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Connections on fibered squares

ABSTRACT. We clarify that the theory of projectable natural bundles over fibered manifolds is essentially related with the idea of fibered square. We deduce the basic properties of the geometrically most interesting kinds of fibered squares and of the corresponding connections. Special attention is paid to linear square connections of order (q, s, r) .

Section 1 of the present paper is devoted to the basic properties of a projectable bundle functor F on the category $\mathcal{FM}_{m,n}$ of fibered manifolds with m -dimensional bases and n -dimensional fibers and their local isomorphisms over a bundle functor \underline{F} on the category $\mathcal{M}f_m$ of m -dimensional manifolds and their local diffeomorphisms. Then we are interested in the fact that FY is a fibered square for every fibered manifold $Y \rightarrow M$. (We prefer this terminology introduced by J. Pradines, [8], to the equivalent notion of fibered fibered manifold by W. Mikulski, [5].) In Section 2 we discuss the most important kinds of fibered squares, namely the principal, associated and vector ones. Special attention is paid to the (q, s, r) -jet prolongation of a fibered square.

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In Section 3 we introduce the general concept of square connection and discuss the principal and linear square connections in more details. In particular, we prove that the operation of associating square connection establishes a bijection between the principal square connections on the linear frame square of a vector square Z and the linear square connections on Z . In the last section we define the linear square connections of order (q, s, r) on a fibered manifold Y and deduce that they are in bijection with the principal square connections on the frame square $P^{q,s,r}Y$ of order (q, s, r) . Finally we remark that the concept of torsion can be extended from the classical case of linear r -th order connections on a manifold M to the linear square connections of order (q, s, r) on a fibered manifold Y .

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notation from the book [3].

1. Natural bundles over (m, n) -manifolds. The concept of a natural bundle F over m -manifolds was introduced by A. Nijenhuis, [6]. Using the terminology of [3], one can say that F is a bundle functor on the category $\mathcal{M}f_m$. It is well known that every natural bundle over m -manifolds has a finite order r and the r -th order bundles are in bijection with the actions of the r -jet group $G_m^r = \text{inv } J_0^r(\mathbb{R}^m, \mathbb{R}^m)_0$, [3]. If we replace $\mathcal{M}f_m$ by the category $\mathcal{FM}_{m,n}$, we can introduce

Definition 1. A natural bundle over (m, n) -manifolds is a bundle functor on $\mathcal{FM}_{m,n}$.

The inclusion $\mathcal{FM}_{m,n} \subset \mathcal{M}f_{m+n}$ implies that every natural bundle over $(m+n)$ -manifolds restricts to a natural bundle over (m, n) -manifolds.

By [3], every bundle functor F on $\mathcal{FM}_{m,n}$ has finite order. However, we shall need a more subtle characterization of the order of F that is based on the concept of (q, s, r) -jet. Write \mathcal{FM} for the category of all fibered manifolds and all their morphisms. Having two fibered manifolds $p : Y \rightarrow M$ and $q : Z \rightarrow N$ and two \mathcal{FM} -morphisms $f, g : Y \rightarrow Z$ with base maps $\underline{f}, \underline{g} : M \rightarrow N$, we say that f and g determine the same (q, s, r) -jet $j_y^{q,s,r} f = j_y^{q,s,r} g$, $s \geq q \leq r$, at $y \in Y$, if

$$(1) \quad j_y^q f = j_y^q g, \quad j_y^s(f|Y_x) = j_y^s(g|Y_x), \quad j_x^r \underline{f} = j_x^r \underline{g}, \quad x = p(y)$$

([1], [3]). We write $J^{q,s,r}(Y, Z)$ for the space of all (q, s, r) -jets of \mathcal{FM} -morphisms of Y into Z . We say that the order of functor F is (q, s, r) , if

$$(2) \quad j_y^{q,s,r} f = j_y^{q,s,r} g \quad \text{implies} \quad Ff|Y_y = Fg|Y_y, \quad y \in Y,$$

for every pair of $\mathcal{FM}_{m,n}$ -morphisms $f, g : Y \rightarrow Z$.

Write $\mathbb{R}^{k,l}$ for the product fibered manifold $\mathbb{R}^k \times \mathbb{R}^l \rightarrow \mathbb{R}^k$. We define the space of (k, l) -dimensional velocities of order (q, s, r) on a fibered manifold

$Y \rightarrow M$ by

$$(3) \quad T_{k,l}^{q,s,r} Y = J_{0,0}^{q,s,r}(\mathbb{R}^{k,l}, Y).$$

Using the jet composition we extend $T_{k,l}^{q,s,r}$ into a bundle functor on the category \mathcal{FM} . In particular, we introduce the principal bundle of all (q, s, r) -frames on Y , $m = \dim M$, $m + n = \dim Y$, by

$$(4) \quad P^{q,s,r} Y = \text{inv } J_{0,0}^{q,s,r}(\mathbb{R}^{m,n}, Y),$$

where inv indicates the invertible (q, s, r) -jets. Its structure group is

$$G_{m,n}^{q,s,r} = \text{inv } J_{0,0}^{q,s,r}(\mathbb{R}^{m,n}, \mathbb{R}^{m,n})_{0,0}$$

and both multiplication in $G_{m,n}^{q,s,r}$ and the action of $G_{m,n}^{q,s,r}$ on $P^{q,s,r} Y$ are given by the jet composition. Analogously to the manifold case, every $\mathcal{FM}_{m,n}$ -morphism $f : Y \rightarrow \bar{Y}$ induces a principal bundle morphism $P^{q,s,r} f : P^{q,s,r} Y \rightarrow P^{q,s,r} \bar{Y}$.

We are going to the concept of projectable natural bundle over (m, n) -manifolds. This requires the following idea.

Definition 2. A fibered square is a commutative diagram

$$(5) \quad \begin{array}{ccc} Z & \xrightarrow{\varphi} & Y \\ q \downarrow & & \downarrow p \\ N & \xrightarrow{\psi} & M \end{array}$$

where all arrows are surjective submersions and even the induced map $Z \rightarrow Y \times_M N$, $z \mapsto (\varphi(z), q(z))$ is a surjective submersion.

Fibered square (5) can be also denoted by $(Z, q, N, \varphi, \psi, Y, p, M)$. In short, we write (Z, N, Y, M) or (Z, N) or Z . We say that M is the base of Z . The arrow φ or ψ or q or p in (5) can be called the upper or lower or left or right bundle, respectively.

Let x^i, y^p or x^i, v^a be some local fiber coordinates on Y or N , respectively. The assumption $Z \rightarrow Y \times_M N$ is a fibered manifold implies there are some additional fiber coordinates z^s on Z . The local coordinates

$$(6) \quad \begin{aligned} x^i, y^p, v^a, z^s, \quad & i = 1, \dots, m, \quad p = 1, \dots, n, \\ & a = 1, \dots, k, \quad s = 1, \dots, l, \end{aligned}$$

on Z express the fact that fibered squares are locally isomorphic to the products $\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^l$.

Let \underline{F} be a natural bundle over m -manifolds. By a natural transformation $t : F \rightarrow \underline{F}$ we mean a system $t_Y : FY \rightarrow \underline{F}M$ of \mathcal{FM} -morphisms over id_M such that

$$t_{\bar{Y}} \circ Ff = \underline{F}f \circ t_Y$$

for every $\mathcal{FM}_{m,n}$ -morphism $f : Y \rightarrow \bar{Y}$ over $\underline{f} : M \rightarrow \bar{M}$.

Definition 3. A natural bundle F over (m, n) -manifolds is called projectable, if there is a natural bundle \underline{F} over m -manifolds and a natural transformation $t : F \rightarrow \underline{F}$ such that

$$(7) \quad \begin{array}{ccc} FY & \longrightarrow & Y \\ t_Y \downarrow & & \downarrow p \\ \underline{F}M & \longrightarrow & M \end{array}$$

is a fibered square for every fibered manifold $p : Y \rightarrow M$.

For example, the functor $T_{k,l}^{q,s,r}$ is projectable over T_k^r . The functor $P^r Y$ of the classical r -th order frames on a fibered manifold Y is not projectable. In general, a bundle functor G on the category $\mathcal{M}f$ of all manifolds induces a natural bundle over (m, n) -manifolds transforming $p : Y \rightarrow M$ into

$$(8) \quad \begin{array}{ccc} GY & \longrightarrow & Y \\ Gp \downarrow & & \downarrow p \\ GM & \longrightarrow & M \end{array}$$

The fact (8) is a fibered square is proved in Section 38 of [3]. In particular, the tangent bundle TY of a fibered manifold Y is a very important example of fibered square. Let F be a projectable natural bundle over (m, n) -manifolds of order (q, s, r) over \underline{F} , so that \underline{F} is of the order r . From the manifold case we know $\underline{F}M = P^r M[\underline{F}_0, \mu]$, where $\underline{F}_0 = \underline{F}_0 \mathbb{R}^m$ and μ is the action of G_m^r on \underline{F}_0 induced by \underline{F} . Analogously we obtain $FY = P^{q,s,r} Y[F_{0,0}, \lambda]$, where $F_{0,0} = F_{0,0} \mathbb{R}^{m,n}$ and the left action λ of $G_{m,n}^{q,s,r}$ on $F_{0,0}$ is induced by F . Moreover, the natural transformation t induces a surjective submersion $\tau : F_{0,0} \rightarrow \underline{F}_0$ that is ϱ -equivariant, where $\varrho : G_{m,n}^{q,s,r} \rightarrow G_m^r$ is the canonical group homomorphism.

Conversely, consider a left action λ of $G_{m,n}^{q,s,r}$ on a manifold S , a left action μ of G_m^r on a manifold W and a surjective ϱ -equivariant submersion $\tau : S \rightarrow W$. Then we define $FY = P^{q,s,r} Y[S, \lambda]$, $\underline{F}M = P^r M[W, \mu]$ and $t_Y = \{\pi_Y, \tau\}$, where $\pi_Y : P^{q,s,r} Y \rightarrow P^r M$ is the canonical projection. Moreover, for every $\mathcal{F}\mathcal{M}_{m,n}$ -morphism $f : Y \rightarrow \bar{Y}$ over $\underline{f} : M \rightarrow \bar{M}$ we define $Ff = \{P^{q,s,r} f, \text{id}_S\} : FY \rightarrow F\bar{Y}$ and $\underline{F}\underline{f} = \{P^r \underline{f}, \text{id}_W\}$. In the same way as in the manifold case, one verifies

Proposition 1. *The projectable natural bundles over (m, n) -manifolds of order (q, s, r) are in bijection with the above triples $((S, \lambda), (W, \mu), \tau)$.*

2. Fibered squares. Consider another fibered square

$$(9) \quad \begin{array}{ccc} \bar{Z} & \xrightarrow{\bar{\varphi}} & \bar{Y} \\ \bar{q} \downarrow & & \downarrow \bar{p} \\ \bar{N} & \xrightarrow{\bar{\psi}} & \bar{M} \end{array}$$

A fibered square morphism of (5) into (9) is a quadruple of maps $f : Z \rightarrow \bar{Z}$, $f_1 : N \rightarrow \bar{N}$, $f_2 : Y \rightarrow \bar{Y}$, $f_0 : M \rightarrow \bar{M}$ such that all squares of the cube in question commute. This defines the category \mathcal{FS} of fibered squares.

The concept of principal bundle is modified to the square case as follows. Consider a fibered manifold $p : Y \rightarrow M$, two principal bundles $P(Y, G)$, $Q(M, H)$ and a surjective group homomorphism $\varrho : G \rightarrow H$.

Definition 4. The fibered square

$$(10) \quad \begin{array}{ccc} P & \longrightarrow & Y \\ q \downarrow & & \downarrow p \\ Q & \longrightarrow & M \end{array}$$

is called a principal square (or a fibered principal bundle), if q is a principal bundle morphism with the associated group homomorphism ϱ .

In other words, $q(ug) = q(u)\varrho(g)$ for all $u \in P$, $g \in G$.

For example

$$(11) \quad \begin{array}{ccc} P^{q,s,r}Y & \longrightarrow & Y \\ \pi_Y \downarrow & & \downarrow p \\ P^r M & \longrightarrow & M \end{array}$$

is a principal square with respect to the canonical group homomorphism $\varrho : G_{m,n}^{q,s,r} \rightarrow G_m^r$.

In general, let λ or μ be a left action of G or H on a manifold S or W , respectively, and $\tau : S \rightarrow W$ be a ϱ -equivariant surjective submersion. Construct the associated bundles $P[S, \lambda]$ and $Q[W, \mu]$. Then

$$(12) \quad \begin{array}{ccc} P[S, \lambda] & \longrightarrow & Y \\ \{q, \nu\} \downarrow & & \downarrow p \\ Q[W, \mu] & \longrightarrow & M \end{array}$$

is a fibered square, which is said to be associated to (10).

In particular, in the situation of Section 1, one can say that FY is an associated square

$$(13) \quad \begin{array}{ccc} P^{q,s,r}Y[S, \lambda] & \longrightarrow & Y \\ \{\pi_Y, \tau\} \downarrow & & \downarrow p \\ P^r M[W, \mu] & \longrightarrow & M \end{array}$$

The concept of vector bundle can be modified to the square case as follows.

Definition 5. A fibered square (5) is called a vector square, if both $\varphi : Z \rightarrow Y$ and $\psi : N \rightarrow M$ are vector bundles and $q : Z \rightarrow N$ is a linear morphism with the base map $p : Y \rightarrow M$.

An important example of a vector square is the tangent square TY of a fibered manifold Y .

The linear frames in the individual fibers of vector bundle $N \rightarrow M$ form a principal bundle PN with structure group $GL(k, \mathbb{R})$, which is called the linear frame bundle of N . Clearly, $PTM = P^1M$ is the first order frame bundle of M . A linear frame $(A_1, \dots, A_k, A_{k+1}, \dots, A_{k+l})$ in a fiber Z_y of vector bundle $Z \rightarrow Y$ will be called projectable, if $(q(A_1), \dots, q(A_k))$ is a linear frame in $N_{p(y)}$. Write ΠZ for the space of all projectable linear frames in the fibers of Z . Let $GL(k, l, \mathbb{R}) \subset GL(k+l, \mathbb{R})$ be the subgroup of all linear isomorphisms projectable with respect to the projection $\mathbb{R}^k \times \mathbb{R}^l \rightarrow \mathbb{R}^k$. The canonical coordinates on $G(k, l, \mathbb{R})$ are a_b^a, a_a^s, a_t^s . Then $\Pi Z \rightarrow Z$ is a principal bundle with structure group $G(k, l, \mathbb{R})$. Our construction yields a canonical projection $\kappa : \Pi Z \rightarrow PN$ and a group homomorphism $\varrho : G(k, l, \mathbb{R}) \rightarrow GL(k, \mathbb{R})$. Clearly,

$$(14) \quad \begin{array}{ccc} \Pi Z & \longrightarrow & Z \\ \kappa \downarrow & & \downarrow q \\ PN & \longrightarrow & N \end{array}$$

is a principal square with the associated group homomorphism ϱ .

Definition 6. (14) will be called the linear frame square of the vector square Z .

Proposition 2. For every manifold Y , we have an identification

$$(15) \quad \Pi(TY) \approx P^{1,1,1}Y.$$

Proof. Any local fiber coordinates x^i, y^p on Y induces the additional coordinates x_j^i, y_i^p, y_q^p on $P^{1,1,1}Y$. Then we obtain (15) by the same identification as in the manifold case. \square

The (q, s, r) -jet prolongation of a fibered square was introduced by W. Mikulski, [5]. We recall his ideas.

Definition 7. A section of fibered square Z is an \mathcal{FM} -morphism

$$\sigma : (Y \rightarrow M) \rightarrow (Z \rightarrow N)$$

satisfying $\varphi \circ \sigma = \text{id}_Y$.

Clearly, the base map $\underline{\sigma} : M \rightarrow N$ satisfies $\psi \circ \underline{\sigma} = \text{id}_M$. For example, the sections of the tangent square TY are the projectable vector fields on fibered manifold Y .

Definition 8. The space $J^{q,s,r}Z$ of (q, s, r) -jets of local sections of a fibered square Z is called the (q, s, r) -jet prolongation of Z .

Clearly,

$$(16) \quad \begin{array}{ccc} J^{q,s,r}Z & \longrightarrow & Z \\ \downarrow & & \downarrow \\ J^rN & \longrightarrow & N \end{array}$$

is a fibered square. If (Z, N) is a vector square, then (16) is a vector square, too.

We underline that $J^{q,s,r}TY$ is used as an important idea e.g. in [4].

3. Square connections. We recall that, on an arbitrary fibered manifold $Y \rightarrow M$, a connection can be considered either as the lifting map $Y \times_M TM \rightarrow TY$ or as a section $Y \rightarrow J^1Y$, [3].

Definition 9. A square connection on fibered square (5) is a pair of q -related connections Γ on $Z \rightarrow Y$ and Δ on $N \rightarrow M$.

If we consider both Γ and Δ in the lifting form, then the q -relatedness means that the following diagram commutes

$$(17) \quad \begin{array}{ccc} Z \times_Y TY & \xrightarrow{\Gamma} & TZ \\ q \times_p Tp \downarrow & & \downarrow Tq \\ N \times_M TM & \xrightarrow{\Delta} & TN \end{array}$$

Of course, Δ is determined by Γ .

In the local coordinates (6), the equations of Γ are

$$(18) \quad \begin{aligned} dv^a &= F_i^a(x, v) dx^i, \\ dz^s &= F_i^s(x, y, v, z) dx^i + F_p^s(x, y, v, z) dy^p. \end{aligned}$$

The first line are the equations of Δ .

The equations (18) imply that a square connection Γ over Δ is equivalent to a section of the $(1, 1, 1)$ -jet prolongation of Z , i. e. to a commutative diagram

$$(19) \quad \begin{array}{ccc} J^{1,1,1}Z & \xleftarrow{\Gamma} & Z \\ \downarrow & & \downarrow \\ J^1N & \xleftarrow{\Delta} & N \end{array}$$

where Γ and Δ are sections.

A linear square connection on a vector square is defined by the requirement that both Γ and Δ are linear. In the linear fiber coordinates (6), the equations of Γ are

$$(20) \quad \begin{aligned} dv^a &= \Gamma_{bi}^a(x) v^b dx^i, \\ dz^s &= (\Gamma_{ai}^s(x, y) v^a + \Gamma_{ti}^s(x, y) z^t) dx^i + (\Gamma_{ap}^s(x, y) v^a + \Gamma_{tp}^s(x, y) z^t) dy^p. \end{aligned}$$

Given a principal square (P, Q, Y, M) , a square connection (Γ, Δ) is said to be principal, if both Γ and Δ are principal connections. Analogously to the manifold case, every principal square connection induces a square connection on every associated square.

In particular, a vector square Z is an associated square to its linear frame square ΠZ . One verifies easily that every principal square connection on ΠZ induces a linear square connection on Z . Using (20), one proves by direct evaluation

Proposition 3. *The construction of associated square connections establishes a bijection between the principal square connections on ΠZ and the linear square connections on a vector square Z .*

4. Linear square connections of order (q, s, r) . It is remarkable that several other properties of connections on fibered manifolds can be extended to fibered squares. We are going to discuss an interesting special case in details.

Every fibered manifold $p : Y \rightarrow M$ induces a vector square

$$(21) \quad \begin{array}{ccc} J^{q,s,r}TY & \xrightarrow{\varphi} & Y \\ q \downarrow & & \downarrow p \\ J^rTM & \xrightarrow{\psi} & M \end{array}$$

where q , φ and ψ are the canonical projections. Analogously to the manifold case, we introduce

Definition 10. A linear square connection of order (q, s, r) on a fibered manifold Y is a linear splitting

$$(22) \quad \begin{array}{ccc} TY & \xrightarrow{\Gamma} & J^{q,r,s}TY \\ Tp \downarrow & & \downarrow q \\ TM & \xrightarrow{\Delta} & J^rTM, \end{array}$$

i.e. Γ or Δ is a linear morphism of vector bundles over Y or M , respectively, $\varphi \circ \Gamma = \text{id}_{TY}$, $\psi \circ \Delta = \text{id}_{TM}$ and diagram (22) commutes.

The underlying map Δ is a classical linear r -th order connection on M ([2], [7]).

Since $P^{q,s,r}$ is a bundle functor on $\mathcal{FM}_{m,n}$, every projectable vector field η on Y induces the flow prolongation $\mathcal{P}^{q,s,r}\eta$, which is a vector field on $P^{q,s,r}Y$. Since $P^{q,s,r}$ is a functor of the order (q, s, r) , the value of $\mathcal{P}^{q,s,r}\eta$ at every $u \in P_y^{q,s,r}Y$ depends on $j_y^{q,s,r}\eta$ only, [4]. This defines a map

$$(23) \quad i : J^{q,s,r}TY \times_Y P^{q,s,r}Y \rightarrow TP^{q,s,r}Y.$$

Lemma. *i is a diffeomorphism.*

Proof. We have $P^{q,s,r}Y \subset T_{m,n}^{q,s,r}Y$. Even in the fibered case there is an exchange isomorphism

$$(24) \quad \kappa_Y : T_{m,n}^{q,s,r}(TY \rightarrow TM) \rightarrow T(T_{m,n}^{q,s,r}Y)$$

with the property

$$(25) \quad \mathcal{T}_{m,n}^{q,s,r}\eta = \kappa_Y \circ T_{m,n}^{q,s,r}\eta,$$

where $\mathcal{T}_{m,n}^{q,s,r}\eta$ is the flow prolongation of η . For every $Q \in T_u P^{q,s,r}Y$ we have $\kappa_Y^{-1}(Q) \in T_{m,n}^{q,s,r}(TY \rightarrow TM) = J_{0,0}^{q,s,r}(\mathbb{R}^{m,n}, TY \rightarrow TM)$. Since $u \in J_{0,0}^{q,s,r}(\mathbb{R}^{m,n}, Y)$, we can construct the jet composition $\kappa_Y^{-1}(Q) \circ u^{-1} \in J_y^{q,s,r}(Y, TY)$. Let $\pi : TY \rightarrow Y$ be the bundle projection. We have $T_{m,n}^{q,s,r}\pi(\kappa_Y^{-1}(Q) \circ u^{-1}) = (T_{m,n}^{q,s,r}\pi)(\kappa_Y^{-1}(Q)) \circ u^{-1} = u \circ u^{-1} = j_y^{q,s,r}\text{id}_Y$, so that $\kappa_Y^{-1}(Q) \circ u^{-1} \in J_y^{q,s,r}TY$. By (25), $i(\kappa_Y^{-1}(Q) \circ u^{-1}, u) = Q$. This proves our claim. \square

Since $P^{q,s,r}$ is a functor with values in the category of principal bundles, $\mathcal{P}^{q,s,r}\eta$ is a right-invariant vector field on $P^{q,s,r}Y$. Thus, every linear splitting $\Gamma : TY \rightarrow J^{q,s,r}TY$ defines a principal connection $\tilde{\Gamma}$ on $P^{q,s,r}Y$ in the lifting form $\tilde{\Gamma} : P^{q,s,r}Y \times_Y TY \rightarrow TP^{q,s,r}Y$ by

$$(26) \quad \tilde{\Gamma}(u, V) = i(\Gamma(V), u),$$

$u \in P_y^{q,s,r}Y$, $V \in T_yY$. In the same way, $\Delta : TM \rightarrow J^rTM$ induces a principal connection on P^rM . Thus, we have proved

Proposition 4. *The rule (26) establishes a bijection between the linear square connections of order (q, s, r) on Y and the principal square connections on $(P^{q,s,r}Y, P^rM)$.*

Remark. The torsion of a linear r -th order connection Δ on M can be defined in two different ways. The first one, due to A. Zajtz, [7], uses the truncated bracket of vector fields. The second one, due to P. C. Yuen, [9], takes into account the associated principal connection $\tilde{\Delta}$ on P^rM and constructs the covariant exterior differential of the solder form of P^rM with respect to $\tilde{\Delta}$. In [2] we clarified that both definitions are naturally equivalent. We remark that each of these approaches can be generalized to the square case in a direct way.

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