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A coefficient product estimate for bounded univalent functions

*Dedicated to Professor Zdzisław Lewandowski
on his 70-th birthday*

Abstract. For given integers m and n, $2 < m < n$, we consider the problem $\max|a_m a_n|$ in the class $S(M)$ of holomorphic, univalent and bounded functions $f(z) = z + a_2z^2 + \dots$ in the unit disk $|z| < 1$. We prove that $\max|a_m a_n|$ is realized by the Pick function for M close to 1 iff $(m-1)$ and $(n-1)$ are relatively prime.

1. Introduction. In what follows we deal with the class $S(M)$, $M > 1$, of holomorphic functions f in the unit disk $D = \{z : |z| < 1\}$ which have the form

$$f(z) = z + a_2z^2 + \dots, \quad z \in D,$$

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and are univalent and bounded by M in D, i.e. $|f(z)| < M,\ z \in D$.

Very little is known about the coefficient problem within the class $S(M)$. Among others we suggest [3], [13], [14], [11], [12] as the references.

The important role in this class plays the so-called Pick function $P_M(z)$ given by the formula

$$P_M(z) = 2z\left[(1-z)^2 + \frac{2}{M}z + (1-z)\sqrt{(1-z)^2 + \frac{4}{M}z}\right]^{-1}, \quad z \in D,$$

which maps the unit disk D onto the disk $D_M = \{w : |w| < M\}$ slit along the segment $[-M, -M(2M - 1 - 2\sqrt{M(M-1)})]$.

If $M \to \infty$, then the class $S(M)$ reduces to the well known class S and the Pick function reduces to the Koebe function $K(z) = z(1-z)^{-2}$ which is known to be extremal for $\max|a_n|,\ f \in S,\ n = 2, 3, \ldots$, by the famous de Branges result [2].

Contrary to that, in the class $S(M)$ the Pick function is not extremal for $\max|a_3|$ if $M \in (1, e)$. However, for the functional $|a_2 a_n|,\ n \geq 3$, the Pick function is extremal if M is close to 1, [6]. The last result proved the conjecture [5] characterizing the extremal property of the Pick function for M close to 1.

In this note we extend the result from [6] and we will prove that

$$\max_{f \in S(M)} |a_m a_n|, \quad 2 < m < n, \tag{1}$$

is attained by the Pick function for M close to 1 iff $(m-1)$ and $(n-1)$ are relatively prime.

We have

Theorem 1. *For every given integers m, n, $2 < m < n$, such that $(m-1)$ and $(n-1)$ are relatively prime there exists $M_{m,n} > 1$ such that for all $M \in (1, M_{m,n})$ the functional $|a_m a_n|$ is maximized in $S(M)$ only by the Pick function $P_M \in S(M)$ or its rotations.*

Theorem 2. *Let integers m, n, $2 < m < n$, be such that $(m-1)$ and $(n-1)$ are not relatively prime. Then the functional $|a_m a_n|$ is not maximized by rotations of the Pick function, when M is close to 1.*

In order to prove the above Theorems, we will apply the Loewner equation for the class $S(M)$ and optimal control theory results adjusted to univalent functions problems as developed in [8]. This method has been successfully applied to other coefficient problems in the class $S(M)$ (e.g. [9], [10]).

2. Auxiliary Theorems and Lemmas.

Theorem A (Loewner equation) [4]. *Let $w = w(z,t)$ be the solution of the Loewner equation*

$$\frac{dw}{dt} = -w\frac{e^{iu}+w}{e^{iu}-w}, \quad w|_{t=0} = z, \quad 0 \le t \le \log M, \tag{2}$$

with a piecewise continuous function $u = u(t)$.

Then

$$w(z,t) = e^{-t}[z + a_2(t)z^2 + a_3(t)z^3 + \dots], \quad z \in D, \quad t \ge 0, \tag{3}$$

is holomorphic and univalent with respect to $z \in D$ for every $t \ge 0$. Moreover, the functions given by the formula

$$f(z) := Mw(z, \log M) \in S(M), \tag{4}$$

form a dense subclass of $S(M)$.

Remark 1. In the case $u(t) = const.$, the function f given by (4) is the Pick function $P_M(z)$ or its rotations and in the case when $u(t)$ is a smooth function on $[0, \log M]$, the corresponding f maps D onto D_M minus a smooth slit.

In general, piecewise smooth functions $u(t)$ correspond to mappings f of D onto D_M with a finite number of smooth slits [8].

The function $u(t)$ will be called *the control function.*

Remark 2. By the generalized Loewner equation [8] we mean the differential equation of the form

$$\frac{dw}{dt} = -w\sum_{j=1}^{n}\lambda_j\frac{e^{iu_j}+w}{e^{iu_j}-w}, \quad w|_{t=0} = z, \quad 0 \le t \le \log M, \tag{5}$$

where $\lambda_j \ge 0$, $j = 1,\dots,n$, and $\sum_{j=1}^{n}\lambda_j = 1$. The functions $u_j(t)$, $j = 1,\dots,n$, are again piecewise continuous functions [8].

In the case when $u_j(t)$, $j = 1,\dots,n$, are smooth functions on $[0, \log M]$ the functions f of the form (4) obtained from the equation (5) map D onto D_M with a finite number of smooth slits.

Moreover, in the case when $f^* \in S(M)$ is a boundary function of the coefficient region

$$V_n^M = \{a = (a_2,\dots,a_n) : f \in S(M)\}$$

there exists the unique system of continuous functions $u_1, \dots, u_{n-1}$ and non-negative constants $\lambda_1, \dots, \lambda_{n-1}$, $\sum_{j=1}^{n-1} \lambda_j = 1$, such that $f^*(z) = Mw(z, \log M)$, where $w = w(z,t)$ is the solution of (5).

The parametric representation for the coefficients obtained from the Loewner equation allows us to apply the classical variational methods [1] or the Pontryagin maximum principle [7] to solve extremal problems in the class $S(M)$.

Indeed, let $a_k(t)$ be given by (3), $a_k(t) = x_{2k-1}(t) + ix_{2k}(t)$, $k = 2, \dots, n$, and $a(t) = (a_1(t), a_2(t), \dots, a_n(t))^T$, $a_1(t) = 1$, $a^0 = (1, 0, \dots, 0)^T$, $x(t) = (x_3(t), \dots, x_{2n}(t))$. Substituting (3) into (2) we obtain the following differential equation for $a(t)$ [8]:

$$\frac{da(t)}{dt} = -2 \sum_{s=1}^{n-1} e^{-s(t+iu)} A^s(t) a(t), \quad a(0) = a^0, \tag{6}$$

where

$$A(t) = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ a_1(t) & 0 & \dots & 0 & 0 \\ a_2(t) & a_1(t) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1}(t) & a_{n-2}(t) & \dots & a_1(t) & 0 \end{pmatrix}.$$

The k-th row in the formula (6) for vector $a(t)$ is a system of two equations for x_{2k-1} and x_{2k}. We will write equations for coordinates of $x(t)$ as follows

$$\frac{dx_k}{dt} = g_k(t, x, u), \; x_k(0) = 0, \; x_{2k-1}(\log M) + ix_{2k}(\log M) = a_k. \tag{7}$$

Note that

$$g_{2k-1}(0,0,u) = -2\cos(k-1)u, \quad g_{2k}(0,0,u) = 2\sin(k-1)u, \quad k \geq 2. \tag{8}$$

The coefficient region V_n^M is the reachable set for the system (7), i.e. the set of all possible values of $x(\log M)$ which can be obtained as solutions of (7) with arbitrary piecewise continuous functions $u = u(t)$. To find V_n^M it is sufficient to describe its boundary ∂V_n^M. Every boundary point $a \in \partial V_n^M$ is represented by a solution of (7) with $u(t)$ satisfying corresponding variational equations. As proved in [8], in order to describe all the boundary functions $f \in S(M)$ which correspond to the boundary points of the coefficient region V_n^M we have to consider the following Hamilton function

$$H(t, x, \psi, u) = \sum_{k=3}^{2n} g_k(t, x, u) \psi_k, \tag{9}$$

where $\psi = \psi(t) = (\psi_3(t), \dots, \psi_{2n}(t))$ is the nonzero conjugate vector which satisfies the conjugate hamiltonian system

$$\frac{d\psi_k}{dt} = -\frac{\partial H}{\partial x_k}, \quad \psi_k(0) = \xi_k, \quad k = 3, \dots, 2n. \tag{10}$$

Theorem B [8]. *Let $x(t)$ be a solution of the system (7) with a piecewise continuous control function $u^*(t)$. If $x = x(\log M)$ is a boundary point of V_n^M, then there exists a solution $\psi = \psi(t)$ of the system (10) with the same control function $u^*(t)$ such that*

$$\max_u H(t, x(t), \psi(t), u) = H(t, x(t), \psi(t), u^*(t)) \tag{11}$$

for all $t \in [0, \log M]$ for which $u^*(t)$ is continuous.

The condition (11) is called *the Pontryagin maximum principle.* Evidently $u^*(t)$ is a root of the equation

$$H_u(t, x, \psi, u) = 0 \tag{12}$$

at the continuity points of $u^*(t)$.

Denote $\xi = (\xi_3, \dots, \xi_{2n})$. In particular, at $t = 0$ we have

$$H(0, 0, \xi, u) = -2\sum_{k=2}^{n}(\xi_{2k-1}\cos(k-1)u - \xi_{2k}\sin(k-1)u). \tag{13}$$

Varying the initial data ξ in (10) and solving the systems (7) and (10) with the control functions u satisfying the Pontryagin maximum principle, we obtain all the boundary points of $x(\log M)$ of V_n^M.

Since the conjugate system (10) is linear with respect to ψ, the vector $\psi(t)$ depends linearly on the initial data ξ in (10). The maximizing property of a control function u satisfying the Pontryagin maximum principle is also preserved if ψ is multiplied by a positive number. This allows us to normalize the initial vector ξ in a suitable form. Such a suitable normalization will be introduced and explained later.

For certain values ξ the Hamilton function may have several maximum points $u \in (-\pi, \pi]$. In this case instead of (2) we have to use the generalized Loewner differential equation (5) and the corresponding system of differential equations for $x_k(t), \psi(t)$ instead of (7) and (10). More precisely, if at $t = 0$ the Hamilton function $H(0, 0, \xi, u)$ attains its maximum at m different points $u_1, \dots, u_m$ in $(-\pi, \pi]$, then we have to use the generalized Loewner differential equation (5) with index m.

Theorem C [8]. *Let $x(t)$ be a solution of the system* (7) *with a control function $u^*(t)$ and $x = x(\log M)$ be a boundary point of V_n^M. If $H(0,0,\xi,u)$ attains its maximal value at exactly one point in $(-\pi,\pi]$ for which $H_{uu}(0,0,\xi,u) \neq 0$, then $u^*(t)$ is continuous on $[0,\log M]$.*

Note that because of possible rotation in $S(M)$ the extremal problem (1) can be reduced to the following:

$$\Re a_m a_n \to \max . \tag{14}$$

Suppose $f \in S(M)$ maximizes $\Re a_m a_n$ in $S(M)$. Then $f_\alpha(z) = e^{-i\alpha} f(e^{i\alpha} z)$ with $\alpha = 2\pi k/(m+n-2)$, $k = 0,1,\dots,m+n-3$, also maximizes $\Re a_m a_n$ in $S(M)$. Evidently $a_m \neq 0$ for the extremal function. Hence, if $m-1$ and $n-1$ are relatively prime, there exists an extremal function $f \in S(M)$ for which

$$-\frac{\pi}{m+n-2} < \pi - \arg a_m \le \frac{\pi}{m+n-2}.$$

Denote such a function by f_0^M. It is known [3] that f_0^M maps D onto D_M minus finitely many piecewise analytic curves. Hence, it may be represented by the formula (4), where $w(z,t)$ is the solution of the Loewner differential equation (2).

Lemma 1. *Assume that the extremal function $f_0^M \in S(M)$ is given by the formula* (4) *where $w(z,t)$ has the expansion* (3) *and is a solution of the Loewner differential equation* (2). *Then*

$$a_m(t) = -2t + o(t), \quad a_n(t) = -2t + o(t), \quad a_m(t)a_n(t) = 4t^2 + o(t^2), \tag{15}$$

$$a_m = -2(M-1) + o(M-1), \quad a_n = -2(M-1) + o(M-1),$$

$$a_m a_n = 4(M-1)^2 + o((M-1)^2), \tag{16}$$

as $t \to 0^+$ and $M \to 1^+$.

Proof. Denote

$$I(t) = \Re a_m(t)a_n(t) = x_{2m-1}(t)x_{2n-1}(t) - x_{2m}(t)x_{2n}(t).$$

Using (8) after differentiating $I(t)$ we have

$$\frac{dI}{dt}\Big|_{t=0} = 0, \tag{17}$$

$$\begin{aligned}(18)\quad \frac{d^2 I}{dt^2}|_{t=0} &= 2\left(\frac{dx_{2m-1}}{dt}\frac{dx_{2n-1}}{dt}\right)_{t=0} - 2\left(\frac{dx_{2m}}{dt}\frac{dx_{2n}}{dt}\right)_{t=0} \\ &= 8(\cos(m-1)u(0)\cos(n-1)u(0) - \sin(m-1)u(0)\sin(n-1)u(0)) \\ &= 8\cos(m+n-2)u(0).\end{aligned}$$

The formulas (17) and (18) imply the asymptotic expansion

$$(19)\quad \Re a_m a_n = I(\log M) = 4\cos(m+n-2)u(0)(M-1)^2 + o((M-1)^2).$$

Solving the extremal problem (14) for M close to 1 we have to maximize the first nonzero term in the asymptotic expansion (19). Hence, we have to put
$\cos(m+n-2)u(0) = 1$ in (19) or equivalently

$$u(0) = u_j(0) = \frac{2\pi j}{m+n-2}, \quad j = 0, 1, \dots, m+n-3.$$

Therefore, for the extremal function we have

$$\Re a_m a_n = 4(M-1)^2 + o((M-1)^2).$$

From (8) we obtain the asymptotic expansion for a_m and a_n

$$\begin{aligned} a_m &= -2\cos(m-1)u_j(0)(M-1) + o(M-1), \\ a_n &= -2\cos(n-1)u_j(0)(M-1) + o(M-1).\end{aligned}$$

For the extremal function f_0^M we can choose $j = 0$ or equivalently $u_j(0) = 0$, which implies in this case

$$a_m(t) = -2t + o(t), \quad a_n(t) = -2t + o(t),$$

and hence

$$a_m(t)a_n(t) = 4t^2 + o(t^2).$$

For $t = \log M$ this gives

$$a_m = -2(M-1) + o(M-1), \quad a_n = -2(M-1) + o(M-1),$$

which ends the proof of Lemma 1. □

The coefficient region V_n^M is the closed set in $\mathbb{R}^{2n-2}$ whose points have coordinates $(x_3, \dots, x_{2n})$. For $c \in \mathbb{R}$, let us define the surface Q_c in $\mathbb{R}^{2n-2}$ by the equation

$$(20)\quad x_{2m-1}x_{2n-1} - x_{2m}x_{2n} = c.$$

In order to solve the extremal problem (14) we have to find the maximal value c such that the surface Q_c has a nonempty intersection with V_n^M. Denote $Q_{c=\max} = Q$ and assume that

$$x \in Q \cap V_n^M, \text{ where } x = x(\log M) = (x_3(\log M), \dots, x_{2n}(\log M)).$$

Without loss of generality we assume that the point $x \in V_n^M$ which corresponds to the extremal function f_0^M is such that $x_{2m-1} < 0$ and $x_{2n-1} < 0$ when M is close to 1.

Lemma 2. *The normal vector $\overline{n}$ to the surface Q at the point x has the form*

$$\overline{n} = \left[0, \dots, 0, -\frac{\Re a_n}{\Re a_m}, \frac{\Im a_n}{\Re a_m}, 0, \dots, 0, -1, \frac{\Im a_m}{\Re a_m}\right], \tag{21}$$

where the first two nonzero coordinates correspond to the indices $2m-1$ and $2m$.

Lemma 2 has been proved in [6] for $m = 2$. Its generalization for arbitrary $m > 2$ is evident.

Note that the representation formula (21) for $\overline{n}$ follows from the normalization condition $x_{2n-1} = -1$. Since we have $x_{2m-1} < 0$ and $x_{2n-1} < 0$, the chosen direction of $\overline{n}$ corresponds to increasing parameter c of the level surfaces Q_c.

Remark that functions $g_3, \dots, g_{2n}$ on the right-hand side of (7) do not depend on x_{2n-1}, x_{2n}. Therefore,

$$\frac{d\psi_{2n-1}}{dt} = \frac{d\psi_{2n}}{dt} = 0$$

and we may assume that $\psi_{2n-1}(t) = -1$.

When c is the maximal value in (20), the normal vector $\overline{n}$ is orthogonal to the tangent or support hyperplane to V_n^M at the point x. The boundary point x is delivered by the extremal function $f_0^M \in S(M)$ represented by an integral of the Loewner equation (2) or the generalized Loewner equation (5). It is known from the calculus of variations [1] or from the optimal control theory [7] that the conjugate vector $\psi(\log M)$ of the system (10) or its analogue which corresponds to (5) is also orthogonal to the tangent or support hyperplane to V_n^M at the point x. Therefore, we can normalize $\psi(\log M)$ in such a way that it coincides with $\overline{n}$.

The condition

$$\psi(\log M) = \overline{n} \tag{22}$$

is called *the transversality condition* at the point x. This is the necessary condition for our extremal problem.

Lemma 3. *If M is close to 1, then the extremal function f_0^M is given by the formula* (4) *where $w(z,t)$ is a solution of the Loewner differential equation* (2) *with a continuous control function $u(t)$.*

Proof. Let us observe that since $(m-1)$ and $(n-1)$ are relatively prime, the function

$$h(u) = 2(\cos(m-1)u + \cos(n-1)u)$$

attains its maximum on $[-\pi,\pi]$ only at $u=0$. Indeed, $h(u)\le 4$ and $h(u)=4$ only if $\cos(m-1)u=1$ and $\cos(n-1)u=1$, which is possible on $[-\pi,\pi]$ only for $u=0$.

From (16) we see that the nonzero coordinates of the normal vector $\overline{n}$ have the asymptotic expansions

$$(23)\qquad -\frac{\Re a_n}{\Re a_m}=-1+o(M-1),\quad \frac{\Im a_n}{\Re a_m}=o(M-1),\quad \frac{\Im a_m}{\Re a_m}=o(M-1).$$

The functions $-\frac{\partial H}{\partial x_k}$ on the right-hand side of (10) are bounded for $0\le t\le \log M$, which means that $\psi_k(t)$ are close to ξ_k if t is close to 0. Therefore, according to the transversality conditions the initial data ξ for the extremal function f_0^M are close to the vector $\xi^0=(0,\dots,0,-1,0,\dots,0,-1,0)$ where -1 first appears at the $(2m-1)$-th place.

Notice that the function

$$H(0,0,\xi^0,u)=h(u)=2(\cos(m-1)u+\cos(n-1)u)$$

has only one absolute maximum in $(-\pi,\pi]$ at $u=0$ and $H_{uu}(0,0,\xi^0,0)=h''(0)=-2((m-1)^2+(n-1)^2)<0$. This property is preserved for ξ close to ξ^0, i.e. $H(0,0,\xi,u)$ attains its maximal value at exactly one point in $(-\pi,\pi]$ for which $H_{uu}(0,0,\xi,u)<0$ for all ξ from the neighborhood of ξ^0 including the point ξ corresponding to the extremal function f_0^M. Applying Theorem C we end the proof of Lemma 3. □

Lemma 3 guarantees that the control function u in the right-hand side of (7) and (10) is the analytic branch of the implicit function $u=u(t,x,\psi)$ determined by the equation (12) with the initial value $u(0,0,\xi^0)=0$.

Vectors x, ψ being the solution of the systems (7) and (10) with $u=u(t,x,\psi)$ on their right-hand sides depend only on t and ξ, i.e. $x=x(t,\xi)$, $\psi=\psi(t,\xi)$. Put

$$u(t,\xi)=u(t,x(t,\xi),\psi(t,\xi)).$$

Remark 3. In particular, the initial value for $u(0,0,\xi^0)$ means that $u(0,\xi^0)=0$.

Every control function $u(t)$ corresponding to the extremal function f of the extremal problem (14) has to satisfy two necessary conditions: the Pontryagin maximum principle (11) and the transversality condition (22). Now we are able to show that the control function $u(t)=0$ which corresponds to the rotation: $-P_M(-z)$ of the Pick function satisfies both of these necessary conditions.

Lemma 4. *The control function $u(t)=0$ satisfies the Pontryagin maximum principle* (11) *and the transversality condition* (22) *if M is close to 1.*

Proof. The control function $u(t) = 0$ in the Loewner differential equation (2) corresponds to the function $-P_M(-z)$ for all $M > 1$. It is well known that this function has real coefficients. Hence $x_{2k}(t) = 0, \quad k = 2, \dots, n$, in (7) as well as $g_{2k}(t, x, u) = 0, \ k = 2, \dots, n$.

From the explicit formulas for the right-hand side of the system (10) (see e.g. [8]) it follows that in this case

$$-\frac{\partial H}{\partial x_{2k}} = 0, \quad k = 2, \dots, n.$$

Therefore, the initial equations $\xi_{2k} = 0, \ k = 2, \dots, n$, are preserved for the whole trajectory $(x(t), \psi(t))$, namely, $\psi_{2k}(t) = 0, \ k = 2, \dots, n, \ 0 \le t \le \log M$.

Let $x(t)$ be the solution of the system (7) with $u = 0$ on its right-hand side and $\psi(t)$ be the solution of the Cauchy problem for the system (10) with $x(t)$ given by (7) and $u = 0$ on its right-hand side. Instead of the initial data at $t = 0$ in (10) we consider the initial data

$$\psi(\log M) = \left(0, \dots, 0, -\frac{x_{2n-1}(\log M)}{x_{2m-1}(\log M)}, 0, \dots, 0, -1, 0\right),$$

which is equivalent to the transversality condition (22). In this case we integrate the system (10) from $\log M$ to 0. Write $\psi(0) = \xi^P = (\xi_3^P, \dots, \xi_{2n}^P)$. The choice of the initial data $\xi = \xi^P$ at $t = 0$ guarantees that the transversality condition (22) is satisfied. Note that $\psi(\log M)$ and $\psi(0)$ are close to $\xi^0 = (0, \dots, 0, -1, 0, \dots, 0, -1, 0)$. Moreover, $\psi_{2k} = 0, \ \xi_{2k}^P = 0, \ k = 2, \dots, n$.

The Hamilton function $H(t, x, \psi, u)$ as a function of u is a polynomial of the $(n-1)$-th degree with respect to $\cos u$. It is "close" to $h(u) = 2(\cos(m-1)u + \cos(n-1)u)$ if M is close to 1. As we have pointed out, this is a polynomial with respect to $\cos u$ and polynomials $h_\epsilon(u)$ which are close to it have only one absolute maximum at $\cos u = 1$ and $h''_\epsilon(0) < 0$. This means that $u(t) = 0$ satisfies the Pontryagin maximum principle (11), which ends the proof of Lemma 4. $\square$

Recall that the maximum principle generates the function $u(t, \xi)$ by the equation (12). It was shown in [6] that $u(t, \xi)$ has bounded partial derivatives in a neighborhood of the point $(t, \xi) = (0, \xi^0)$.

3. Proofs of the theorems.

Proof of Theorem 1. We wish to show that there exists the unique point ξ^* in the neighborhood of ξ^0 for which the solution of the systems (7) and (10) satisfies the maximum principle (11) and the transversality condition (22). As soon as the point ξ^P corresponding to the rotation $-P_M(-z)$ of the Pick function also generates the solution of (7) and (10) satisfying these necessary extremum conditions, we obtain $\xi^* = \xi^P$.

The problem is difficult due to the fact that the value $\psi(\log M)$ in the transversality condition (22) depends on unknown coefficients a_m, a_n. To avoid this difficulty we require that according to Lemma 1, $x_{2m-1}(t,\xi) < 0$ and introduce the vector

$$\psi^*(t,\xi) = \psi(t,\xi) - \overline{m(t,\xi)}, \tag{24}$$

where

$$\overline{m(t,\xi)} = \left(0,\dots,0,-\frac{x_{2n-1}(t,\xi)}{x_{2m-1}(t,\xi)},\frac{x_{2n}(t,\xi)}{x_{2m-1}(t,\xi)},0,\dots,0,\frac{x_{2m}(t,\xi)}{x_{2m-1}(t,\xi)}\right). \tag{25}$$

The transversality condition can now be written in the form

$$\psi^*(\log M, \xi^*) = (0,\dots,0,-1,0). \tag{26}$$

From (7) we obtain the differential equations and from (8) the initial data for $(-\overline{m(t,\xi)})$

$$\frac{d}{dt}\frac{x_{2n-1}}{x_{2m-1}} = \frac{g_{2n-1}(t,x,u)x_{2m-1} - g_{2m-1}(t,x,u)x_{2n-1}}{x_{2m-1}^2} = G_{2m-1}(t,x,u),$$

$$\frac{x_{2n-1}}{x_{2m-1}}\Big|_{t=0} = \frac{\cos(n-1)u(0,\xi)}{\cos(m-1)u(0,\xi)},$$

$$-\frac{d}{dt}\frac{x_{2n}}{x_{2m-1}} = -\frac{g_{2n}(t,x,u)x_{2m-1} - g_{2m-1}(t,x,u)x_{2n}}{x_{2m-1}^2} = G_{2m}(t,x,u),$$

$$-\frac{x_{2n}}{x_{2m-1}}\Big|_{t=0} = \frac{\sin(n-1)u(0,\xi)}{\cos(m-1)u(0,\xi)},$$

$$-\frac{d}{dt}\frac{x_{2m}}{x_{2m-1}} = -\frac{g_{2m}(t,x,u)x_{2m-1} - g_{2m-1}(t,x,u)x_{2m}}{x_{2m-1}^2} = G_{2n}(t,x,u),$$

$$-\frac{x_{2m}}{x_{2m-1}}\Big|_{t=0} = \frac{\sin(m-1)u(0,\xi)}{\cos(m-1)u(0,\xi)}.$$

This implies that the function $\psi^*(t) = (\psi_3^*(t), \dots, \psi_{2n}^*(t))$ is the solution of the system of differential equations

$$\frac{d\psi_{2m-1}^*}{dt} = -\frac{\partial H}{\partial x_{2m-1}} + G_{2m-1}, \quad \psi_{2m-1}^*(0) = \xi_{2m-1} + \frac{\cos(n-1)u(0,\xi)}{\cos(m-1)u(0,\xi)}, \tag{27}$$

$$\frac{d\psi_{2m}^*}{dt} = -\frac{\partial H}{\partial x_{2m}} + G_{2m}, \quad \psi_{2m}^*(0) = \xi_{2m} + \frac{\sin(n-1)u(0,\xi)}{\cos(m-1)u(0,\xi)}, \tag{28}$$

$$\frac{d\psi_k^*}{dt} = -\frac{\partial H}{\partial x_k}, \quad \psi_k^*(0) = \xi_k, \quad k = 3, \dots, 2m-2, 2m+1, \dots, 2n-2, \tag{29}$$

$$\frac{d\psi_{2n}^*}{dt} = G_{2n}, \quad \psi_{2n}^*(0) = \xi_{2n} + \frac{\sin(m-1)u(0,\xi)}{\cos(m-1)u(0,\xi)}. \tag{30}$$

Let us consider the mapping

$$F : \xi \to y = \psi^*(\log M, \xi),$$

where ξ is from a neighborhood of ξ^0.

The function $y = F(\xi)$ maps the initial data analytically depending on ξ onto the solution of the Cauchy problem (27) - (30). Hence, F is analytic and its derivative F_ξ is the Jacobi matrix consisting of elements

$$\psi_{jk}^*(\log M, \xi) = \frac{\partial \psi_j^*(\log M, \xi)}{\partial \xi_k}, \quad j, k = 3, \dots, 2n-2, 2n.$$

To determine $\psi_{jk}^*(t, \xi)$ we differentiate the equations (27)-(30) with respect to $\xi_k,\ k = 3, \dots, 2n-2, 2n$. The derivatives x_ξ and ψ_ξ can be found in the following way. Differentiating the systems (7) and (10) with respect to ξ gives the system of differential equations

$$\frac{dx_\xi}{dt} = L(t, x, u, x_\xi, u_\xi), \quad x_\xi(0, \xi) = 0, \tag{31}$$

$$\frac{d\psi_\xi}{dt} = N(t, x, \psi, u, x_\xi, \psi_\xi, u_\xi), \quad \psi_\xi(0, \xi) = 1. \tag{32}$$

The initial data 0 and 1 in (31) and (32) are the zero matrix and the unit matrix respectively. The right-hand sides L and N in (31) and (32) are linear with respect to x_ξ, ψ_ξ and u_ξ. To find u_ξ we differentiate the equation (12) with respect to ξ, and obtain

$$H_{ux}(t,x,\psi,u)x_\xi + H_{u\psi}(t,x,\psi,u)\psi_\xi + H_{uu}(t,x,\psi,u)u_\xi = 0$$

which implies the formula

$$u_\xi = -(H_{ux}(t,x,\psi,u)x_\xi + H_{u\psi}(t,x,\psi,u)\psi_\xi)/H_{uu}(t,x,\psi,u). \tag{33}$$

Substituting u_ξ from (33) to (31) and (32) we solve the Cauchy problem for the obtained system of differential equations. The solution $(x_\xi(t,\xi), \psi_\xi(t,\xi))$ of the Cauchy problem is bounded for ξ close to ξ^0 and t close to 0. The obtained system of differential equations with respect to ψ_{jk} has a solution as the Cauchy problem smoothly depending on initial data.

There remains to determine the initial data $\eta_{jk} = \psi^*_{jk}(0,\xi)$, $j,k = 3,\dots,2n-2,\ 2n$. Differentiating the initial data in (27)-(30) with respect to $\xi_3,\dots,\xi_{2n-2},\xi_{2n}$, we obtain

$$\eta_{(2m-1)k}(\xi) = \delta_{(2m-1)k} - [(n-1)\sin(n-1)u(0,\xi)\cos(m-1)u(0,\xi) -$$

$$(m-1)\cos(n-1)u(0,\xi)\sin(m-1)u(0,\xi)]\frac{u_{\xi_k}(0,\xi)}{\cos^2(m-1)u(0,\xi)}, \tag{34}$$

$$k = 3,\dots,2n-2,2n,$$

$$\eta_{(2m)k}(\xi) = \delta_{(2m)k} + [(n-1)\cos(n-1)u(0,\xi)\cos(m-1)u(0,\xi) +$$

$$(m-1)\sin(n-1)u(0,\xi)\sin(m-1)u(0,\xi)]\frac{u_{\xi_k}(0,\xi)}{\cos^2(m-1)u(0,\xi)}, \tag{35}$$

$$k = 3,\dots,2n-2,2n,$$

(36)
$$\eta_{jk}(\xi) = \delta_{jk}, \quad j = 3,\dots,2m-2,2m+1,\dots,2n-2,2n,\ k = 3,\dots,2n-2,2n,$$

(37)
$$\eta_{(2n)k}(\xi) = \delta_{(2n)k} + \frac{m-1}{\cos^2(m-1)u(0,\xi)}u_{\xi_k}(0,\xi), \quad k = 3,\dots,2n-2,2n.$$

To determine $u_{\xi_k}(0,\xi)$ we notice that $H_{u\psi_k}(0,0,\xi,u) = (g_k)_u(0,0,u)$, $k = 3,\dots,2n$, $x_\xi|_{t=0} = 0$, and obtain from (33) at $\xi = \xi^0$ that

$$u_{\xi_k}(0,\xi^0) = \frac{(g_k)_u(0,0,u)}{2((m-1)^2+(n-1)^2)}, \quad k = 3,\dots,2n-2,2n.$$

In particular, taking into account (8) and Remark 3 we have

$$u_{\xi_{2m-1}}(0,\xi^0)=\frac{(m-1)\sin(m-1)u(0,\xi^0)}{(m-1)^2+(n-1)^2}=0, \tag{38}$$

$$u_{\xi_{2m}}(0,\xi^0)=\frac{(m-1)\cos(m-1)u(0,\xi^0))}{(m-1)^2+(n-1)^2}=\frac{m-1}{(m-1)^2+(n-1)^2}, \tag{39}$$

$$u_{\xi_{2n}}(0,\xi^0)=\frac{(n-1)\cos(n-1)u(0,\xi^0)}{(m-1)^2+(n-1)^2}=\frac{n-1}{(m-1)^2+(n-1)^2}. \tag{40}$$

Substitution of (38) - (40) to (34), (35) and (37) at $\xi=\xi^0$ gives

$$\eta_{(2m-1)k}(\xi^0)=\delta_{(2m-1)k},\quad k=3,\dots,2n-2,2n, \tag{41}$$

$$\begin{aligned}\eta_{(2m)(2m)}(\xi^0)&=1+\frac{(m-1)(n-1)}{(m-1)^2+(n-1)^2},\\ \eta_{(2m)(2n)}(\xi^0)&=\frac{(n-1)^2}{(m-1)^2+(n-1)^2},\end{aligned} \tag{42}$$

$$\begin{aligned}\eta_{(2n)(2m)}(\xi^0)&=\frac{(m-1)^2}{(m-1)^2+(n-1)^2},\\ \eta_{(2n)(2n)}(\xi^0)&=1+\frac{(m-1)(n-1)}{(m-1)^2+(n-1)^2}.\end{aligned} \tag{43}$$

Let $B(t,\xi)$ be the Jacobi matrix consisting of elements $\psi^*_{jk}(t,\xi)$. According to (36), (41) - (43) we have

$$\det B(0,\xi^0)=1+\frac{2(m-1)(n-1)}{(m-1)^2+(n-1)^2}>0.$$

Hence, $\det B(\log M,\xi^0)>0$ if M is close to 1. This means that $B(\log M,\xi^0)=F_\xi(\xi^0)$ is invertible and F maps a neighborhood $U_\epsilon(\xi^0)=\{\xi:\|\xi-\xi^0\|<\epsilon\}$ of ξ^0 one-to-one onto a neighborhood of $y^0=F(\xi^0)$. Therefore, there exists the unique $\xi\in Q_\epsilon(\xi^0)$ for which the maximum principle (11) and the transversality condition (22) in the form (26) are satisfied. According to Lemma 4 this is $\xi=\xi^P$ which correponds to the function $-P_M(-z)$. This ends the proof of Theorem 1. $\square$

Remark 4. We have proved that in the extremal problem (1) or (14) the necessary extremum conditions in the form of the Pontryagin maximum principle (11) - (12) and the transversality conditions (22) or (26) are in fact the sufficient extremum conditions if M is close to 1.

Proof of Theorem 2. Let $j \geq 2$ be the common divisor of $(m-1)$ and $(n-1)$ and the function $f_1^M(z)$ be given by the formula (4) where $w(z,t)$ is a solution of the generalized Loewner equation (5) with index 2, $\lambda_1 = \lambda_2 = 1/2$, $u_1(t) = 0$ and $u_2(t) = 2\pi/j$. The idea of the proof of Theorem 2 is to compare the asymptotic expansions of $I(t) = \Re a_m(t)a_n(t)$ for two functions: $-P_M(-z)$ and $f_1^M(z)$. Denote $I(t)$ corresponding to $-P_M(-z)$ and $f_1^M(z)$ by $I_P(t)$ and $I_1(t)$ respectively. Note that $-P_M(-z)$ can be represented by formula (4) if the solution $w(z,t)$ of the equation (5) corresponds to $u_1(t) = u_2(t) = 0$.

The differential equation for $a(t)$ generated by the generalized Loewner equation (5) with index 2 has,according to [8], the form

$$\frac{da(t)}{dt} = -2\sum_{j=1}^{2}\lambda_j \sum_{s=1}^{n-1} e^{-s(t+iu_j)}A^s(t)a(t), \quad a(0) = a^0. \tag{44}$$

Recall that according to (17) $I'_P(0) = I'_1(0) = 0$. For given $u_1,\ u_2$ and $\lambda_1,\ \lambda_2$ we have from (44)

$$a'_k(0) = -(e^{-i(k-1)u_1} + e^{-i(k-1)u_2}) = -(1+e^{-i(k-1)2\pi/j}),\ k = 2,\dots,n. \tag{45}$$

In particular, according to (8) and (45)

$$a'_m(0) = -2, \quad a'_n(0) = -2,$$

for both functions $-P_M(-z)$ and $f_1^M(z)$. From here we immediately obtain

$$I''_P(0) = I''_1(0) = 2\Re a'_m(0)a'_n(0) = 8. \tag{46}$$

We now establish the equality

$$I'''_1(0) = 3\Re(a'_m(0)a''_n(0) + a''_m(0)a'_n(0)). \tag{47}$$

Differentiate (44) with respect to t at $t = 0$ with $\lambda_1 = \lambda_2 = 1/2$ and $u_1 = 0$ to obtain

(48)

$$\frac{d^2a(t)}{dt^2}|_{t=0} = -\sum_{s=1}^{n-1}(1+e^{-isu_2})((-s)A^s(0)a^0 + sA^{s-1}(0)A'(0)a^0 + A^s(0)a'(0)).$$

The specific triangular form of the matrices $A(0)$ and $A'(0)$ implies

$$A^s(0)a^0 = (0,\dots,0,1,0,\dots,0)^T,$$

where 1 appears at the $(s+1)$-th place, and

$$A^{s-1}(0)A'(0)a^0 = A^s(0)a'(0) = (0,\dots,0,a_2'(0),\dots,a_{n-s}'(0))^T.$$

Therefore, it follows from (48) that

$$a_m''(0) = 2(m-1) - \sum_{s=1}^{m-2}(1+e^{-isu_2})(s+1)a_{m-s}'(0), \tag{49}$$

$$a_n''(0) = 2(n-1) - \sum_{s=1}^{n-2}(1+e^{-isu_2})(s+1)a_{n-s}'(0). \tag{50}$$

Substituting (45) into (49) and (50) and taking into account that $e^{-i(m-1)u_2} = e^{-i(n-1)u_2} = 1$ we have

$$\begin{aligned} a_m''(0) &= 2(m-1) + \sum_{s=1}^{m-2}(s+1)(2+e^{-isu_2}+e^{-i(m-s-1)u_2}) \\ &= 2(m-1) + \sum_{s=1}^{m-2}(s+1)(2+e^{-isu_2}+e^{isu_2}), \end{aligned} \tag{51}$$

$$\begin{aligned} a_n''(0) &= 2(n-1) + \sum_{s=1}^{n-2}(s+1)(2+e^{-isu_2}+e^{-i(n-s-1)u_2}) \\ &= 2(n-1) + \sum_{s=1}^{n-2}(s+1)(2+e^{-isu_2}+e^{isu_2}). \end{aligned} \tag{52}$$

Finally, (45), (51) and (52) being substituted into (47) give the required formula for $I_1'''(0)$

$$\begin{aligned} I_1'''(0) = &-6\Re(a_m''(0)+a_n''(0)) = -12(m+n-2) \\ &- 12\left[\sum_{s=1}^{m-2}(s+1)(1+\cos su_2) + \sum_{s=1}^{n-2}(s+1)(1+\cos su_2)\right]. \end{aligned} \tag{53}$$

To obtain $I_P'''(0)$ it is sufficient to substitute $u_2 = 0$ instead of u_2 into (53). Hence we have

$$I_P'''(0) = -12(m+n-2) - 24\left[\sum_{s=1}^{m-2}(s+1) + \sum_{s=1}^{n-2}(s+1)\right]. \tag{54}$$

Evidently

$$I_1'''(0) > I_P'''(0).$$

From the asymptotic expansions

$$I_1(t) = \frac{1}{2!}I_1''(0)t^2 + \frac{1}{3!}I_1'''(0)t^3 + o(t^3),$$

$$I_P(t) = \frac{1}{2!}I_P''(0)t^2 + \frac{1}{3!}I_P'''(0)t^3 + o(t^3),$$

and from (46) we claim that $I_1(t) > I_P(t)$ for t close to 0. Therefore the Pick function does not maximize $\Re a_m a_n$ in the class $S(M)$ for M close to 1 which ends the proof of Theorem 2. □

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