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On certain subclasses of multivalent functions involving Cho–Kwon–Srivastava operator

Abstract. By making use of the method of differential subordination, we investigate inclusion relationships among certain subclasses of analytic and p-valent functions, which are defined here by means of Cho–Kwon–Srivastava operator $\mathcal{I}_p^\lambda(a,c)$. The integral preserving properties in connection with this operator are also studied.

1. Introduction. Let $\mathcal{A}_p$ be the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}, \quad p \in \mathbb{N} = \{1, 2, \dots\}, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. We write $\mathcal{A}_1 = \mathcal{A}$. If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g, written symbolically as $f \prec g$ or $f(z) \prec g(z)$, $z \in \mathbb{U}$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$, $z \in \mathbb{U}$.

For fixed parameters A, B $(-1 \le B < A \le 1)$, we denote by $P(A,B)$ the class of functions of the form

$$\phi(z) = 1 + c_1 z + c_2 z^2 + \cdots$$

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which are analytic in $\mathbb{U}$ and satisfy the condition

$$\phi(z) \prec \frac{1+Az}{1+Bz}, \quad z \in \mathbb{U}.$$

The class $P(A,B)$ was investigated by Janowski [6]. By $\mathcal{S}_p^\star(A,B)$ we mean the class of functions $f \in \mathcal{A}_p$ such that $zf'(z)/pf(z) \in P(A,B)$. Similarly, $\mathcal{K}_p(A,B)$ is the class of functions $f \in \mathcal{A}_p$ satisfying $(zf'(z))'/pf'(z) \in P(A,B)$.

It is easily seen that $\mathcal{S}_p^\star(1-(2\eta/p),-1) = \mathcal{S}_p^\star(\eta)$, $\mathcal{K}_p(1-(2\eta/p),-1) = \mathcal{K}_p(\eta)$ $(0 \le \eta < p)$, the subclasses of functions in $\mathcal{A}_p$ which are respectively p-valently starlike of order η and p-valently convex of order η in $\mathbb{U}$.

In our present investigation, we shall also make use of the *Gauss hypergeometric function* ${}_2F_1$ defined by

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n} \frac{z^n}{(1)_n}, \tag{1.2}$$

where $a,b,c \in \mathbb{C}$, $c \notin \mathbb{Z}_0^- = \{0,-1,-2,\dots\}$ and $(\kappa)_n$ denotes the Pochhammer symbol (or the *shifted factorial*) given, in terms of the Gamma function Γ, by

$$(\kappa)_n = \frac{\Gamma(\kappa+n)}{\Gamma(\kappa)} = \begin{cases} \kappa(\kappa+1)\cdots(\kappa+n-1), & n \in \mathbb{N}, \\ 1, & n = 0. \end{cases}$$

We note that the series defined by (1.2) converges absolutely for $z \in \mathbb{U}$ and hence ${}_2F_1$ represents an analytic function in the open unit disk $\mathbb{U}$ (see, for details [17, Chapter 14]).

We now define a function $\phi_p(a,c;z)$ by

$$\phi_p(a,c;z) = z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{p+k}, \quad z \in \mathbb{U},$$

where $a \in \mathbb{R}$ and $c \in \mathbb{R} \setminus \mathbb{Z}_0^-$. With the aid of $\phi_p(a,c;z)$, we consider a function $\phi_p^{(+)}(a,c;z)$ defined by

$$\phi_p(a,c;z) \star \phi_p^{(+)}(a,c;z) = \frac{z^p}{(1-z)^{\lambda+p}}, \quad z \in \mathbb{U},$$

where $\lambda > -p$. This function yields the following family of linear operators

$$\mathcal{I}_p^\lambda(a,c)f(z) = \phi_p^{(+)}(a,c;z) \star f(z), \quad z \in \mathbb{U}, \tag{1.3}$$

where $a,c \in \mathbb{R}\setminus\mathbb{Z}_0^-$. (Here the symbol "$\star$" stands for the Hadamard product (or convolution)). For a function $f \in \mathcal{A}_p$, given by (1.1), it follows from (1.3) that for $\lambda > -p$ and $a, c \in \mathbb{R} \setminus \mathbb{Z}_0^-$

$$\begin{aligned} \mathcal{I}_p^\lambda(a,c)f(z) &= z^p + \sum_{k=1}^{\infty} \frac{(c)_k\,(\lambda+p)_k}{(a)_k\,(1)_k} a_{p+k} z^{p+k} \\ &= z^p\,{}_2F_1\,(c, \lambda+p;\, a;\, z) \star f(z), \quad z \in \mathbb{U}. \end{aligned} \tag{1.4}$$

From (1.4), we deduce that

$$z\left(\mathcal{I}_p^\lambda(a,c)f(z)\right)' = (\lambda+p)\mathcal{I}_p^{\lambda+1}(a,c)f(z) - \lambda \mathcal{I}_p^\lambda(a,c)f(z) \tag{1.5}$$

and

$$z\left(\mathcal{I}_p^\lambda(a+1,c)f(z)\right)' = a\mathcal{I}_p^\lambda(a,c)f(z) - (a-p)\mathcal{I}_p^\lambda(a+1,c)f(z). \tag{1.6}$$

We also note that

$$\mathcal{I}_p^0(p+1,1)f(z) = p\int_0^z \frac{f(t)}{t}\,dt,$$

$$\mathcal{I}_p^0(p,1)f(z) = \mathcal{I}_p^1(p+1,1)f(z) = f(z),$$

$$\mathcal{I}_p^1(p,1)f(z) = \frac{zf'(z)}{p},$$

$$\mathcal{I}_p^2(p,1)f(z) = \frac{2zf'(z) + z^2f''(z)}{p(p+1)},$$

$$\mathcal{I}_p^2(p+1,1)f(z) = \frac{f(z) + zf'(z)}{p+1},$$

$$\mathcal{I}_p^n(a,a)f(z) = D^{n+p-1}f(z), \quad n \in \mathbb{N},\ n > -p,$$

the Ruscheweyh derivative of $(n+p-1)$th order [5] and

$$\mathcal{I}_p^\mu(\mu+p+1,1)f(z) = \mathcal{F}_{\mu,p}(f)(z), \quad \mu > -p,$$

where $\mathcal{F}_{\mu,p}(f)$ denotes a familiar integral operator defined by (2.10) below (see Section 2).

The operator $\mathcal{I}_p^\lambda(a,c)$ $(\lambda > -p,\ a, c \in \mathbb{R} \setminus \mathbb{Z}_0^-)$ was recently introduced by Cho et al. [1], who investigated (among other things) some inclusion relationships and properties of various subclasses of multivalent functions in $\mathcal{A}_p$, which were defined by means of the operator $\mathcal{I}_p^\lambda(a,c)$. For $\lambda = c = 1$ and $a = n+p$, the Cho–Kwon–Srivastava operator $\mathcal{I}_p^\lambda(a,c)$ yields the Noor integral operator $\mathcal{I}_p^1(n+p,1) = \mathcal{I}_{n,p}$ $(n > -p)$ of $(n+p-1)$th order, studied by Liu and Noor [7] (see also [11], [12]). The linear operator $\mathcal{I}_1^\lambda(\mu+2,1)$ $(\lambda > -1,\ \mu > -2)$ was also recently introduced and studied by Choi et al. [3]. For relevant details about further special cases of the Choi–Saigo–Srivastava operator $\mathcal{I}_1^\lambda(\mu+2,1)$, the interested reader may refer to the works by Cho et al. [1] and Choi et al. [3] (see also [2]).

Using the Cho–Kwon–Srivastava operator $\mathcal{I}_p^\lambda(a,c)$, we now define a subclass of $\mathcal{A}_p$ as follows:

Definition. For fixed parameters A, B $(-1 \le B < A \le 1)$ and $\alpha \ge 0$, we say that a function $f \in \mathcal{A}_p$ is in the class $T^{\lambda}_{p,\alpha}(a, c, A, B)$ if

$$(1.7) \qquad (1-\alpha)\frac{\mathcal{I}^{\lambda+1}_p(a,c)f(z)}{\mathcal{I}^{\lambda}_p(a,c)f(z)} + \alpha\frac{\mathcal{I}^{\lambda+2}_p(a,c)f(z)}{\mathcal{I}^{\lambda+1}_p(a,c)f(z)} \in P(A,B), \quad z \in \mathbb{U},$$

where $\lambda > -p$ and $a, c \in \mathbb{R} \setminus \mathbb{Z}^{-}_0$.

It is readily seen that

$$T^{0}_{p,0}(p, 1, A, B) = \mathcal{S}^{\star}_p(A, B)$$

and

$$T^{1}_{p,0}\left(p, 1, \frac{pA+B}{p+1}, B\right) = T^{0}_{p,1}\left(p, 1, \frac{pA+B}{p+1}, B\right) = \mathcal{K}_p(A, B).$$

In the present paper, we obtain inclusion relationships among the classes $T^{\lambda}_{p,\alpha}(a, c, A, B)$. The integral preserving properties in connection with the operator $\mathcal{I}^{\lambda}_p(a, c)$ are considered. Relevant connections of the results presented here with those obtained in earlier works are also pointed out.

2. Main results. Unless otherwise mentioned, we assume throughout the sequel that $-1 \le B < A \le 1$, $\lambda > -p$ and $p \in \mathbb{N}$.

Theorem 1. *Let $f \in T^{\lambda}_{p,\alpha}(a, c, A, B)$ and $0 < \alpha < \lambda + p + 1$ satisfy*

$$(2.1) \qquad (\lambda + p + 1)(1 - A) - \alpha(1 - B) \ge 0.$$

(i) *Then*

$$T^{\lambda}_{p,\alpha}(a, c, A, B) \subset T^{\lambda}_{p,0}\left(a, c, \widetilde{A}, B\right),$$

where

$$(2.2) \qquad \widetilde{A} = 1 - \frac{1}{\lambda + p + 1 - \alpha}\left\{(\lambda + p + 1)(1 - A) - \alpha(1 - B)\right\}.$$

Further for $f \in T^{\lambda}_{p,\alpha}(a, c, A, B)$, we also have

$$(2.3) \qquad \frac{\mathcal{I}^{\lambda+1}_p(a,c)f(z)}{\mathcal{I}^{\lambda}_p(a,c)f(z)} \prec \frac{\alpha}{\lambda+p+1-\alpha}\left(\frac{1}{Q(z)}\right) = q(z), \quad z \in \mathbb{U},$$

where

$$(2.4) \qquad Q(z) = \begin{cases} \displaystyle\int_0^1 t^{\frac{\lambda+p+1}{\alpha}}\left(\frac{1+Btz}{1+Bz}\right)^{\frac{\lambda+p+1}{\alpha}\left(\frac{A-B}{B}\right)} dt, & B \ne 0, \\ \displaystyle\int_0^1 t^{\frac{\lambda+p+1}{\alpha}} \exp\left(\frac{\lambda+p+1}{\alpha}(t-1)Az\right) dt, & B = 0. \end{cases}$$

and $q(z)$ is the best dominant of (2.3).

(ii) *If, in addition to* (2.1) *one has $-1 \le B < A \le 0$, then*

$$T^{\lambda}_{p,\alpha}(a, c, A, B) \subset T^{\lambda}_{p,0}(a, c, 1 - 2\rho, -1),$$

where $\rho = \left[{}_2F_1\left(1, \frac{\lambda+p+1}{\alpha}\left(\frac{B-A}{B}\right); \frac{\lambda+p+1}{\alpha}; \frac{B}{B-1}\right)\right]^{-1}$. *The result is the best possible.*

Proof. Let $f \in T_{p,\alpha}^{\lambda}(a, c, A, B)$ and suppose that the function g is defined by

$$g(z) = z\left(\frac{\mathcal{I}_p^{\lambda}(a,c)f(z)}{z^p}\right)^{1/(\lambda+p)} \tag{2.5}$$

and $r_1 = \sup\{r : g(z) \neq 0,\ 0 < |z| < r < 1\}$. Taking logarithmic differentiation in (2.5) and using the identity (1.5) in the resulting equation, it follows that

$$\varphi(z) = \frac{zg'(z)}{g(z)} = \frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{\mathcal{I}_p^{\lambda}(a,c)f(z)} \tag{2.6}$$

is analytic in $|z| < r_1$ and $\varphi(0) = 1$. Carrying out logarithmic differentiation in (2.6) followed by the use of (1.5) and (1.7) easily lead to

$$P(z) + \frac{zP'(z)}{\beta P(z) + \gamma} \prec \frac{1+Az}{1+Bz}, \quad |z| < r_1,$$

where

$$P(z) = \left(1 - \frac{1}{\beta}\right)\varphi(z) + \frac{1}{\beta}, \quad \beta = \frac{\lambda+p+1}{\alpha} \quad \text{and} \quad \gamma = -1.$$

Hence by applying a result [8, Corollary 3.2], we get

$$\varphi(z) \prec \frac{\alpha}{\lambda+p+1-\alpha}\left(\frac{1}{Q(z)}\right) = q(z) \prec \frac{1+\widetilde{A}z}{1+Bz}, \quad |z| < r_1,$$

where $\widetilde{A}$ is given by (2.2), Q is given by (2.4) and q is the best dominant of (2.3). The remaining part of the proof can now be deduced on the same lines as in Theorem 1 [15, p. 325] (see also [8]). This completes the proof of Theorem 1. $\square$

Putting $a = p,\ c = \alpha = 1,\ \lambda = 0$ and replacing A by $(pA + B)/(p + 1)$ in Theorem 1, we obtain the following result which, in turn yields the corresponding work of Srivastava et al. [16, Corollary 7] for $A = 1 - (2\eta/p)$ $(0 \leq \eta < p)$ and $B = -1$ (see also [15]).

Corollary 1. *For* $-1 \leq B < 0$ *and* $B < A \leq -(B/p)$, *we have*

$$\mathcal{K}_p(A, B) \subset \mathcal{S}_p^{\star}(\rho_1),$$

where $\rho_1 = p\left[{}_2F_1\left(1, \frac{p(B-A)}{B}; p+1; \frac{B}{B-1}\right)\right]^{-1}$. *The result is the best possible.*

Setting $a = \mu + p + 1$, $c = \alpha = 1$, $\lambda = \mu$ and replacing A by $\{pA + (\mu + 1)B\}/(\mu + p + 1)$ in Theorem 1, we get

Corollary 2. *If $\mu > -p$, $-1 \le B < 0$ and*

$$B < A \le \min\left\{1 + \frac{\mu(1-B)}{p}, -\frac{(\mu+1)B}{p}\right\},$$

then for $f \in \mathcal{S}_p^\star(A,B)$ we have

$$\Re\left(\frac{z^\mu f(z)}{\int_0^z t^{\mu-1} f(t)\,dt}\right) > \rho_2, \quad z \in \mathbb{U},$$

where $\rho_2 = (p+\mu)\left[{}_2F_1\left(1, \frac{p(B-A)}{B}; \mu+p+1; \frac{B}{B-1}\right)\right]^{-1}$. The result is the best possible.

Substituting $A = 1-(2\eta/p)$ $(0 \le \eta < p)$ and $B = -1$ in Theorem 1, we get

Corollary 3. *If $0 < \alpha < \lambda + p + 1$ and $\max\left\{\frac{p\alpha}{\lambda+p+1}, \frac{p}{2}\right\} \le \eta < p$, then*

$$T_{p,\alpha}^{\lambda}(a,c,1-(2\eta/p),-1) \subset T_{p,0}^{\lambda}(a,c,1-2\rho_3,-1),$$

where $\rho_3 = \left[{}_2F_1\left(1, \frac{2(\lambda+p+1)(p-\eta)}{p\alpha}; \frac{\lambda+p+1}{\alpha}; \frac{1}{2}\right)\right]^{-1}$. The result is the best possible.

Remarks. (i) In the case $A = 1-(2\eta/p)$ $(0 \le \eta < p)$ and $B = -1$, Corollary 2 gives the result contained in [14, Corollary 3.5].

(ii) Letting $a = p$, $c = 1$, $\lambda = 0$, $\alpha = \{(p+1)\delta\}/(p+\delta)$, $\eta = p\delta/(p+\delta)$ in Corollary 3 and using the well-known identity

$${}_2F_1\left(a, b; \frac{a+b+1}{2}; \frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{a+b+1}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)},$$

we observe that if $f \in \mathcal{A}_p$ satisfies

$$\Re\left\{(1-\delta)\frac{zf'(z)}{f(z)} + \delta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > 0, \quad z \in \mathbb{U},$$

for $\delta \ge p$, then $f \in \mathcal{S}_p^\star(\sigma)$ which in turn implies that $f \in \mathcal{K}_p\left(\frac{(\delta-1)\sigma}{\delta}\right)$, where $\sigma = \frac{p\Gamma((2p+\delta)/2\delta)}{\sqrt{\pi}\,\Gamma((p+\delta)/\delta)}$. This for $p = 1$ reduces to a result of Miller et al. [10].

Theorem 2. *If $0 < \alpha < \lambda + p + 1$ and $0 \le \eta < p$, then*

$$f \in T_{p,0}^{\lambda}(a,c,1-(2\eta/p),-1) \Longrightarrow f \in T_{p,\alpha}^{\lambda}(a,c,1-2\rho_4,-1)$$

in $|z| < R(p,\alpha,\lambda,\eta)$, where $\rho_4 = \{\alpha(p-\eta)+\eta(\lambda+p+1)\}/p(\lambda+p+1)$ and

$$R(p,\alpha,\lambda,\eta) = \begin{cases} \frac{p-\eta}{p-2\eta} + \frac{p\alpha - \sqrt{(p\alpha)^2 + (\lambda+p+1-\alpha)\{(\lambda+p+1-\alpha)\eta^2 + 2p(p-\eta)\alpha\}}}{(\lambda+p+1-\alpha)(p-2\eta)}, & \eta \ne \frac{p}{2}, \\ \frac{\lambda+p+1-\alpha}{\lambda+p+\alpha+1}, & \eta = \frac{p}{2}. \end{cases} \tag{2.7}$$

The result is the best possible.

Proof. We have

$$\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{\mathcal{I}_p^{\lambda}(a,c)f(z)} = \frac{\eta}{p} + \left(1-\frac{\eta}{p}\right)u(z), \tag{2.8}$$

where $u(z) = 1 + u_1 z + u_2 z^2 + \cdots$ is analytic and has a positive real part in $\mathbb{U}$. Taking logarithmic differentiation in (2.8) followed by the use of the identity (1.5) and after simplifications, we deduce that

$$\begin{aligned}
&\Re\left\{(1-\alpha)\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{\mathcal{I}_p^{\lambda}(a,c)f(z)} + \alpha\frac{\mathcal{I}_p^{\lambda+2}(a,c)f(z)}{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}\right\} - \frac{\alpha(p-\eta)+\eta(\lambda+p+1)}{p(\lambda+p+1)} \\
&\geq \frac{(p-\eta)(\lambda+p+1-\alpha)}{p(\lambda+p+1)} \\
&\quad \times \left\{\Re(u(z)) - \frac{\alpha p|zu'(z)|}{(\lambda+p+1-\alpha)|\eta+(p-\eta)u(z)|}\right\}.
\end{aligned} \tag{2.9}$$

Now using the well-known [8] estimates

$$|zu'(z)| \leq \frac{2r}{1-r^2}\Re(u(z)) \quad \text{and} \quad \Re(u(z)) \geq \frac{1-r}{1+r}, \quad |z| = r < 1$$

in (2.9), we get

$$\begin{aligned}
&\Re\left\{(1-\alpha)\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{\mathcal{I}_p^{\lambda}(a,c)f(z)} + \alpha\frac{\mathcal{I}_p^{\lambda+2}(a,c)f(z)}{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}\right\} - \frac{\alpha(p-\eta)+\eta(\lambda+p+1)}{p(\lambda+p+1)} \\
&\quad \geq \frac{(p-\eta)(\lambda+p+1-\alpha)}{p(\lambda+p+1)}\Re\left(u(z)\right) \\
&\qquad \times \left\{1 - \frac{2\alpha p r}{(\lambda+p+1-\alpha)\left[\eta(1-r^2)+(p-\eta)(1-r)^2\right]}\right\}
\end{aligned}$$

which is certainly positive if $r < R(p,\alpha,\lambda,\eta)$, where $R(p,\alpha,\lambda,\eta)$ is given by (2.7).

It is easily seen that the bound $R(p,\alpha,\lambda,\eta)$ is the best possible for the function $f \in \mathcal{A}_p$ defined by

$$\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{\mathcal{I}_p^{\lambda}(a,c)f(z)} = \left(1-\frac{\eta}{p}\right)\frac{1+z}{1-z} + \frac{\eta}{p},$$

where $0 \leq \eta < p$ and $z \in \mathbb{U}$. This completes the proof of the theorem. $\square$

Remark. For $a = p$, $c = \alpha = 1$ and $\lambda = 0$, we get Corollary 3.2 in [14].

For a function $f \in \mathcal{A}_p$ and $\mu > -p$, the integral operator $\mathcal{F}_{\mu,p} : \mathcal{A}_p \longrightarrow \mathcal{A}_p$ is defined by [3]

$$\begin{aligned} \mathcal{F}_{\mu,p}(f)(z) &= \frac{\mu+p}{z^p}\int_0^z t^{\mu-1} f(t)\,dt \\ &= \left(z^p + \sum_{k=1}^{\infty} \frac{\mu+p}{\mu+p+k} z^{p+k}\right) \star f(z) \\ &= z^p\,{}_2F_1\left(1, \mu+p; \mu+p+1; z\right) \star f(z), \quad z \in \mathbb{U}. \end{aligned} \tag{2.10}$$

It follows from (2.10) that

$$\begin{aligned} & z\left(\mathcal{I}_p^{\lambda}(a,c)\mathcal{F}_{\mu,p}(f)(z)\right)' \\ &\quad = (\mu+p)\mathcal{I}_p^{\lambda}(a,c)f(z) - \mu\mathcal{I}_p^{\lambda}(a,c)F_{\mu,p}(f)(z), \quad z \in \mathbb{U}. \end{aligned} \tag{2.11}$$

We now prove

Theorem 3. *Let μ be a complex number satisfying*

$$\Re(\mu) \geq \frac{\lambda(A-B) + p(A-1)}{(1-B)}.$$

(i) *If $f \in T_{p,0}^{\lambda}(a,c,A,B)$, then the function $\mathcal{F}_{\mu,p}(f)$ defined by* (2.10) *belongs to the class $T_{p,0}^{\lambda}(a,c,A,B)$. Furthermore,*

$$\frac{\mathcal{I}_p^{\lambda+1}(a,c)\mathcal{F}_{\mu,p}(f)(z)}{\mathcal{I}_p^{\lambda}(a,c)\mathcal{F}_{\mu,p}(f)(z)} \prec \frac{1}{\lambda+p}\left(\frac{1}{Q_1(z)} - (\mu-\lambda)\right) = q_1(z), \quad z \in \mathbb{U},$$

where

$$Q_1(z) = \begin{cases} \displaystyle\int_0^1 t^{\mu+p-1}\left(\frac{1+Btz}{1+Bz}\right)^{\frac{(\lambda+p)(A-B)}{B}} dt, & B \neq 0, \\ \displaystyle\int_0^1 t^{\mu+p-1}\exp\left((\lambda+p)(t-1)Az\right) dt, & B = 0 \end{cases}$$

and q_1 is the best dominant.

(ii) *If $-1 \leq B < 0$, μ is real and satisfies*

$$\mu \geq \max\left\{\frac{(\lambda+p)(B-A)}{B} - p - 1, -\frac{(p+\lambda)(1-A)}{(1-B)} + \lambda\right\},$$

then

$$f \in T_{p,0}^{\lambda}(a,c,A,B) \Longrightarrow \mathcal{F}_{\mu,p}(f) \in T_{p,0}^{\lambda}\left(a,c,1-2\rho',-1\right),$$

where

$$\rho' = \frac{1}{\lambda+p}\left\{(\mu+p)\left[{}_2F_1\left(1, \frac{(\lambda+p)(B-A)}{B}; \mu+p+1; \frac{B}{B-1}\right)\right]^{-1} - (\mu-\lambda)\right\}.$$

The result is the best possible.

Proof. We put

$$g(z) = z\left(\frac{\mathcal{I}_p^{\lambda}(a,c)\mathcal{F}_{\mu,p}(f)(z)}{z^p}\right)^{1/(\lambda+p)} \tag{2.12}$$

and $r_1 = \sup\{r : g(z) \neq 0,\ 0 < |z| < r < 1\}$. Then g is single valued and analytic in $|z| < r_1$. By carrying out logarithmic differentiation in (2.12) and using the identity (1.5) for the function $\mathcal{F}_{\mu,p}(f)$, it follows that

$$\varphi(z) = \frac{zg'(z)}{g(z)} = \frac{\mathcal{I}_p^{\lambda+1}(a,c)\mathcal{F}_{\mu,p}(f)(z)}{\mathcal{I}_p^{\lambda}(a,c)\mathcal{F}_{\mu,p}(f)(z)} \tag{2.13}$$

is analytic in $|z| < r_1$ and $\varphi(0) = 1$. Now, (1.5) and (2.11) easily lead to

$$(\lambda+p)\frac{\mathcal{I}_p^{\lambda+1}(a,c)\mathcal{F}_{\mu,p}(f)(z)}{\mathcal{I}_p^{\lambda}(a,c)\mathcal{F}_{\mu,p}(f)(z)} + (\mu-\lambda) = (\mu+p)\frac{\mathcal{I}_p^{\lambda}(a,c)f(z)}{\mathcal{I}_p^{\lambda}(a,c)\mathcal{F}_{\mu,p}(f)(z)}. \tag{2.14}$$

Since $f \in T_{p,0}^{\lambda}(a,c,A,B)$, it is clear that $\mathcal{I}_p^{\lambda}(a,c)f(z) \neq 0$ in $0 < |z| < 1$. So, (2.13) and (2.14) give

$$\frac{\mathcal{I}_p^{\lambda}(a,c)\mathcal{F}_{\mu,p}(f)(z)}{\mathcal{I}_p^{\lambda}(a,c)f(z)} = \frac{\mu+p}{(\lambda+p)\varphi(z)+(\mu-\lambda)}. \tag{2.15}$$

Taking logarithmic differentiation in the above expression and using (2.11) in the resulting equation, we get

$$\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{\mathcal{I}_p^{\lambda}(a,c)f(z)} = \varphi(z) + \frac{z\varphi'(z)}{(\lambda+p)\varphi(z)+(\mu-\lambda)}, \quad |z| < r_1. \tag{2.16}$$

Hence by the hypothesis and (2.16) that

$$\varphi(z) + \frac{z\varphi'(z)}{\beta\varphi(z)+\gamma} \prec \frac{1+Az}{1+Bz}, \quad |z| < r_1,$$

where $\beta = \lambda + p$ and $\gamma = \mu - \lambda$.

Proceeding on the same lines as in Theorem 2 [15, p. 328], we can prove the assertions (i) and (ii) of the theorem. □

Letting $a = p$, $c = 1$ and $\lambda = 0$ (or $a = p$, $c = 1$, $\lambda = 1$ and replacing A by $(pA + B)/(p + 1)$) in Theorem 3, we deduce the following corollary.

Corollary 4. *If $-1 \leq B < 0$, μ is real and satisfies*

$$\mu \geq \max\left\{\frac{p(B-A)}{B} - p - 1,\ -\frac{p(1-A)}{(1-B)}\right\},$$

then

$$f \in \mathcal{S}_p^{\star}(A,B) \Longrightarrow \mathcal{F}_{\mu,p}(f) \in \mathcal{S}_p^{\star}(\tau)$$

and

$$f \in \mathcal{K}_p(A,B) \Longrightarrow \mathcal{F}_{\mu,p}(f) \in \mathcal{K}_p(\tau),$$

where $\tau = (\mu+p)\left[{}_2F_1\left(1, \frac{p(B-A)}{B}; \mu+p+1; \frac{B}{B-1}\right)\right]^{-1} - \mu$. *The result is the best possible.*

Remark. Taking $A = 1 = (2\eta/p)$ $(0 \leq \eta < p)$ and $B = -1$ in Corollary 4, we obtain the results contained in [15, Remark 2] which also improves the corresponding work of Fukui et al. [4] for $\eta = 0$ and $p = 1$.

To establish our next result, we need the following lemma.

Lemma A ([13])**.** *Let ϕ be analytic in $\mathbb{U}$ with $\phi(0) = 1$ and $\phi(z) \neq 0$ for $0 < |z| < 1$.*

(i) *Let $B \neq 0$ and $\gamma \in \mathbb{C} \setminus \{0\}$ satisfy either*

$$\left|\frac{\gamma(A-B)}{B} - 1\right| \leq 1 \quad \textit{or} \quad \left|\frac{\gamma(A-B)}{B} + 1\right| \leq 1.$$

If ϕ satisfies

$$1 + \frac{z\phi'(z)}{\gamma\phi(z)} \prec \frac{1+Az}{1+Bz}, \quad z \in \mathbb{U},$$

then

$$\phi(z) \prec (1+Bz)^{\gamma(A-B)/B}, \quad z \in \mathbb{U}$$

and this is the best dominant.

(ii) *Let $B = 0$ and $\gamma \in \mathbb{C} \setminus \{0\}$ be such that $|\gamma A| < \pi$. If ϕ satisfies*

$$1 + \frac{z\phi'(z)}{\gamma\phi(z)} \prec 1 + Az, \quad z \in \mathbb{U},$$

then

$$\phi(z) \prec e^{\gamma A z}, \quad z \in \mathbb{U}$$

and this is the best dominant.

Theorem 4. *Assume that $B \neq 0$, $\lambda > -p$ and $\nu \in \mathbb{C} \setminus \{0\}$ satisfies either*

$$\left|\frac{\nu(\lambda+p)(A-B)}{B} - 1\right| \leq 1 \quad \textit{or} \quad \left|\frac{\nu(\lambda+p)(A-B)}{B} + 1\right| \leq 1.$$

If $f \in T^{\lambda}_{p,0}(a, c, A, B)$, then

$$\left(\frac{\mathcal{I}^{\lambda}_p(a,c)f(z)}{z^p}\right)^{\nu} \prec q_2(z) = (1+Bz)^{\nu(\lambda+p)(A-B)/B}, \quad z \in \mathbb{U}$$

and q_2 is the best dominant. In the case $B = 0$, i.e., for $f \in T^{\lambda}_{p,0}(a, c, A, 0)$, we have

$$\left(\frac{\mathcal{I}^{\lambda}_p(a,c)f(z)}{z^p}\right)^{\nu} \prec e^{\nu(\lambda+p)Az}, \quad z \in \mathbb{U},$$

where $\nu \neq 0$, $|\nu| < \pi/(\lambda+p)A$ and this is the best dominant.

Proof. Let us put

$$\varphi(z) = \left(\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^p}\right)^\nu, \quad z \in \mathbb{U}. \tag{2.17}$$

Then φ is analytic in $\mathbb{U}$, $\varphi(0) = 1$ and $\varphi(z) \neq 0$ for $z \in \mathbb{U}$. By making use of (1.5) in the logarithmic differentiation of (2.17), we deduce that

$$1 + \frac{z\varphi'(z)}{\nu(\lambda+p)\varphi(z)} \prec \frac{1+Az}{1+Bz}, \quad z \in \mathbb{U}.$$

Now the assertions of the theorem follows by using Lemma A with $\gamma = \nu(\lambda+p)$. This completes the proof of Theorem 4. $\square$

Upon setting $a = p$, $c = 1$ and $\lambda = 0$ (or $a = p$, $c = \lambda = 1$ and replacing A by $(pA+B)/(p+1)$) in Theorem 4, we obtain

Corollary 5. *Assume that $B \neq 0$ and $\nu \in \mathbb{C} \setminus \{0\}$ satisfies either*

$$\left|\frac{\nu p(A-B)}{B} - 1\right| \leq 1 \quad or \quad \left|\frac{\nu p(A-B)}{B} + 1\right| \leq 1.$$

Then

$$\text{(i) } f \in \mathcal{S}_p^\star(A,B) \Longrightarrow \left(\frac{f(z)}{z^p}\right)^\nu \prec \frac{1}{(1+Bz)^{\nu p(B-A)/B}}, \quad z \in \mathbb{U}$$

and

$$\text{(ii) } f \in \mathcal{K}_p(A,B) \Longrightarrow \left(\frac{f'(z)}{z^{p-1}}\right)^\nu \prec \frac{p^\nu}{(1+Bz)^{\nu p(B-A)/B}}, \quad z \in \mathbb{U}.$$

The above implications are the best possible.

Remark. In the special case when $A = 1 - 2\xi$ $(0 \leq \xi < 1)$, $B = -1$ and $\nu = p = 1$, Corollary 5 gives the following best possible results.

If $f \in \mathcal{A}$, then

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \xi \Longrightarrow \Re\left(\frac{f(z)}{z}\right) > \frac{1}{2^{2(1-\xi)}}, \quad z \in \mathbb{U}$$

and

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \xi \Longrightarrow \Re\left(f'(z)\right) > \frac{1}{2^{2(1-\xi)}}, \quad z \in \mathbb{U}.$$

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