ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LX, 2006 SECTIO A 65–73

ANDRZEJ MIERNOWSKI and WITOLD RZYMOWSKI

On homogeneous distributions

Abstract. Any homogeneous function is determined by its values on the unit sphere. We shall prove that an analogous fact is true for homogeneous distributions.

1. Test functions on the unit sphere. For $x, y \in \mathbb{R}^n$ we will write

$$x \cdot y = \sum_{i=1}^{n} x_i y_i$$

and

$$|x| = \sqrt{x \cdot x} = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

By S^{n-1} we denote the unit sphere in $\mathbb{R}^n$, i.e.

$$S^{n-1} = \{x \in \mathbb{R}^n : \ |x| = 1\}.$$

Let X be a linear space and $f : S^{n-1} \to X$. For any $\alpha \in \mathbb{R}$ we define the extension of f, of degree α, by the formula

$$(\mathcal{E}_\alpha f)(x) = |x|^\alpha f\left(\frac{x}{|x|}\right), \quad x \in \mathbb{R}^n \setminus \{0\}.$$

2000 *Mathematics Subject Classification.* 60E05.

Key words and phrases. Homogeneous distribution, spherical derivative, distribution on the sphere.

In the case of $\alpha = 0$ we have

$$(\mathcal{E}_0 f)(x) = f\left(\frac{x}{|x|}\right), \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Definition 1. Let $(X, \|\cdot\|)$ be a normed space and $f : S^{n-1} \to X$. We say that f is differentiable at $x_0 \in S^{n-1}$ if there exists a linear operation $A : \mathbb{R}^n \to X$ such that

$$Ax_0 = 0 \quad \text{and} \quad \lim_{S^{n-1} \ni x \to x_0} \frac{f(x) - f(x_0) - Ax}{|x - x_0|} = 0.$$

It is not very hard to check that such an operation is unique, so that we call it the spherical derivative of f at the point x_0. The spherical derivative of f at the point x_0 will be denoted by $\partial^S f(x_0)$. The mapping $f : S^{n-1} \to X$ is called differentiable if $\partial^S f(x)$ exists for all $x \in S^{n-1}$.

The notion of the spherical derivative agrees with the usual derivative in the following sense. If $f : U \to X$, where U is an open neighbourhood of S^{n-1}, then f is differentiable at $x_0 \in S^{n-1}$ if and only if there exists $\partial^S f(x_0)$. Moreover, for any $\xi \in \mathbb{R}^n$ with $\xi \cdot x_0 = 0$, we have then

$$\partial^S f(x_0)\,\xi = f'(x_0)\,\xi = (\mathcal{E}_0 f)'(x_0)\,\xi.$$

The symbol $C^k\left(S^{n-1}, X\right)$ will stand for the space of all $f : S^{n-1} \to X$ having continuous spherical derivatives $\partial^S f, \left(\partial^S\right)^{(2)} f, \ldots, \left(\partial^S\right)^{(k)} f$ up to degree k. For $f \in C^k\left(S^{n-1}, X\right)$ we define

$$\|f\|_{C^k} = \max_{j=1,2,\ldots,k} \max_{x \in S^{n-1}} \left\| \left(\partial^S\right)^{(j)} f(x) \right\|,$$

where $\left\| \left(\partial^S\right)^{(j)} f(x) \right\|$ denotes the norm of linear operation $\left(\partial^S\right)^{(j)} f(x)$. In the case of $k = 0$ the symbol $C^0\left(S^{n-1}, X\right)$ denotes the space of all continuous $f : S^{n-1} \to X$ with the norm

$$\|f\|_{C^0} = \max_{x \in S^{n-1}} \|f(x)\|.$$

In the sequel we will consider the space

$$C^\infty\left(S^{n-1}, X\right) \stackrel{\text{def}}{=} \bigcap_{k=0}^{\infty} C^k\left(S^{n-1}, X\right),$$

being the space of test functions for distributions on the sphere S^{n-1}, equipped with the sequence of semi-norms $\|\cdot\|_{C^k}$, $k = 0, 1, \ldots$. Clearly, the space $C^\infty\left(S^{n-1}, X\right)$ is locally convex and complete.

It can be shown that any distribution on the sphere S^{n-1} in the sense of [2], see Section 6.3, is a distribution in the following sense.

Definition 2. Any linear continuous functional $u : C^\infty\left(S^{n-1}, \mathbb{R}\right) \to \mathbb{R}$ we call the distribution on the sphere. The space of all distributions on the sphere we denote by $\mathcal{D}'\left(S^{n-1}, \mathbb{R}\right)$.

Since the topology in $C^{\infty}\left(S^{n-1},\mathbb{R}\right)$ is given by the sequence of seminorms $\|\cdot\|_{C^k}$, $k\in\mathbb{N}_0$, a linear functional $u: C^{\infty}\left(S^{n-1},\mathbb{R}\right)\to\mathbb{R}$ is continuous if and only if there exist $k\in\mathbb{N}_0$ and $C\geq 0$ such that

$$|\langle u,\varphi\rangle|\leq C\,\|\varphi\|_{C^k}\,,\quad \varphi\in C^{\infty}\left(S^{n-1},\mathbb{R}\right).$$

Each distribution on the sphere is thus of finite degree.

Any continuous function $f: S^{n-1}\to\mathbb{R}$ is a regular distribution $\{f(x)\}$ given by

$$\langle\{f(x)\},\varphi\rangle=\int_{S^{n-1}} f(x)\,\varphi(x)\,\mathcal{H}^{n-1}(dx)\,,$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$.

Let $u_m\in\mathcal{D}'\left(S^{n-1},\mathbb{R}\right)$, $m\in\mathbb{N}$, and $u\in\mathcal{D}'\left(S^{n-1},\mathbb{R}\right)$ be given. We say that

$$u=\lim_{m\to\infty} u_m$$

if, for each $\varphi\in C^{\infty}\left(S^{n-1},\mathbb{R}\right)$,

$$\langle u,\varphi\rangle=\lim_{m\to\infty}\langle u_m,\varphi\rangle\,.$$

Let us recall that $u\in\mathcal{D}'\left(\mathbb{R}^n\setminus\{0\},\mathbb{R}\right)$ is homogeneous of degree α if

$$u(\psi)=r^{\alpha+n}u(\psi_r)\,,$$

for all $\psi\in C_0^{\infty}\left(\mathbb{R}^n\setminus\{0\},\mathbb{R}\right)$ and $r>0$, where

$$\psi_r(x)=\psi(rx)\,,\quad x\in\mathbb{R}^n\setminus\{0\}\,.$$

We will denote by $\mathcal{D}'_{\alpha}\left(\mathbb{R}^n\setminus\{0\},\mathbb{R}\right)$ the space of all distributions $u\in\mathcal{D}'\left(\mathbb{R}^n\setminus\{0\},\mathbb{R}\right)$ homogeneous of degree α. For any $u\in\mathcal{D}'\left(S^{n-1},\mathbb{R}\right)$ and any $\alpha\in\mathbb{R}$ we define $\mathcal{E}_{\alpha}u\in\mathcal{D}'\left(\mathbb{R}^n\setminus\{0\},\mathbb{R}\right)$, being the extension of order α of u, by the formula, see formula (3) of [1], p. 387,

$$\text{(1)}\qquad \langle\mathcal{E}_{\alpha}u,\psi\rangle=\int_0^{\infty} r^{\alpha+n-1}\langle u,\psi_r\rangle\,dr,\quad \psi\in C_0^{\infty}\left(\mathbb{R}^n\setminus\{0\},\mathbb{R}\right).$$

It is easy to prove that $\mathcal{E}_{\alpha}u$ is homogeneous of degree α and

$$\mathcal{E}_{\alpha}:\mathcal{D}'\left(S^{n-1},\mathbb{R}\right)\to\mathcal{D}'_{\alpha}\left(\mathbb{R}^n\setminus\{0\},\mathbb{R}\right)$$

is a linear continuous and univalent mapping.

2. Main result. We are going to prove in this section that for any homogeneous $u\in\mathcal{D}'\left(\mathbb{R}^n\setminus\{0\},\mathbb{R}\right)$, of degree α, there exists a unique $\mathcal{R}_{\alpha}u\in\mathcal{D}'\left(S^{n-1},\mathbb{R}\right)$ such that

$$\mathcal{E}_{\alpha}\mathcal{R}_{\alpha}u=u.$$

In other words $\mathcal{E}_{\alpha}$ is a continuous linear isomorphism between $\mathcal{D}'\left(S^{n-1},\mathbb{R}\right)$ and $\mathcal{D}'_{\alpha}\left(\mathbb{R}^n\setminus\{0\},\mathbb{R}\right)$.

Theorem 1. *For any* $u \in \mathcal{D}'_\alpha(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ *we have*

$$u = \mathcal{E}_\alpha \mathcal{R}_\alpha u,$$

where $\mathcal{R}_\alpha : \mathcal{D}'_\alpha(\mathbb{R}^n \setminus \{0\}, \mathbb{R}) \to \mathcal{D}'(S^{n-1}, \mathbb{R})$ *is a linear continuous mapping given by the formula*

$$\langle \mathcal{R}_\alpha u, \varphi \rangle = \left\langle u, \left\{ \varphi\left(\frac{x}{|x|}\right) \psi_0(|x|) \right\} \right\rangle, \quad u \in \mathcal{D}'_\alpha(\mathbb{R}^n \setminus \{0\}, \mathbb{R}),$$

with a fixed $\psi_0 \in C_0^\infty((0,\infty), \mathbb{R})$ *such that*

$$\psi_0 \geq 0, \ \int_0^\infty r^{n+\alpha-1} \psi_0(r)\, dr = 1.$$

The proof will be divided into a few steps.

Claim 1. *If* $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ *and* $a \in \mathbb{R}$ *then the equation*

$$a\Phi(x) + x \cdot \Phi'(x) = f(x), \quad x \in \mathbb{R}^n \setminus \{0\},$$

has exactly one solution $\Phi \in C^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ *given by the formula*

$$\Phi(x) = \int_0^1 t^{a-1} f(tx)\, dt, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Moreover, if $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ *and, for each* $x \in \mathbb{R}^n \setminus \{0\}$,

$$\int_0^\infty t^{a-1} f(tx)\, dt = 0$$

then $\Phi \in C_0^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$.

Proof of Claim 1. Let us define

$$\Phi(x) = \int_0^1 t^{a-1} f(tx)\, dt, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Clearly $\Phi \in C^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$. For any $x \in \mathbb{R}^n \setminus \{0\}$ we have

$$\begin{aligned} x \cdot \Phi'(x) &= \int_0^1 t^a x \cdot f'(tx)\, dt = \int_0^1 t^a \frac{d}{dt} f(tx)\, dt \\ &= \left[t^a f(tx)\right]_{t=0}^{t=1} - a \int_0^1 t^{a-1} f(tx)\, dt \\ &= f(x) - a\Phi(x), \end{aligned}$$

so that Φ satisfies the equation.

Let us suppose that $\psi \in C^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ satisfies the equation. Let $x \in \mathbb{R}^n \setminus \{0\}$ be fixed arbitrarily. Define

$$v(t) = \psi(tx), \ w(t) = f(tx), \quad t \in (0,\infty).$$

For all $t > 0$ we have

$$\begin{aligned}\frac{d}{dt}\left(t^{a}v\left(t\right)\right) &= at^{a-1}v\left(t\right)+t^{a}v'\left(t\right)=at^{a-1}\psi\left(tx\right)+t^{a}x\cdot\psi'\left(tx\right)\\ &= at^{a-1}\psi\left(tx\right)+t^{a-1}tx\cdot\psi'\left(tx\right)\\ &= t^{a-1}\cdot\left(a\psi\left(tx\right)+tx\cdot\psi'\left(tx\right)\right)\\ &= t^{a-1}\cdot f\left(tx\right)=t^{a-1}\cdot w\left(t\right),\end{aligned}$$

thus

$$\psi\left(x\right)=v\left(1\right)=\int_{0}^{1}t^{a-1}w\left(t\right)dt=\int_{0}^{1}t^{a-1}f\left(tx\right)dt=\Phi\left(x\right).$$

Suppose now that $f\in C_{0}^{\infty}\left(\mathbb{R}^{n}\setminus\{0\},\mathbb{R}\right)$ and, for each $x\in\mathbb{R}^{n}\setminus\{0\}$,

$$\int_{0}^{\infty}t^{a-1}f\left(tx\right)dt=0.$$

Since $\operatorname{supp}\left(f\right)\subset\mathbb{R}^{n}\setminus\{0\}$ there exist $a,b\in\mathbb{R}$ such that $0<a<b$ and

$$|x|\notin\left(a,b\right)\Rightarrow f\left(x\right)=0.$$

Let us fix arbitrarily an $x\in\mathbb{R}^{n}\setminus\{0\}$. If $|x|\leq a$ then

$$\Phi\left(x\right)=\int_{0}^{1}t^{a-1}f\left(tx\right)dt=0.$$

If $|x|\geq b$ then

$$\Phi\left(x\right)=\int_{0}^{1}t^{a-1}f\left(tx\right)dt=\int_{0}^{\infty}t^{a-1}f\left(tx\right)dt=0.$$

□

Claim 2. *If $u\in\mathcal{D}'_{\alpha}\left(\mathbb{R}^{n}\setminus\{0\},\mathbb{R}\right)$ then*

$$\left\langle u,\left\{\varphi\left(\frac{x}{|x|}\right)\psi\left(|x|\right)\right\}\right\rangle=0$$

for all $\varphi\in C^{\infty}\left(S^{n-1},\mathbb{R}\right)$ and $\psi\in C_{0}^{\infty}\left(\left(0,\infty\right),\mathbb{R}\right)$ such that

$$\int_{0}^{\infty}t^{n+\alpha-1}\psi\left(t\right)dt=0.$$

Proof of Claim 2. Let us define $a=n+\alpha$. Using the Euler's identity

$$\sum_{i=1}^{n}x_{i}\frac{\partial}{\partial x_{i}}u=au,$$

for any $\Phi\in C_{0}^{\infty}\left(\mathbb{R}^{n}\setminus\{0\},\mathbb{R}\right)$ we obtain

$$\left\langle u,a\Phi+\sum_{i=1}^{n}x_{i}\frac{\partial}{\partial x_{i}}\Phi\right\rangle=0.$$

By Claim 1, there exists a $\Phi \in C_0^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ such that

$$\varphi\left(\frac{x}{|x|}\right)\psi(|x|) = a\Phi(x) + \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}\Phi(x).$$

□

Claim 3. *If $u \in \mathcal{D}'_\alpha(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ then there exists a distribution $\mathcal{R}_\alpha u \in \mathcal{D}'\left(S^{n-1}, \mathbb{R}\right)$ such that*

$$\left\langle u, \varphi\left(\frac{x}{|x|}\right)\psi(|x|)\right\rangle = \langle \mathcal{R}_\alpha u, \varphi\rangle \cdot \int_0^\infty r^{n+\alpha-1}\psi(r)\,dr,$$

for all $\varphi \in C^\infty\left(S^{n-1}, \mathbb{R}\right)$ and $\psi \in C_0^\infty((0,\infty), \mathbb{R})$. Moreover,

$$\mathcal{R}_\alpha : \mathcal{D}'_\alpha(\mathbb{R}^n \setminus \{0\}, \mathbb{R}) \to \mathcal{D}'\left(S^{n-1}, \mathbb{R}\right)$$

is a linear continuous mapping.

Proof of Claim 3. Let us fix a $\psi_0 \in C_0^\infty((0,\infty), \mathbb{R})$ such that

$$\psi_0 \geq 0, \int_0^\infty r^{n+\alpha-1}\psi_0(r)\,dr = 1.$$

Define, for all $\varphi \in C^\infty\left(S^{n-1}, \mathbb{R}\right)$,

$$\langle \mathcal{R}_\alpha u, \varphi\rangle = \left\langle u, \left\{\varphi\left(\frac{x}{|x|}\right)\psi_0(|x|)\right\}\right\rangle.$$

Clearly $\mathcal{R}_\alpha u \in \mathcal{D}'\left(S^{n-1}, \mathbb{R}\right)$. For each $\psi \in C_0^\infty((0,\infty), \mathbb{R})$ and each $r > 0$ define

$$\psi_1(r) = \psi(r) - \left(\int_0^\infty \varrho^{n+\alpha-1}\psi(\varrho)\,d\varrho\right) \cdot \psi_0(r).$$

Since $\psi_1 \in C_0^\infty((0,\infty), \mathbb{R})$ and

$$\int_0^\infty r^{n+\alpha-1}\psi_1(r)\,dr = 0,$$

we have

$$\left\langle u, \left\{\varphi\left(\frac{x}{|x|}\right)\psi_1(|x|)\right\}\right\rangle = 0.$$

Consequently, for all $\varphi \in C^\infty\left(S^{n-1}, \mathbb{R}\right)$ and $\psi \in C_0^\infty((0,\infty), \mathbb{R})$, we obtain

$$\begin{aligned}\left\langle u, \left\{\varphi\left(\frac{x}{|x|}\right)\psi(|x|)\right\}\right\rangle &= \int_0^\infty r^{n+\alpha-1}\psi(r)\,dr \cdot \left\langle u, \left\{\varphi\left(\frac{x}{|x|}\right)\psi_0(|x|)\right\}\right\rangle \\ &= \langle \mathcal{R}_\alpha u, \varphi\rangle \cdot \int_0^\infty r^{n+\alpha-1}\psi(r)\,dr.\end{aligned}$$

The linearity and continuity of the mapping

$$\mathcal{R}_\alpha : \mathcal{D}'_\alpha(\mathbb{R}^n \setminus \{0\}, \mathbb{R}) \to \mathcal{D}'\left(S^{n-1}, \mathbb{R}\right)$$

are obvious. □

It is easy to check that in the case of regular homogeneous distribution

$$u = \{f(x)\} \in \mathcal{D}'_\alpha(\mathbb{R}^n \setminus \{0\}, \mathbb{R}),$$

the restriction $\mathcal{R}_\alpha u$ coincides with f restricted to S^{n-1}.

Claim 4. *Given a homogeneous $u \in \mathcal{D}'_\alpha(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$. Then, for all $\varphi \in C^\infty(S^{n-1}, \mathbb{R})$ and $\psi \in C_0^\infty((0,\infty), \mathbb{R})$ we have*

$$\left\langle \mathcal{E}_\alpha \mathcal{R}_\alpha u, \left\{ \varphi\left(\frac{x}{|x|}\right) \psi(|x|) \right\} \right\rangle = \left\langle u, \left\{ \varphi\left(\frac{x}{|x|}\right) \psi(|x|) \right\} \right\rangle.$$

Proof of Claim 4. Let us fix arbitrarily $\varphi \in C^\infty(S^{n-1}, \mathbb{R})$ and $\psi \in C_0^\infty((0,\infty), \mathbb{R})$. According to the extension formula (1), by Claim 3, we obtain

$$\begin{aligned} \left\langle \mathcal{E}_\alpha \mathcal{R}_\alpha u, \left\{ \varphi\left(\frac{x}{|x|}\right) \psi(|x|) \right\} \right\rangle &= \int_0^\infty r^{n+\alpha-1} \langle \mathcal{R}_\alpha u, \{\varphi(\omega)\psi(r)\}\rangle \, dr \\ &= \langle \mathcal{R}_\alpha u, \varphi \rangle \cdot \int_0^\infty r^{n+\alpha-1} \psi(r) \, dr \\ &= \left\langle u, \left\{ \varphi\left(\frac{x}{|x|}\right) \psi(|x|) \right\} \right\rangle. \end{aligned}$$

□

Let us define, for $f \in C_0^\infty((0,\infty), \mathbb{R})$ and $g \in C_0^\infty(S^{n-1}, \mathbb{R})$,

$$(f \otimes g)(x) = f(|x|) \cdot g\left(\frac{x}{|x|}\right), \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Claim 5. *For each $\varphi \in C_0^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ there exists a sequence*

$$\varphi_m = \sum_{k=1}^{k_m} t_{m,k} f_{m,k} \otimes g_{m,k}, \quad m \in \mathbb{N},$$

such that $t_{m,k} \in \mathbb{R}$,

$$f_{m,k} \in C_0^\infty((0,\infty), \mathbb{R}), \ g_{m,k} \in C_0^\infty(S^{n-1}, \mathbb{R}), \quad k = 1, 2, \ldots, k_m$$

and

$$\varphi = \lim_{m\to\infty} \sum_{k=1}^{k_m} t_{m,k} f_{m,k} \otimes g_{m,k}$$

(in the space $C_0^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$).

Proof of Claim 5. Since $\operatorname{supp}(\varphi) \subset \mathbb{R}^n \setminus \{0\}$, there exist $0 < a < b < \infty$ such that

$$|x| \notin (a,b) \Rightarrow \varphi(x) = 0.$$

Let us define

$$F(r,\omega) = \varphi(r\omega), \quad t \in (0,\infty), \ \omega \in S^{n-1}.$$

There exists an $\widetilde{F} \in C_0^\infty\left((0,\infty)\times(\mathbb{R}^n\setminus\{0\}),\mathbb{R}\right)$ such that

$$\widetilde{F}(w) = \begin{cases} F\left(r, \frac{w}{\|w\|}\right) & \text{if } \frac{2}{3} \le \|w\| \le \frac{4}{3}, \\ 0 & \text{if } \|w\| < \frac{1}{3} \text{ or } \|w\| > \frac{5}{3}. \end{cases}$$

Let us define, for $0 < \alpha < \beta < \infty$,

$$R_{\alpha,\beta} = \{x \in \mathbb{R}^n : \ \alpha < |x| < \beta\}.$$

By Lemma 1 of [3], p. 48, one can find a sequence

$$\sum_{k=1}^{k_m} t_{m,k} f_{m,k} \cdot g_{m,k}$$

such that $t_{m,k} \in \mathbb{R}$,

$$\begin{aligned} &f_{m,k} \in C_0^\infty\left((0,\infty),\mathbb{R}\right), \ \operatorname{supp}\left(f_{m,k}\right) \subset \left(\frac{1}{2}, 2b\right), \\ &g_{m,k} \in C_0^\infty\left(\mathbb{R}^n\setminus\{0\},\mathbb{R}\right), \ \operatorname{supp}\left(g_{m,k}\right) \subset R_{\frac{2}{3},\frac{4}{3}}, \quad k = 1,2,\dots,k_m \end{aligned}$$

and

$$\widetilde{F} = \lim_{m\to\infty} \sum_{k=1}^{k_m} t_{m,k} f_{m,k} \cdot g_{m,k}$$

(in the space $C_0^\infty\left(\left(\frac{1}{2}, 2b\right) \times R_{\frac{2}{3},\frac{4}{3}}, \mathbb{R}\right)$). Since

$$\widetilde{F}\left(|x|, \frac{x}{|x|}\right) = \varphi(x), \quad x \in \mathbb{R}^n\setminus\{0\},$$

we obtain

$$\varphi = \lim_{m\to\infty} \sum_{k=1}^{k_m} t_{m,k} f_{m,k} \otimes g_{m,k}$$

(in the space $C_0^\infty\left(\mathbb{R}^n\setminus\{0\},\mathbb{R}\right)$). □

Proof of Theorem 1. By Claim 4, $u = \mathcal{E}_\alpha\mathcal{R}_\alpha u$ in the set Z being the linear hull of the set

$$C_0^\infty\left((0,\infty),\mathbb{R}\right) \otimes C^\infty\left(S^{n-1},\mathbb{R}\right)$$

of all $f \otimes g$ where $f \in C_0^\infty\left((0,\infty),\mathbb{R}\right)$ and $g \in C_0^\infty\left(S^{n-1},\mathbb{R}\right)$. Since, by Claim 5, the set Z is dense in the space $C_0^\infty\left(\mathbb{R}^n\setminus\{0\},\mathbb{R}\right)$, we obtain

$$u = \mathcal{E}_\alpha\mathcal{R}_\alpha u.$$

□

Corollary 1. *Any homogeneous distribution $u \in \mathcal{D}'\left(\mathbb{R}^n\setminus\{0\},\mathbb{R}\right)$ is of finite order.*

References

[1] Gelfand, I. M., Shilov, G. E., *Generalized Functions and Operations on Them*, Second Edition, Gos. Izd. Fiz.-Mat. Lit., Moscow, 1959 (Russian).
[2] Hörmander, L., *The Analysis of Linear Partial Differential Operators I*, Springer-Verlag, Berlin–Heidelberg–New York–Tokyo, 1983.
[3] Shilov, G. E., *Mathematical Analysis, Second Special Lecture*, MGU, Moscow, 1984 (Russian).

Andrzej Miernowski
Institute of Mathematics
M. Curie-Skłodowska University
pl. Marii Curie-Skłodowskiej 1
20-031 Lublin, Poland
e-mail: mierand@golem.umcs.lublin.pl

Witold Rzymowski
Department of Quantitative Methods
in Management
Lublin University of Technology
ul. Nadbystrzycka 38
20-618 Lublin, Poland
e-mail: w.rzymowski@pollub.pl

Department of Applied Mathematics
The John Paul II Catholic University of Lublin
ul. Konstantynów 1 H
20-708 Lublin, Poland

Received July 24, 2006