ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LVIII, 2004 SECTIO A 17–25

WALDEMAR CIEŚLAK and ROBERT STĘPNICKI

Equipower curves

Abstract. In this paper we consider the family of equipower curves. It is proved that each equipower oval has $4n+2$ vertices ($n \geq 1$) and an example of an equipower oval with exactly six vertices is given. Moreover, it is shown that two vertices lie at ends of one equipower chord. The last sections are devoted to Crofton-type integral formula and estimations of the area and the length of an equipower curve.

1. Introduction. We consider the family K of all ovals, i.e. the family of all plane closed curves with the positive curvature, Leichtweiss [2].

Let a curve $C \in K$ be of the form

$$t \to z(t) = r(t)e^{it},\ t \in [0, 2\pi], \tag{1}$$

where the radius function $r : R \to R$ satisfies the following conditions:

$$\begin{cases} r \in C^3(R) \\ r(t + 2\pi) = r(t) \\ r(t) > 0 \end{cases} \tag{2}$$

2000 *Mathematics Subject Classification.* 52A39, 53A04.
Key words and phrases. Convex bodies, convex curves, ovals, equipower point.

for all $t \in R$. The curvature k of C is given by the formula:

$$k = \frac{2\dot{r}^2 + r^2 - r\ddot{r}}{\left(\sqrt{r^2 + \dot{r}^2}\right)^3} > 0, \tag{3}$$

where the dot denotes the differentiation with respect to t.

We denote by $\alpha(t)$ the oriented angle between the radius vector of the point $z(t)$ and the tangent line to C at $z(t)$.

We note that

$$\cot\alpha = \frac{\dot{r}}{r}\,. \tag{4}$$

Making use of (3) and (4) we get

$$\begin{aligned}
k\left(\sqrt{\dot{r}^2 + r^2}\right)^3 &= 2\,\dot{r}^2 - r\,\ddot{r} + r^2,\\
k\frac{r^3}{\sin^3\alpha} &= 2r^2\cot^2\alpha - r\left(\dot{r}\cot\alpha - r\frac{\dot{\alpha}}{\sin^2\alpha}\right) + r^2,\\
k\frac{r^3}{\sin^3\alpha} &= 2r^2\cot^2\alpha - r^2\cot^2\alpha + r^2\frac{\dot{\alpha}}{\sin^2\alpha} + r^2,\\
k\frac{r^3}{\sin^3\alpha} &= r^2\cot^2\alpha + r^2\frac{\dot{\alpha}}{\sin^2\alpha} + r^2,\\
k\frac{r}{\sin^3\alpha} &= \cot^2\alpha + \frac{\dot{\alpha}}{\sin^2\alpha} + 1,\\
\frac{kr}{\sin\alpha} &= 1 + \dot{\alpha},
\end{aligned}$$

and

$$kr = (1 + \dot{\alpha})\sin\alpha, \tag{5}$$

for all $t \in R$.

2. Equipower ovals. In this paper we consider the subfamily K_p of K containing of all equipower ovals. An equipower oval C is a curve such that there exists a point P in the region bounded by C with the following property:

if a chord of C passes through P and it joins points P_1, P_2 of C then

$$|PP_1||PP_2| = c = \text{const}. \tag{6}$$

and the product does not depend on the choice of a chord.
The point P is called the equipower point of C.

The equipower curves were introduced by Yanagihara [4], [5] and next considered by Kelly [1], Rosenbaum [3] and Zuccheri [6].

We may assume that the origin O is the equipower point of an equipower curve $C \in K_p$. If $C \in K_p$ is of the form (1) then we have

$$r(t)\, r(t+\pi) = c, \tag{7}$$

for all $t \in R$.

In the sequel we will write down $f_\pi(t) = f(t+\pi)$, for all $t \in R$. Thus the formula (7) can be written as follows

$$r r_\pi = c, \tag{8}$$

for all $t \in R$.

We have the following theorems.

Theorem 1. *A curve $C \in K_p$ is an equipower curve if and only if the condition*

$$\alpha + \alpha_\pi = \pi, \quad for\ all\ t \in R \tag{9}$$

is satisfied.

Proof. Differentiating (8) and using (4) we obtain

$$(r r_\pi)^{\cdot} = r r_\pi \left(\frac{\dot{r}}{r} + \frac{\dot{r_\pi}}{r_\pi} \right) = r r_\pi \left(\cot\alpha + \cot\alpha_\pi \right),$$

$$\cot\alpha + \cot\alpha_\pi = 0 \Longleftrightarrow \alpha + \alpha_\pi = \pi.$$

□

Theorem 2. *If an equipower oval $C \in K_p$ satisfies the relation (8) then its curvature k satisfies the equality*

$$\dot{k} r^2 + \dot{k}_\pi c = 0. \tag{10}$$

Proof. From (5) with respect to (9) we get

$$k_\pi r_\pi = (1 - \dot{\alpha}) \sin\alpha. \tag{11}$$

The formulae (5) and (11) lead us to the following formula

$$k r + k_\pi r_\pi = 2 \sin\alpha. \tag{12}$$

Differentiating (12), substituting $\dot{\alpha}$ from (5) and next using (12), we obtain

$$\dot{k} r + k \dot{r} + \dot{k}_\pi r_\pi + k_\pi \dot{r}_\pi = 2\dot{\alpha} \cos\alpha,$$

$$\dot{k} r + \dot{k}_\pi r_\pi + r k \cot\alpha + r_\pi k_\pi \cot\alpha_\pi = 2 \left(\frac{k r}{\sin\alpha} - 1 \right) \cos\alpha,$$

$$\dot{k}r + \dot{k}_\pi r_\pi = 2rk\cot\alpha - 2\cos\alpha - rk\cot\alpha - r_\pi k_\pi \cot\alpha_\pi,$$

$$\dot{k}r + \dot{k}_\pi r_\pi = rk\cot\alpha + r_\pi k_\pi \cot\alpha - 2\cos\alpha,$$

$$\dot{k}r + \dot{k}_\pi r_\pi = (rk + r_\pi k_\pi - 2\sin\alpha)\cot\alpha$$

and

$$\dot{k}r + \dot{k}_\pi r_\pi = 0. \tag{13}$$

Thus, using (8), we have

$$\dot{k}r^2 + \dot{k}_\pi c = 0.$$

□

The formula (10) has the following geometric interpretation:

Theorem 3. *Let $C \in K_p$ and let an equipower chord of C pass through A, B. If A is the vertex of C, then B is also a vertex.*

Theorem 2 and the four vertices theorem [2], imply immediately:

Corollary 1. *An equipower curve $C \in K_p$ has $4n+2$ vertices, $n \geq 1$.*

3. Equipower ovals with exactly six vertices. According to Corollary 1 an equipower oval has at least six vertices. We construct an oval with exactly six vertices.

Let

$$r(t) = \exp(b\sin t), \tag{14}$$

for all $t \in R$, for some fixed $b \in (0,1)$. It is clear that $rr_\pi = 1$ and $r(t) > 0$ for all $t \in R$. Moreover, we have

$$k = \frac{1 + b\sin t + b^2\cos^2 t}{(1+b^2\cos^2 t)^{\frac{3}{2}}} \exp(-b\sin t) > 0 \tag{15}$$

and

$$\frac{1}{c}\dot{k}\left(1+b^2\cos^2 t\right)^{\frac{5}{2}} \exp(b\sin t) = -b^3\left[b^2\sin^4 t - 2\left(2+b^2\right)\sin^2 t + \left(1+b^2\right)\right]. \tag{16}$$

From (16) we note that $\dot{k}$ has exactly six zeros in the interval $[0, 2\pi)$.

4. Convex equipower curves. In this section we consider the class K. Let $C \in K$ be of the form (1). We define the function $\lambda : R \to R$ by the formula

$$\lambda = k|\dot{z}|, \tag{17}$$

i.e.

$$\lambda = \frac{2\dot{r}^2 + r^2 - r\ddot{r}}{r^2 + \dot{r}^2}. \tag{18}$$

It is clear that the function λ is a non-negative, continuous and 2π periodic one. With respect to (4) we have

$$\begin{aligned}
\lambda &= 1 - \frac{\ddot{r}r - \dot{r}^2}{r^2 + \dot{r}^2}, \\
\lambda &= 1 - \frac{\left(\frac{\dot{r}}{r}\right)^{\cdot}}{1 + \left(\frac{\dot{r}}{r}\right)^2}, \\
\lambda &= 1 + \left(\operatorname{arccot}\frac{\dot{r}}{r}\right)^{\cdot}, \\
\lambda &= 1 + \dot{\alpha}.
\end{aligned}$$

Thus we have a simple relation between λ and α, namely

$$\lambda = 1 + \dot{\alpha}. \tag{19}$$

Hence we have

$$\int_0^{2\pi} \lambda(t)\, dt = 2\pi. \tag{20}$$

The function

$$\Lambda(t) = \int_0^t \lambda(s)\, ds \tag{21}$$

has the following geometric meaning:

$\Lambda(t)$ is an oriented angle between a tangent line at $z(0)$ and the tangent line at $z(t)$.

We note that the function λ satisfies the following properties:

1° λ is a non-negative function,
2° λ is a 2π-periodic function,
3° λ is a continuous function,

4° $\int_0^{2\pi} \lambda(t)\,dt = 2\pi$,

5° $0 < \int_0^t \lambda(s)\,ds < t + \pi, \quad$ for $t \in [0, 2\pi)$.

Moreover, we note that the function α satisfies the following properties:

1° α is a C^1-function,

2° α is a 2π-periodic function,

3° $\dot{\alpha} + 1 \geq 0$,

4° $\int_0^{2\pi} \cot\alpha(t)\,dt = 0$.

Since we have $\cot\alpha = \frac{\dot{r}}{r}$, so

$$r(t) = r_o \exp \int_0^t \cot\alpha(s)\,ds \tag{22}$$

where $r_o = r(0)$. We may assume that $\alpha(0) = \frac{\pi}{2}$. The relation $\dot{\alpha} = \lambda - 1$ implies the formula

$$r(t) = r_o \exp \int_0^t \tan \int_0^\tau (1 - \lambda(s))\,ds d\tau. \tag{23}$$

Periodicity of r implies the equality

$$\int_0^{2\pi} \tan \int_0^\tau (\lambda(s) - 1)\,ds d\tau = 0. \tag{24}$$

We note that the curvature of C is given by the formula $k = \frac{\lambda}{r}\sin\alpha$,

$$k = \frac{1}{r}(1 + \dot{\alpha})\sin\alpha. \tag{25}$$

Hence we have

$$\dot{k} = \frac{1}{r}\left[\ddot{\alpha} + \left(\dot{\alpha}^2 - 1\right)\cot\alpha\right]\sin\alpha. \tag{26}$$

The functions λ and α allows us to construct special examples of equipower curves.

Example 4.1. Let $\alpha(t) = \frac{\pi}{2} + \arctan\sin t$ for $t \in [0, 2\pi]$. Obviously, α is a regular, 2π-periodic function and $\alpha(t) + \alpha(t + \pi) = \pi$. It is easy to see that $\int_0^{2\pi} \cot\alpha(t)\,dt = 0$ and $1 + \dot{\alpha}(t) = 1 + \frac{\cos t}{1+\sin^2 t} \geq 0$ with the equality at

π only. According to (25) it means that $k \geq 0$ and the curve generated by α is convex but it is not an oval.

Next we have

$$\ddot{\alpha} + \left(\dot{\alpha}^2 - 1\right) \cot \alpha = \frac{\sin t}{\left(1 + \sin^2 t\right)^2} \left(\sin^4 t + 4 \sin^2 t - 3\right),$$

$$\ddot{\alpha} + \left(\dot{\alpha}^2 - 1\right) \cot \alpha = \frac{\sin t}{\left(1 + \sin^2 t\right)^2} \left(\sin^2 t + 2 + \sqrt{7}\right) \left(\sin^2 t + 2 - \sqrt{7}\right).$$

We note that with respect to the above relations and formula (26) the equation $\dot{k}(t) = 0$ has exactly six solutions. It follows from (22) that the polar equation of our curve has the following form

$$r(t) = r_o \exp(\cos t - 1).$$

5. Crofton-type integral formula for equipower curves. Let C_j , $t \to r_j(t) e^{it}$ $(j = 1, 2)$ be different equipower regular curves and $r_j(t) r_j(t + \pi) = c_j$. We assume that C_1 lies in the domain bounded by C_2 (then $c_2 > c_1$). We denote by C_1C_2 the domain bounded by C_1 and C_2, and by D the interior of C_1C_2 with deleted some line segment.

We consider the mapping $F^* : [0, 1] \times [0, 1] \to D$ given by

$$F^*(s, t) = r_2(t)^s r_1(t)^{1-s} e^{it}. \tag{27}$$

For each fixed s_0 the curve $t \to F(s_0, t)$ is an equipower one. Let $E = \{(s, t) : 0 < s < 1,\ 0 < t < 2\pi\}$ and let F denote the restriction of F^* to E.

We note that F is a $1 : 1$ mapping and the jacobian $F'(s, t)$ of F at (s, t) is given by the formula

$$F'(s, t) = \left(r_2(t)^s r_1(t)^{1-s}\right)^2 \ln \frac{r_2(t)}{r_1(t)}. \tag{28}$$

Let $x \in R^2$. We denote by $\|x\|$ the length of the segment joining the origin O and x. Using the diffeomorphism F we get the following theorem.

Theorem 4. *The following Crofton-type integral formula holds*

$$\iint\limits_{C_1C_2} \frac{1}{\|x\|^2} dx = \pi \ln \frac{c_2}{c_1}. \tag{29}$$

Proof. Let $x \in C_1C_2$. We denote by t the oriented angle between the x_1-axe and the segment Ox. Then $x = r_2(t)^s r_1(t)^{1-s} e^{it}$ for some $s \in (0, 1)$

and making use of F' we get

$$\begin{aligned}\iint\limits_{C_1C_2}\frac{1}{\|x\|^2}dx &= \int\limits_0^1\int\limits_0^{2\pi}\frac{1}{\left(r_2(t)^s r_1(t)^{1-s}\right)^2}\left(r_2(t)^s r_1(t)^{1-s}\right)^2\ln\frac{r_2(t)}{r_1(t)}dtds\\ &= \int\limits_0^{2\pi}\ln\frac{r_2(t)}{r_1(t)}dt = \int\limits_0^{\pi}\ln\frac{r_2(t)}{r_1(t)}dt+\int\limits_0^{\pi}\ln\frac{r_2(t+\pi)}{r_1(t+\pi)}dt\\ &= \int\limits_0^{\pi}\ln\frac{c_2}{c_1}dt = \pi\ln\frac{c_2}{c_1}.\end{aligned}$$

□

6. Estimations of the area and the length. Let C, $t\to r(t)e^{it}$ be an equipower curve such that $r(t)r(t+\pi)\equiv c$. Let

$$r_M = \max\{r(t) : t\in[0,2\pi]\} \tag{30}$$

and

$$r_m = \min\{r(t) : t\in[0,2\pi]\}. \tag{31}$$

We denote by S_mC the domain bounded by C and the circle S_m with the center at the origin and the radius r_m. Similarly, we introduce the domain CS_M.

Making use of the Crofton-type integral formula (29) we obtain

$$\pi\ln\frac{c}{r_m^2} = \iint\limits_{S_mC}\frac{1}{\|x\|^2}dx \le \frac{1}{r_m^2}\operatorname{area}S_mC = \frac{1}{r_m^2}\left(\operatorname{area}C-\pi r_m^2\right)$$

and

$$\pi\ln\frac{r_M^2}{c} = \iint\limits_{CS_M}\frac{1}{\|x\|^2}dx \ge \frac{1}{r_M^2}\operatorname{area}CS_M = \frac{1}{r_M^2}\left(\pi r_M^2-\operatorname{area}C\right).$$

Hence we have

$$\operatorname{area}C \ge \max\left\{\pi r_m^2\left(1+\ln\frac{c}{r_m^2}\right),\pi r_M^2\left(1-\ln\frac{c}{r_M^2}\right)\right\}. \tag{32}$$

On the other hand we have

$$2\operatorname{area}C = \int\limits_0^{2\pi}r(t)^2\,dt = \int\limits_0^{\pi}\left(r(t)^2+\frac{c^2}{r(t)^2}\right)dt \ge 2\pi c,$$

and we obtain the well-known inequality

$$\operatorname{area} C \geq \pi c. \tag{33}$$

By similar way we obtain the inequality for the length of an equipower curve. Namely, using (9) and (7) we obtain

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{r(t)^2 + \dot{r}(t)^2}\, dt = \int_0^{2\pi} \frac{r(t)}{\sin\alpha(t)}\, dt \\ &= \int_0^{\pi} \frac{1}{\sin\alpha(t)} \left(r(t) + \frac{c}{r(t)} \right) dt \geq 2\pi\sqrt{c}, \end{aligned}$$

i.e.

$$L \geq 2\pi\sqrt{c}. \tag{34}$$

References

[1] Kelly, J.B., *Power points*, Amer. Math. Monthly **53** (1946), 395–396.

[2] Laugwitz, D., *Differential and Riemannian Geometry*, Academic Press, New York–London, 1965.

[3] Rosenbaum, J., *Power points (Discussion of Problem E 705)*, Amer. Math. Monthly **54** (1947), 164–165.

[4] Yanagihara, K., *On a characteristic property of the circle and the sphere*, Tŏhoku Math. J. **10** (1916), 142–143.

[5] Yanagihara, K., *Second note on a characteristic property of the circle and the sphere*, Tŏhoku Math. J. **11** (1917), 55–57.

[6] Zuccheri, L., *Characterization of the circle by equipower properties*, Arch. Math. **58** (1992), 199–208.

Waldemar Cieślak
Technical University of Lublin
Department of Mathematics
and Engineering Geometry
20-618 Lublin
ul. Nadbystrzycka 40
Poland

Wyższa Szkoła Ekonomiczna
w Stalowej Woli
37-450 Stalowa Wola
ul. 1-go Sierpnia 26
Poland

Robert Stępnicki
Technical University of Lublin
Department of Mathematics
and Engineering Geometry
20-618 Lublin
ul. Nadbystrzycka 40
Poland
e-mail: roberts@enterpol.pl

Received March 14, 2003