathrm{e}^{\mathrm{i}\gamma})$ gives the result for other values of γ. □

Proof of Theorem 2. From (13) it follows that

$$\rho\,\Re\,\frac{f'(\rho)}{f(\rho)} \ge n - \frac{n}{1-\rho|\varphi(\rho)|}\,,$$

and so in view of (15), we have

$$\rho\,\Re\,\frac{f'(\rho)}{f(\rho)} \ge n - \frac{n}{1-(\rho^2+\lambda\rho)/(\lambda\rho+1)} = -n\frac{\rho^2+\lambda\rho}{1-\rho^2}\,.$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\log|f(\rho)| = \Re\,\frac{f'(\rho)}{f(\rho)} \ge -n\,\frac{\rho+\lambda}{1-\rho^2}\,,$$

which implies that for $0 \le r < R \le 1$, we have

$$\begin{aligned}
\log\frac{|f(R)|}{|f(r)|} &= \int_r^R \frac{\mathrm{d}}{\mathrm{d}\rho}\log|f(\rho)|\,\mathrm{d}\rho \\
&\ge -\int_r^R n\frac{\rho+\lambda}{1-\rho^2}\,\mathrm{d}\rho \\
&= \frac{n}{2}\left[\log(1-\rho^2)\right]_r^R - \lambda\frac{n}{2}\int_r^R \left(\frac{1}{1+\rho}+\frac{1}{1-\rho}\right)\mathrm{d}\rho \\
&= \log\left\{\frac{(1+R)^{(1-\lambda)n/2}(1-R)^{(1+\lambda)n/2}}{(1+r)^{(1-\lambda)n/2}(1-r)^{(1+\lambda)n/2}}\right\}.
\end{aligned}$$

This proves (8) in the case where γ is zero. The same argument applied to the polynomial $f(z\mathrm{e}^{\mathrm{i}\gamma})$ gives the result for other values of γ. □

Proof of Corollary 1. Let T_k denote the Chebyshev polynomial of the first kind of degree k. Then

$$T_k(z) = 2^{k-1}z^k + t_{k-2}(z) \qquad (k \geq 2)\,,$$

where t_{k-2} is a polynomial of degree $k-2$. Hence,

$$P(z) = \frac{1}{2^{n-1}}a_n T_n(z) + \frac{1}{2^{n-2}}a_{n-1}T_{n-1}(z) + \sum_{\nu=2}^{n} b_\nu T_{n-\nu}(z)\,.$$

Since

$$T_k\left(\frac{z+z^{-1}}{2}\right) = \frac{z^k + z^{-k}}{2} \qquad (0 \leq k < \infty)\,,$$

we see that

$$P\left(\frac{z+z^{-1}}{2}\right) = \frac{1}{2^{n-1}}a_n\frac{z^n+z^{-n}}{2} + \frac{1}{2^{n-2}}a_{n-1}\frac{z^{n-1}+z^{-n+1}}{2} + \frac{1}{2}\sum_{\nu=2}^{n} b_\nu\left(z^{n-\nu}+z^{-n+\nu}\right)\,.$$

Thus

$$f(z) := z^n P\left(\frac{z+z^{-1}}{2}\right) = \frac{1}{2^n}a_n + \frac{1}{2^{n-1}}a_{n-1}z + \cdots + \frac{1}{2^n}a_n z^{2n}$$

is a polynomial of degree $2n$ having all its zeros on $|z| = 1$. Applying Theorem 1 with $2n$ instead of n, we obtain

$$\frac{\left|f\left(r\mathrm{e}^{\mathrm{i}\gamma}\right)\right|}{\left|f\left(\mathrm{e}^{\mathrm{i}\gamma}\right)\right|} \geq \left(\frac{1+2\Lambda r+r^2}{2+2\Lambda}\right)^n \qquad \left(0 \leq r < 1\,,\: \gamma \in \mathbb{R}\,,\: \Lambda := \left|\frac{a_{n-1}}{na_n}\right|\right)\,,$$

which leads us to the estimate

$$\frac{\left|P\left((r^{-1}\mathrm{e}^{-\mathrm{i}\gamma}+r\mathrm{e}^{\mathrm{i}\gamma})/2\right)\right|}{|P(\cos\gamma)|} \geq \left(\frac{(r^{-1}+r)/2+\Lambda}{1+\Lambda}\right)^n \quad (0 \leq r < 1\,,\: \gamma \in \mathbb{R})\,.$$

For any $\gamma \in \mathbb{R}$, the point $\zeta := \left(r^{-1}\mathrm{e}^{-\mathrm{i}\gamma} + r\mathrm{e}^{\mathrm{i}\gamma}\right)/2$ lies on the ellipse $\mathcal{E}_{r^{-1}}$ whose foci are -1 and 1, and whose semi-axes are $A := \left(r^{-1}+r\right)/2$ and $B := \left(r^{-1}+r\right)/2$. Since $\cos\gamma = \xi/A$, where $\xi := \Re\zeta$, the preceding inequality is equivalent to

$$|P(\zeta)| \geq \left(\frac{A+\Lambda}{1+\Lambda}\right)^n \left|P\left(\frac{\xi}{A}\right)\right| \qquad (\zeta \notin [-1,1])\,. \quad \square$$

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