ANNALES
UNIVERSITATIS MARIAE CURIE – SKŁODOWSKA
LUBLIN – POLONIA

VOL. LIV, 4 SECTIO A 2000

ZBIGNIEW J. JAKUBOWSKI

On some conjectures concerning bounded univalent functions

Dedicated to Professor Zdzisław Lewandowski on his 70-th birthday

Abstract. Let S denote the class of all functions of the form $F(z) = z + A_2z^2 + \cdots + A_nz^n + \cdots$ holomorphic and univalent in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, S_R - its subclass consisting of functions with real coefficients $(A_n = \overline{A_n}, n = 2, 3, \ldots)$. Let also $S(M)$ and $S_R(M)$, $M > 1$, denote the corresponding subclasses of functions satisfying the condition $|F(z)| < M$ for $z \in \Delta$. The main aim of the paper is to remind a few conjectures concerning some functionals defined in the classes $S(M)$ or $S_R(M)$ and their solutions. We shall formulate certain new problems as well.

1. Introduction. Let S denote the class of all functions F of the form

$$F(z) = z + A_2z^2 + \cdots + A_nz^n + \cdots \tag{1}$$

holomorphic and univalent in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, S_R - its subclass consisting of functions (1) with real coefficients $(A_n = \overline{A_n}, n =$

1991 *Mathematics Subject Classification.* Primary 30 C 50, Secondary 30 C 70.
Key words and phrases. bounded univalent functions, coefficient problems, Pick function, bounded univalent functions close to identity, univalent functions bounded by sufficiently large constants.

$2,3,\dots$). Let next $S(M)$ and $S_R(M)$, $M>1$, denote subclasses of S and S_R, respectively, consisting of functions satisfying the condition $|F(z)|<M$ for $z\in\Delta$.

Let

$$(2)\quad w=P^{/n-1/}(z;M)=z+\sum_{\nu=2}^{\infty}P_\nu^{/n-1/}(M)z^\nu,\ z\in\Delta, M>1, n=2,3,\dots$$

stand for the function defined by the equation

$$w\left(1-w^{n-1}M^{1-n}\right)^{2/(1-n)}=z\left(1-z^{n-1}\right)^{2/(1-n)}. \tag{3}$$

The function

$$P(z;M)=P^{/1/}(z;M)=z+\sum_{\nu=2}^{\infty}P_\nu(M)z^\nu,\ z\in\Delta, M>1 \tag{4}$$

is called a Pick function (also $P_t(z;M)$, $z\in\Delta$, where $F_t(z)=e^{-it}F(e^{it}z)$, $z\in\Delta$, $t\in(0,2\pi)$). The function (2) is called an $(n-1)$- symmetric Pick function. Of course, $P^{/n-1/}(z;M)\in S_R(M)$, $P_t^{/n-1/}(z;M)\in S(M)$.

In several papers (e.g. [10]-[12], [14], [41], [43]) published earlier, certain problems concerning estimates of some functionals defined on the classes $S(M)$ or $S_R(M)$ were stated. The special attention was paid to extremal problems in the classes of functions bounded by sufficiently large constants or functions close to identity (bounded by sufficiently small constants). In the present paper we shall recall some of these problems; we shall give up to date solutions and also establish some new questions.

2. The Charzyński-Tammi conjecture. It is known that a counterpart of the Bieberbach conjecture for the class $S(M)$ (and $S_R(M)$) seems to be difficult to formulate. Moreover, it seems to be impossible to expect estimates of $|A_n|$ for every $n=5,6,\dots$ and for an arbitrary $M>1$. It initiated a search for stating appropriate conjectures valid for any n and M, suitably chosen and sufficiently close to 1 or sufficiently large.

In the first case, on the turn of the 50's and 60's Z. Charzyński and O. Tammi conjectured that *the estimate*

$$|A_n|\le P_n^{/n-1/}(M)=\frac{2}{n-1}\left(1-M^{1-n}\right) \tag{5}$$

holds in the class $S(M)$ for each $n=4,5,\dots$ and M, suitably chosen and sufficiently close to 1 *($M\in(1,M_n^{(-)})$).* The form of the conjecture (5) followed from the known estimates of $|A_2|$ ([22]) and $|A_3|$ ([17], [31], [38]) and, to some extent, from the investigation relating to the coefficient A_4 in the class $S_R(M)$ ([9]).

This conjecture was answered in the affirmative by L. Siewierski (in 1960 - for odd n ([34]), 1968 - for even n ([35]), 1971 - for every n ([36])) and by M. Schiffer and O. Tammi (1968, [32]), who used a different method. The procedure of Siewierski was based on the variational method of Charzyński ([3]), whereas Schiffer and Tammi used the generalized (by them) Grunsky-Nehari inequality. It does not seem to be possible to obtain the maximal interval $(1, M_n^{(-)})$, where (5) holds, using the methods mentioned above, but we know that $M_n^{(-)}$ was fixed in another way for a few n only. Interesting remarks of its solution can be found in [4] (pp. 111–152).

Since the extremal functions for the problem (5) in the class $S(M)$ have the form $P_t^{/n-1/}(z; M)$, it follows that there hold the sharp estimates

$$-P_n^{/n-1/}(M) \leq A_n \leq P_n^{/n-1/}(M)$$

(for M sufficiently close to 1) in the classes $S_R(M)$. The lower bound is attained if $t = (2k+1)\pi/(n-1)$, $k = 0, 1, \ldots, n-2$.

3. The antipodal conjecture. In the 70's the following conjecture was posed (cf. [10], [41], [43], see also [12]).

For each $n = 6, 8, 10, \ldots$ there exists $M_n^{+} \geq 1$ such that, for all $M \in (M_n^{+}, \infty)$ and any function $F \in S(M)$, the estimate

$$\operatorname{Re} A_n \leq P_n(M) \tag{6}$$

holds.

In 1982 A. Zielińska obtained a partial solution for each $n \geq 6$ ([41] also see [12]). The method of proving the theorem is based on the general theorem on extremal functions ([3]).

A parallel conjecture was also posed for the class $S_R(M)$. Its positive solution was presented in 1978 at the International Congress of Mathematicians in Helsinki ([13]). The proofs were published in [14], [42], [43] (see also [11]). The method of proving the conjecture quoted above is based on the general theorem on extremal functions ([6]), the theory of Γ-structures and a result of Dieudonné ([5]).

V. G. Gordenko ([7]) proved the conjecture (6) in $S(M)$ for $n = 6$. In 1993 D. V. Prokhorov ([26]) demonstrated that the conjecture (6) ($F \in S(M)$) holds for all even n. Here and in other papers, he applied his own research method (the Loewner equation ([20]) for the class $S(M)$ and optimal control theory can be adjusted to problems for univalent functions, as developed in [25]).

Generally, the problem of finding the maximal intervals $(M_n^{(+)}, +\infty)$, where (6) holds, remains still open.

The Pick function is not extremal for $n = 3$ and for all $M > 1$. It was proved in [15] that there exists $M_5^+ \geq 1$ such that $P_5(M) < \max \operatorname{Re} A_5$ for each $M > M_5^+$, $F \in S(M)$. It also turned out that the method used in [14] is ineffective in the case of n odd. The following conjecture seemed to be true ([11], [12]):

For each $n = 7, 9, \ldots$, *there exists* $M_n^+ \geq 1$ *such that for any* $M \in (M_n^+, +\infty)$

$$\max_{F \in S(M)} \operatorname{Re} A_n > P_n(M). \tag{7}$$

In [26] D. V. Prokhorov answered in the affirmative the conjecture (7).

Still there remains open the problem of finding $\max \operatorname{Re} A_n$ for $n = 5, 7, 9, \ldots$, $F \in S(M)$ or $F \in S_R(M)$ and sufficiently large constant M. It is possible that there will be different extremal functions for different odd n.

4. The functional $|A_n A_k|$, n even, k odd. According to the considerations in sections 2 and 3 there arise the questions: (i) Is it possible to modify the functional $|A_n|$ in order to obtain one extremal function (e.g. the Pick function) for all n in the class $S(M)$ (or $S_R(M)$) for M sufficiently close to 1 ? (ii) Is it possible to change the functional $|A_n|$ for $n = 3, 5, 7, \ldots$, in order to obtain one extremal function (e.g. the Pick function), in the class $S(M)$ (or $S_R(M)$) for M sufficiently large?

First, it seems that the simplest functional of this kind is $\operatorname{Re}(A_2 A_n)$ (in case (i) $n = 3, 4, 5, \ldots$, in case (ii) $n = 3, 5, \ldots$). Next A_2 was replaced by a coefficient with an even index.

These problems and information about their partial solutions can be found e.g. in [10]-[12]. We recall some of them and the other recent results.

O. Tammi ([39]) determined the maximum of the functional $|A_2 A_3|$, $F \in S(M)$ for each $M > 1$. When $n = 3$, the conjecture from [10] and [11] :

$$|A_2 A_n| \leq |P_2(M) P_n(M)| \tag{8}$$

was confirmed for $M \in (1, \frac{13}{11})$ and this interval is maximal. When $n = 3$ and $M \in (\frac{13}{3}, +\infty)$, the estimate (8) is also true (the interval $(\frac{13}{3}, +\infty)$ is maximal). Analogous research in the class $S_R(M)$ was done simultaneously by L. Pietrasik ([24]). The results of O. Tammi coincide with the results of L. Pietrasik (Tammi cited them).

In [16] the conjecture (8) was confirmed for each $n = 3, 4, \ldots$ and M sufficiently close to 1.

In 1982, L. Pietrasik ([23]) proved that if n is even and k is odd, then, for sufficiently large M,

$$\max_{F \in S_R(M)} A_n A_k \tag{9}$$

is obtained for the Pick function.

In 1995, D. V. Prokhorov ([28]) showed that a product $|A_n A_k|$ of odd coefficient is not maximized by $P_n(M) \cdot P_k(M)$.

It may be worth to notice that there appeared several papers, where sums, e. g. of the form $A_4 + \alpha A_2$, or products of nonlinear functionals, e. g. $A_2(A_3 - \alpha A_2^2)$, α being a parameter were considered, instead of a coefficient product. It seems that in many of them ([8], [27], [29], [40]) one can find closer or further consequences of the conjectures (5) and (6). On the other hand, this research shows that there are great difficulties in finding $\max \operatorname{Re} A_n$ for any $n = 4, 5, \ldots$ and every $M > 1$ ([18]).

5. The problem of E. Netanyahu. As we know, in 1969 E. Netanyahu ([21]) proved the following theorem:

Let S be the family of functions defined in the Introduction and let $d_F = \inf |\alpha|$, $F(z) \neq \alpha$, $z \in \Delta$. Then

$$\max_{F \in S} \{|A_2| \cdot d_F\} = \frac{2}{3}. \tag{10}$$

It was shown in the proof, among others, that the function F^* of the form (1), extremal in the considered problem, with a coefficient $A_2^* > 0$, maps the disc Δ on the whole w-plane slit along an arc of $|w| = d_{F^*}$, symmetric with respect to the real axis and along a part of the real axis leading from the middle of the mentioned arc to infinity. Consequently, it turned out that we have $A_2^* = \frac{3}{2}$, $d_{F^*} = \frac{4}{9}$ for the function F^*.

The problem of Netanyahu in other classes of functions was investigated by Z. Bogucki ([2], pp. 55–57). It turned out that for the class $S^* \subset S$ of starlike functions the maximum in (10) is equal to $0.61\ldots$ and Z. Bogucki conjectured that, for the class S^c of convex functions, it equals $\frac{1}{2}$.

It seems to be natural to find the maximum in (10) for the class $S(M)$. One can suppose analogously that the extremal function will map the disc Δ onto the disc $|w| < M$ with a slit similar as in the class S. We tried to do it, but without the great engagement.

6. On typically-real functions of Lewandowski-Wajler. In [19] Z. Lewandowski and S. Wajler introduced the class T_M, $M > 1$, of functions F of the form

$$F(z) = \int_{-1}^{1} Q(z; M, t) d\mu(t), \quad z \in \Delta, \tag{11}$$

where μ is the unit mass distributed on $[-1,1]$ (see [33],p. 4),

$$Q(z;M,t) = 2z\Big[\big(P^2 - 4z^2M^{-2}\big)^{1/2} + P\Big]^{-1}, \tag{12}$$

and $P = 1 - 2tz(1-M^{-1}) + z^2$, $\sqrt{1} = 1$.

One can see that $Q(z;M,t)$ is a two-slit Pick function and $Q(z;M,1) = P(z;M)$, $Q(z;M,-1) = -P(-z,M)$. Moreover, each function $F \in T_M$ is typically-real and bounded by M, $T_\infty = T$ ([30]) and $Q(z;M,t) \in S_R(M)$.

The authors obtained many interesting properties of the class T_M. In the considered cases, the functions $Q(z;M,t)$ turned out to be the extremal functions.

In the context of the previous observations and remarks, perhaps the following problems are interesting: (a) What is a relationship between the classes $S_R(M)$ and T_M? (b) Find the form of the limit functions ($M \to 1^+$) in the sense of the paper of H. Śmiałek ([37]), in the class T_M. What conclusions can we get from this? (c) Find a form of a functional defined on T_M such that the extremal function is not univalent.

The Pick function plays an important role in the paper [1] of F. Bogowski and Cz. Bucka on a class S_M^*, $M > 1$, of starlike bounded functions. There we have one more point of view on bounded univalent functions.

References

[1] Bogowski, F., Cz. Bucka, *Sur une classe de fonctions étoillées bornées*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **32,1** (1978), 1–21.

[2] Bogucki, Z., *On some extremal problems in certain families of convex and spirallike functions*, High School of Engineering, Radom, 1987. (Polish)

[3] Charzyński, Z., *Sur les fonctions univalentes bornées*, Dissertationes Math. **2** (1953).

[4] Charzyński, Z., J. Śladkowska, *Methods of modern theory of univalent mappings. Local coefficient problems*, Proceedings of Summer School on Analytic Functions, Uniejów, Łódź, 1969. (Polish)

[5] Dieudonné, J., *Sur les fonctions univalentes*, C. R. Acad. Sci. Paris **192** (1931), 1138–1150.

[6] Dziubiński, I., *L' équation des fonctions extrémales dans la famille des fonctions univalentes symétriques et bornées*, Soc. Sci. Lodziensis, Sec. III **65** (1960).

[7] Gordenko, V.G., *Sixth coefficient estimate for bounded univalent functions*, Theory of Functions and Approximation, Proc. 6th Saratov Winter School, Saratov (to appear). (Russian)

[8] Gromova, L., A. Vasil'ev, *On the estimate of the fourth-order homogeneous coefficient functional for univalent functions*, Ann. Polon. Math. **63** (1996), 7–12.

[9] Jakubowski, Z.J., *Sur le maximum du coefficient A_4 des fonctions univalentes bornées*, Colloq. Math. **7** (1959), 128–129.

[10] ———, *On certain extremal problems in classes of bounded univalent functions*, Folia Sci. Univ. Techn. Resoviensis **2** (1984), 9–16. (Polish)

[11] ———, *Some properties of coefficients of bounded symmetric univalent function*, Complex Analysis and Applications '83 (Varna, 1983), Bulgarian Acad. of Sci., Sofia, 1985, pp. 120–126.

[12] ———, *Some problems concerning bounded univalent functions*, Univalent Functions, Fractional Calculus and their Applications, H.M. Srivastava and S. Owa eds., Ellis Horwood Ser. Math. Appl., Horwood, Chichester: New York, 1989, pp. 75–86.

[13] ———, A. Zielińska and K. Zyskowska, *Sharp estimation of even coefficients of bounded symmetric univalent functions*, Abstracts Internat. Congress of Math., Helsinki (1978), 118.

[14] ———, *Sharp estimation of even coefficients of bounded symmetric univalent functions*, Ann. Polon. Math. **40** (1983), 193–206.

[15] ———, K. Zyskowska, *A few remarks on bounded univalent functions*, Pliska Studia Math. Bulgarica **10** (1989), 77–86.

[16] ———, D.V. Prokhorov and J. Szynal, *Proof of a coefficient product conjecture for bounded univalent functions* (to appear).

[17] Janowski, W., *Le maximum A_2 et A_3 des fonctions univalentes bornées*, Ann. Polon. Math. **2** (1955), 145–160.

[18] Jokinen, O., O. Tammi, *On determining the points of the second coefficient body (a_2, a_3, a_4) for bounded real univalent functions*, Kodai Math. J. **17** (1994), 82–100.

[19] Lewandowski, Z., S. Wajler, *Sur les fonctions typiquement réelles bornées*, Ann. Mariae Curie-Skłodowska Sect. A **28** (1974), 59–63.

[20] Löwner, K., *Untersuchungen über schlichte konforme Abbildungen des Einheitskreises I*, Math. Ann. **89** (1923), 103–121.

[21] Netanyahu, E., *The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$*, Arch. Rational Mech. Anal. **32** (1969), 100–112.

[22] Pick, G., *Über die konforme Abbildung eines Kreises auf ein schlichtes und zugleich beschränktes Gebiet*, Sitzungsber. Akad. Wiss. Wien (1917), 247–263.

[23] Pietrasik, L., *On some property of bounded symmetric univalent functions*, Bull. Soc. Sci. Lett. Łódź **32** (1982), 1–13.

[24] ———, *Estimation of the functional $A_2 \cdot A_3$ in the class of bounded symmetric univalent functions*, Acta Univ. Lodz. **2** (1987), 81–105.

[25] Prokhorov, D.V., *Reachable set methods in extremal problems for univalent functions*, Saratov University Publishing House, 1993.

[26] ———, *Even coefficient estimates for bounded univalent functions*, Ann. Polon. Math. **48** (1993), 267–273.

[27] ———, *Methods of optimization in coefficient estimates for bounded univalent functions*, Ann. Mariae Curie-Skłodowska Sect. A **48** (1994), 106–119.

[28] ———, *Coefficient products for bounded univalent functions*, Complex Variables Theory Appl. **27** (1995), 211–216.

[29] ———, Z. Vasileva, *Linear extremal problems for univalent functions close to identity*, Bull. Soc. Sci. Lett. Łódź **45** (1995), 11–17.

[30] Rogosinski, W.W., *Über positive harmonische Entwicklungen und typisch-reelle Potenzreihen*, Math. Z. **35** (1932), 33–121.

[31] Schaeffer, A.C., D.C. Spencer, *The coefficients of schlicht functions*, Duke Math. J. **12** (1945), 107–125.

[32] Schiffer, M., O. Tammi, *On bounded univalent functions which are close to identity*, Ann. Acad. Sci. Fenn. Ser. A I Math. **435** (1968), 3–26.

[33] Schober, G., *Univalent Functions-Selected Topics*, Lecture Notes in Math. 478, Springer–Verlag, 1975.

[34] Siewierski, L., *The local solution of coefficient problem for bounded schlicht functions*, Soc. Sci. Lodziensis, Sec. III, 68, 1960.

[35] ———, *Sharp estimation of the coefficients of bounded univalent functions near the identity*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. **16** (1968), 575–576.

[36] ———, *Sharp estimation of the coefficients of bounded univalent functions close to identity*, Dissertationes Math., 86, 1971.

[37] Śmiałek, H., *The limit functions*, Bull. Soc. Sci. Lett. Łódź **16** (1965), 1–15.

[38] Tammi, O., *On the maximalization of the coefficient* a_3 *of bounded schlicht functions*, Ann. Acad. Sci. Fenn., Ser. A I Math. **140** (1953), 1-14.

[39] ———, *On the geometry of the coefficient body* (a_2, a_3) *for bounded univalent functions and some related coefficient problems*, J. Anal. Math. **46** (1986), 304–317.

[40] Włodarczyk, K., *On certain non-homogeneous combinations of the coefficients of bounded univalent functions near the identity*, Demonstratio Math. **16** (1983), 919–924.

[41] Zielińska, A., *On some extremal problem in classes of bounded univalent functions*, Demonstratio Math. **15** (1982), 381–396.

[42] ———, K. Zyskowska, *On estimation of the eighth coefficient of bounded univalent functions with real coefficients*, Demonstratio Math. **12** (1979), 231–246.

[43] ———, *Estimation of the sixth coefficient in the class of univalent bounded functions with real coefficients*, Ann. Polon. Math. **40** (1983), 245-257.

Chair of the Special Functions
University of Łódź
ul. S. Banacha 22
90-238 Łódź, Poland
e-mail: zjakub@math.uni.lodz.pl

received November 25, 1999