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## On univalence of an integral operator

Abstract. We consider the problem of univalence of the integral operator

$$
\begin{equation*}
h(z ; \alpha, \beta)=\int_{0}^{z}\left(f^{\prime}(t)\right)^{\alpha}(g(t))^{\beta} t^{-\beta} d t \tag{1}
\end{equation*}
$$

Imposing on functions $f(z), g(z)$ various conditions and making use of a close-to-convexity property of the operator, we establish many sufficient conditions for univalence. Our results extend earlier ones. Some questions remain open.

1. Introduction. Denote by $H$ the class of all functions of the form

$$
f(z)=z+\sum_{n=2}^{\infty} a_{n} z^{n}
$$

that are analytic in the unit disk $\Delta=\{z:|z|<1\}$. For given $\alpha, \beta \in \mathbb{R}$ denote

$$
\begin{equation*}
h(z ; \alpha, \beta)=\int_{0}^{z}\left(f^{\prime}(t)\right)^{\alpha}(g(t))^{\beta} t^{-\beta} d t=z+\sum_{n=2}^{\infty} a_{n} z^{n} \tag{2}
\end{equation*}
$$

provided $f^{\prime}(z) \cdot g(z) \neq 0$ for $z \in \Delta, f(z), g(z) \in H$ and we consider principal branches of the powers. Our objective of this paper is the question of univalence of $h(z ; \alpha, \beta)$ in the unit disk.

This problem was studied earlier by many authors (see [3] for instance). A. Wesołowski [6] proved that if $f(z), g(z)$ are univalent in $\Delta$, then $h(z ; \alpha, \beta)$ is univalent in $\Delta$ provided the condition $\alpha, \beta \in \mathbb{C}, 3|\beta|+2|2 \alpha+\beta| \leq 1$ is

[^0]satisfied. J. Godula [2] proved that if $f(z), g(z)$ are univalent and close-toconvex in $\Delta$, then $h(z ; \alpha, \beta)$ is univalent and close-to-convex in $\Delta$ provided $\alpha, \beta$ are non-negative and $0 \leq \alpha+\beta \leq 1$. E. P. Merkes and D. J. Wright [4] considered cases $h(z ; \alpha, 0), h(z ; 0, \beta)$, where they imposed on $f(z)$ or $g(z)$ respectively various conditions. M. Dorff and J. Szynal [1] considered the case $f(z)=g(z)$ in (1).

In this paper we shall extend cited above results included in [2], [4] and [1]. Our method is based on a property of $h(z ; \alpha, \beta)$ which, according to the best of our knowledge, has not been used in such a context. Let us recall that a function $f(z) \in H$ is said to be:

- convex, $f(z) \in K$, if $\operatorname{Re}\left(1+z f^{\prime \prime}(z) / f(z)\right)>0$ for $z$ in $\Delta$,
- starlike, $f(z) \in S^{*}$, if $\operatorname{Re}\left(z f^{\prime}(z) / f(z)\right)>0$ for $z$ in $\Delta$,
- close-to-convex, $f(z) \in L$, if there exist a starlike function $g(z)$ and a constant $\theta \in(-\pi / 2, \pi / 2)$ such that

$$
\operatorname{Re}\left(e^{i \theta} z f^{\prime}(z) / g(z)\right)>0, \quad z \in \Delta
$$

The last definition can be stated in an equivalent form as follows: $f(z) \in L$ if there exist a constant $\theta \in(-\pi / 2, \pi / 2)$ and a function $g(z) \in K$ such that

$$
\operatorname{Re}\left(e^{i \theta} f^{\prime}(z) / g^{\prime}(z)\right)>0, \quad z \in \Delta
$$

We denote by $S$ the subclass of $H$ that consists of all univalent functions in $\Delta$.
2. Main results. We shall start with a lemma which is the main tool in our considerations.

Lemma 1. Let $h(z ; \alpha, \beta)$ be given by (1). If the function $h(z ; \alpha, \beta)$ is close-to-convex for given fixed points $\left(\alpha_{1}, \beta_{1}\right),\left(\alpha_{2}, \beta_{2}\right)$, then it is also close-toconvex for each pair $\left(\alpha_{t}, \beta_{t}\right)$ given by $\left(\alpha_{t}, \beta_{t}\right)=t\left(\alpha_{1}, \beta_{1}\right)+(1-t)\left(\alpha_{2}, \beta_{2}\right)$ and $0 \leq t \leq 1$.

Proof. We have the identity

$$
\begin{aligned}
h^{\prime}\left(z ; \alpha_{t}, \beta_{t}\right) & =\left(f^{\prime}(z)\right)^{t \alpha_{1}+(1-t) \alpha_{2}}(g(z))^{t \beta_{1}+(1-t) \beta_{2}} z^{-t \beta_{1}-(1-t) \beta_{2}} \\
& =\left(\left(f^{\prime}(z)\right)^{\alpha_{1}}(g(z))^{\beta_{1}} z^{-\beta_{1}}\right)^{t}\left(\left(f^{\prime}(z)\right)^{\alpha_{2}}(g(z))^{\beta_{2}} z^{-\beta_{2}}\right)^{1-t} \\
& =\left(h^{\prime}\left(z ; \alpha_{1}, \beta_{1}\right)\right)^{t}\left(h^{\prime}\left(z ; \alpha_{2}, \beta_{2}\right)\right)^{1-t}
\end{aligned}
$$

Since $h\left(z ; \alpha_{1}, \beta_{1}\right), h\left(z ; \alpha_{2}, \beta_{2}\right)$ are in $L$, there exist functions $X(z), Y(z)$ in $K$ and functions $p(z), q(z)$ of positive real part in $\Delta$ for which we have

$$
h^{\prime}\left(z ; \alpha_{1}, \beta_{1}\right)=X^{\prime}(z) p(z), \quad h^{\prime}\left(z ; \alpha_{2}, \beta_{2}\right)=Y^{\prime}(z) q(z)
$$

It is well known that functions of positive real part in $\Delta$ form a convex family. This implies that the function $D(z)=p(z)^{t} q(z)^{1-t}$ is of positive real part. If we set

$$
W^{\prime}(z)=\left(X^{\prime}(z)\right)^{t}\left(Y^{\prime}(z)\right)^{1-t}
$$

then we have

$$
W^{\prime \prime}(z)=t\left(X^{\prime}(z)\right)^{t-1} X^{\prime \prime}(z)\left(Y^{\prime}(z)\right)^{1-t}+(1-t)\left(X^{\prime}(z)\right)^{t}\left(Y^{\prime}(z)\right)^{-t} Y^{\prime \prime} z
$$

hence

$$
\begin{aligned}
\operatorname{Re}(1 & \left.+z W^{\prime \prime}(z) / W^{\prime}(z)\right)=\operatorname{Re}\left(1+z t X^{\prime \prime}(z) / X^{\prime}(z)+z(1-t) Y^{\prime \prime}(z) / Y^{\prime}(z)\right) \\
& =\operatorname{Re}\left(t+z t X^{\prime \prime}(z) / X^{\prime}(z)\right)+\operatorname{Re}\left(1-t+z(1-t) Y^{\prime \prime}(z) / Y^{\prime}(z)\right)>0
\end{aligned}
$$

It shows that $W(z) \in K$. Hence $h\left(z ; \alpha_{t}, \beta_{t}\right) \in L$.
We shall also need a result of W. Royster [5] (see also [3]) which we state as:

Lemma 2. The function

$$
(1-z)^{\beta}=e^{\beta \ln (1-z)}, \quad \beta \neq 0
$$

is univalent in $\Delta$ if and only if $\beta$ is either in the closed disk $|\beta-1| \leq 1$ or in the closed disk $|\beta+1| \leq 1$.

## Theorem 1.

1. Assume that

$$
A_{1}=\operatorname{conv}(\{(-1 / 2,0),(0,3),(3 / 2,0),(0,-1),(-1 / 2,2),(1,-1)\})
$$

and $f(z), g(z) \in K$. Then for each pair $(\alpha, \beta) \in A_{1}$ the operator $h(z ; \alpha, \beta)$ is in $L$.
2. Assume that

$$
\begin{aligned}
B_{1}=\left\{(x, y) \in \mathbb{R}^{2}:\right. & (y<-1) \vee(y>3) \vee(x<-1 / 2) \vee(x>3 / 2) \\
& \vee(y<-2 x-1) \vee(y>-2 x+3)\}
\end{aligned}
$$

Then for each pair $(\alpha, \beta) \in B_{1}$ there exist functions $f(z)$ and $g(z)$ both convex such that the function $h(z ; \alpha, \beta)=\int_{0}^{z}\left(f^{\prime}(t)\right)^{\alpha}(g(t))^{\beta} t^{-\beta} d t$ is not univalent in $\Delta$.

Proof. The set $A_{1}$ is a convex polygon. In view of Lemma 1 it is sufficient to establish close-to-convexity of $h(z ; \alpha, \beta)$ at the vertices of $A_{1}$. The points $(-1 / 2,0),(0,3),(3 / 2,0),(0,-1)$ were handled in [4]. The corresponding functions $h(z ; \alpha, \beta)$ are in $L$.

For the vertex $(-1 / 2,2)$ we have

$$
h^{\prime}(z ;-1 / 2,2)=\left(f^{\prime}(z)\right)^{-\frac{1}{2}}(g(z))^{2} z^{-2}
$$

It is known (see [3]) that $\operatorname{Re}\left(\left(f^{\prime}(z)\right)^{-\frac{1}{2}}\right)>0$ and $(g(z))^{2} z^{-2}$ is a derivative of a function in $K$. Hence $h(z ;-1 / 2,2) \in L$.

For the vertex $(1,-1)$ there is

$$
h^{\prime}(z ; 1,-1)=f^{\prime}(z)(g(z))^{-1} z .
$$

But $\operatorname{Re}\left((g(z))^{-1} z\right)>0($ see $[3])$ so $h(z ; 1,-1) \in L$. This ends the first part of the proof and gives a sufficient condition for univalence of $h(z ; \alpha, \beta)$.

We now prove the second part of the theorem.

1. Let us set $f(z)=z$ or $g(z)=z$, then we have

$$
h(z ; \alpha, \beta)=\int_{0}^{z}\left(f^{\prime}(t)\right)^{\alpha}(g(t))^{\beta} t^{-\beta} d t=\int_{0}^{z}(g(t))^{\beta} t^{-\beta} d t=h(z ; 0, \beta)
$$

or

$$
h(z ; \alpha, \beta)=\int_{0}^{z}\left(f^{\prime}(t)\right)^{\alpha}(g(t))^{\beta} t^{-\beta} d t=\int_{0}^{z}\left(f^{\prime}(t)\right)^{\alpha} d t=h(z ; \alpha, 0)
$$

and these cases were considered in [4]. We get that $h(z ; \alpha, \beta)$ is not univalent if $(\beta<-1) \vee(\beta>3) \vee(\alpha<-1 / 2) \vee(\alpha>3 / 2)$.
2. Setting $f(z)=z /(1-z)=g(z)$ we have

$$
h(z ; \alpha, \beta)=\int_{0}^{z}(1-t)^{-2 \alpha}(1-t)^{-\beta} d t=k(1-z)^{-2 \alpha-\beta+1}-k,
$$

where $k$ is a constant. From Lemma 2 it follows that $h(z ; \alpha, \beta)$ is not univalent for $-2 \alpha-\beta+1>2$ or $-2 \alpha-\beta+1<-2$, in other words for $\beta<-2 \alpha-1$ or $\beta>-2 \alpha+3$.

Cases 1 and 2 define the set $B_{1}$. This ends the proof.
For $(\alpha, \beta) \in C_{1}=\mathbb{R}^{2} \backslash\left(A_{1} \cup B_{1}\right)=T_{1} \cup T_{2}$ the problem of univalence of $h(z ; \alpha, \beta)$ is open (see Figure 1).


Figure 1.
When $f(z)=g(z)$, then

$$
\begin{equation*}
h(z ; \alpha, \beta)=\int_{0}^{z}\left(f^{\prime}(t)\right)^{\alpha}(f(t))^{\beta} t^{-\beta} d t=H(z ; \alpha, \beta) . \tag{3}
\end{equation*}
$$

M. Dorff and J. Szynal proved in their paper [1], that the function $H(z ; \alpha, \beta)$ defined by (3) is univalent in $\Delta$ if $f(z)$ is a convex, univalent function and $(\alpha, \beta) \in A$, where

$$
\begin{aligned}
A= & \{(\alpha, \beta): \alpha \in[0,3 / 2], \beta \in[-1,3-2 \alpha]\} \\
& \cup\{(\alpha, \beta): \alpha \in[-1 / 2,0], \beta \in[-1-2 \alpha, 3]\} .
\end{aligned}
$$

See Figure 2.


## Figure 2.

We shall extend these results.

## Theorem 2.

1. Assume that

$$
E_{1}=\operatorname{conv}(\{(-1,3),(-1,1),(0,-1),(2,-1),(0,3)\})
$$

and $f(z) \in K$. Then for each pair $(\alpha, \beta) \in E_{1}$ the operator

$$
H(z ; \alpha, \beta)=\int_{0}^{z}\left(f^{\prime}(t)\right)^{\alpha}(f(t))^{\beta} t^{-\beta} d t
$$

is in $L$.
2. Assume that

$$
E_{2}=\left\{(x, y) \in \mathbb{R}^{2}:(y<-2 x-1) \vee(y>-2 x+3)\right\}
$$

Then for each pair $(\alpha, \beta) \in E_{2}$ there exists a function $f(z) \in K$ such that the function

$$
H(z ; \alpha, \beta)=\int_{0}^{z}\left(f^{\prime}(t)\right)^{\alpha}(f(t))^{\beta} t^{-\beta} d t
$$

is not univalent in $\Delta$.

Proof. The set $E_{1}$ is a convex polygon. In view of Lemma 1 it is sufficient to establish close-to-convexity of $H(z ; \alpha, \beta)$ at the vertices of $E_{1}$. The points $(0,3),(0,-1)$ were handled in [4]. The corresponding functions $H(z ; \alpha, \beta)$ are in $L$.

For the vertex $(-1,3)$ we have

$$
\begin{gathered}
H^{\prime}(z ;-1,3)=\left(f^{\prime}(z)\right)^{-1}(f(z))^{3} z^{-3} \\
H^{\prime}(z ;-1,3)=\left(f(z) / z f^{\prime}(z)\right)(f(z))^{2} z^{-2} .
\end{gathered}
$$

It is known that $\operatorname{Re}\left(f(z) / z f^{\prime}(z)\right)>0$ and $(f(z))^{2} z^{-2}$ is a derivative of a function in $K$. Hence $H(z ;-1,3) \in L$.

For the vertex $(-1,1)$ there is

$$
H^{\prime}(z ;-1,1)=\left(f(z) / z f^{\prime}(z)\right)
$$

and so $H^{\prime}(z ;-1,1) \in L$. At the point $(2,-1)$ we have

$$
H^{\prime}(z ; 2,-1)=\left(z f^{\prime}(z) / f(z)\right) f^{\prime}(z)
$$

This ends the first part of the proof and gives a sufficient condition for univalence of $H(z ; \alpha, \beta)$.

We now prove the second part of the theorem. Setting $f(z)=z /(1-z)$ we have

$$
H(z ; \alpha, \beta)=\int_{0}^{z}(1-t)^{-2 \alpha}(1-t)^{-\beta} d t=k(1-z)^{-2 \alpha-\beta+1}-k
$$

where k is a constant. From Lemma 2 it follows that $H(z ; \alpha, \beta)$ is not univalent for $-2 \alpha-\beta+1>2$ or $-2 \alpha-\beta+1<-2$, in other words for $\beta<-2 \alpha-1$ or $\beta>-2 \alpha+3$. It defines the set $E_{2}$. This ends the proof.


Figure 3.

We can prove that $H(z ; \alpha, \beta)$ is not close-to-convex if $\beta>3$ or $\beta<-1$.

## Theorem 3.

## 1. Assume that

$$
A_{2}=\operatorname{conv}(\{(-1 / 2,0),(0,3 / 2),(3 / 2,0),(0,-1 / 2),(-1 / 2,1),(1,-1 / 2)\})
$$

and $f(z) \in K, g(z) \in S^{*}$. Then for each pair $(\alpha, \beta) \in A_{2}$ the operator $h(z ; \alpha, \beta)$ is in $L$.
2. Assume that

$$
\begin{aligned}
B_{2}=\left\{(x, y) \in \mathbb{R}^{2}:\right. & (y<-1 / 2) \vee(y>3 / 2) \vee(x<-1 / 2) \\
& \vee(x>3 / 2) \vee(y<-x-1 / 2) \vee(y>-x+3 / 2)\}
\end{aligned}
$$

Then for each pair $(\alpha, \beta) \in B_{2}$ there exist functions $f(z) \in K$ and $g(z) \in S^{*}$ such that the function

$$
h(z ; \alpha, \beta)=\int_{0}^{z}\left(f^{\prime}(t)\right)^{\alpha}(g(t))^{\beta} t^{-\beta} d t
$$

is not univalent in $\Delta$.

Proof. The set $A_{2}$ is a convex polygon. In view of Lemma 1 it is sufficient to establish close-to-convexity of $h(z ; \alpha, \beta)$ at the vertices of $A_{2}$. The points $(-1 / 2,0),(0,3 / 2),(3 / 2,0),(0,-1 / 2)$ were handled in [4]. The corresponding functions $h(z ; \alpha, \beta)$ are in $L$.

For the vertex $(-1 / 2,1)$ we have

$$
h^{\prime}(z ;-1 / 2,1)=\left(f^{\prime}(z)\right)^{-\frac{1}{2}} g(z) z^{-1}
$$

where $\operatorname{Re}\left(\left(f^{\prime}(z)\right)^{-\frac{1}{2}}\right)>0$ and $g(z) \in S^{*}$. Hence

$$
h(z ;-1 / 2,1) \in L
$$

For the vertex $(1,-1 / 2)$ we have the relation

$$
h(z ; 1,-1 / 2)=f^{\prime}(z)(g(z))^{-\frac{1}{2}} z^{\frac{1}{2}} .
$$

$\operatorname{Re}\left((g(z))^{-\frac{1}{2}} z^{\frac{1}{2}}\right)>0$ for each starlike function $g(z)$ (see [3]) so

$$
h(z ; 1,-1 / 2) \in L
$$

This ends the first part of the proof and gives a sufficient condition for univalence of $h(z ; \alpha, \beta)$.

We now prove the second part of the theorem.

1. Proceeding analogously to the proof of Theorem 1 we obtain that $h(z ; \alpha, \beta)$ is not univalent for $(\beta<-1 / 2) \vee(\beta>3 / 2) \vee(\alpha<-1 / 2) \vee(\alpha>$ $3 / 2$ ) as was shown in [4].
2. Setting $f(z)=z(1-z)^{-1}$ and $g(z)=z(1-z)^{-2}$ we have

$$
h(z ; \alpha, \beta)=\int_{0}^{z}(1-t)^{-2 \alpha}(1-t)^{-2 \beta} d t=k(1-z)^{-2 \alpha-2 \beta+1}-k
$$

where $k$ - constant.
From Lemma 2 it follows that $h(z ; \alpha, \beta)$ is not univalent for $-2 \alpha-2 \beta+1>$ 2 or $-2 \alpha-2 \beta+1<-2$, in other words for $\beta<-\alpha-1 / 2$ or $\beta>-\alpha+3 / 2$.

The cases 1 and 2 define the set $B_{2}$. This ends the proof.

For

$$
(\alpha, \beta) \in C_{2}=\mathbb{R}^{2} \backslash\left(A_{2} \cup B_{2}\right)=T_{3} \cup T_{4}
$$

the problem of univalence of $h(z ; \alpha, \beta)$ is open (see Figure 4).


Figure 4.

Lemma 3. If $f(z) \in L$ and $-1 / 3 \leq b \leq 1 / 3$, then there exists $\alpha \in$ $(-\pi / 2, \pi / 2)$, such that $\operatorname{Re}\left(e^{i \alpha}\left(f^{\prime}(z)\right)^{b}\right)>0$ in $\Delta$.

Proof. If $f(z) \in L$, then there exist a constant $\theta \in(-\pi / 2, \pi / 2)$ and a function $w(z) \in K$ such that

$$
\operatorname{Re}\left(e^{i \theta} f^{\prime}(z) w^{\prime}(z)\right)>0, \quad z \in \Delta
$$

If we set $\alpha=b \theta$, then $\alpha \in(-\pi / 6, \pi / 6)$. Now

$$
\operatorname{Re}\left(e^{i \alpha}\left(f^{\prime}(z)\right)^{b}\right)=\operatorname{Re}\left(e^{i b \theta}\left(f^{\prime}(z)\right)^{b}\right)=\operatorname{Re}\left(\left(e^{i \theta} f^{\prime}(z)\right)^{b}\right)=\operatorname{Re}\left(\left(p(z) w^{\prime}(z)\right)^{b}\right)
$$

where $\operatorname{Re}(p(z))>0$ and $w(z) \in K$.
Next we have the equality

$$
\begin{aligned}
\operatorname{Re}\left(e^{i \alpha}\left(f^{\prime}(z)\right)^{b}\right) & =\operatorname{Re}\left(\left(p(z) w^{\prime}(z)\right)^{b}\right)=\operatorname{Re}\left((p(z))^{b}\left(\left(w^{\prime}(z)\right)^{\frac{1}{2}}\right)^{2 b}\right) \\
& =\operatorname{Re}\left((p(z))^{b}(q(z))^{2 b}\right)
\end{aligned}
$$

where $\operatorname{Re}(q(z))>0$. Hence

$$
\operatorname{Re}\left(e^{i \alpha}\left(f^{\prime}(z)\right)^{b}\right)=\operatorname{Re}\left((p(z))^{b}(q(z))^{2 b}\right)>0
$$

because $p(z)$ and $q(z)$ are both such that $\operatorname{Re}(p(z))>0, \operatorname{Re}(q(z))>0$, and $|b|+|2 b| \leq 1$.

The lemma above implies that if $f(z)$ is in $L$ and $-1 / 3 \leq b \leq 1 / 3$, then $h(z ; \alpha, 0)=\int_{0}^{z}\left(f^{\prime}(t)\right)^{\alpha} d t \in L$. This fact was first proved in [4]. The proof presented here is simpler.

## Theorem 4.

1. Assume that

$$
A_{3}=\operatorname{conv}(\{(0,3),(0,-1),(1,0),(-1 / 3,0),(-1 / 3,2)\})
$$

and $f(z) \in S^{*}, g(z) \in K$. Then for each pair $(\alpha, \beta) \in A_{3}$ the operator $h(z ; \alpha, \beta)$ is in $L$.
2. Assume that

$$
\begin{aligned}
B_{3}=\{ & (x, y) \in \mathbb{R}^{2}:(y<-1) \vee(y>3) \vee(x<-1 / 2) \vee(y<-2 x-1) \\
& \vee(y>-2 x+3) \vee(y>4 x+4) \vee(y>-4 x+4) \vee(y<4 x-4)\} .
\end{aligned}
$$

Then for each pair $(\alpha, \beta) \in B_{3}$ there exist functions $f(z) \in S^{*}, g(z) \in K$ such that the function $h(z ; \alpha, \beta)=\int_{0}^{z}\left(f^{\prime}(t)\right)^{\alpha}(g(t))^{\beta} t^{-\beta} d t$ is not univalent in $\Delta$.

Proof. The set $A_{3}$ is a convex polygon. In view of Lemma 1 it is sufficient to establish close-to-convexity of $h(z ; \alpha, \beta)$ at the vertices of $A_{3}$. The points $(0,3),(0,-1)$ were considered in [4]. The corresponding functions $h(z ; \alpha, \beta)$ are in $L$.

For the vertex $(1,0)$ we have

$$
h(z ; 1,0)=\int_{0}^{z} f^{\prime}(t) d t=f(z) \in S^{*} \subset L .
$$

Univalence at the vertex $(-1 / 3,0)$ follows directly from Lemma 3. For the vertex $(-1 / 3,2)$ we have the relation

$$
h^{\prime}(z ;-1 / 3,2)=\left(f^{\prime}(z)\right)^{-\frac{1}{3}}(g(z))^{2} z^{-2} .
$$

From Lemma 3 we know that there exists $\theta \in(-\pi / 2, \pi / 2)$, such that $\operatorname{Re}\left(e^{i \theta}\left(f^{\prime}(z)\right)^{-\frac{1}{3}}\right)>0$ and $(g(z))^{2} z^{-2}$ is a derivative of a function in $K$. We have

$$
h^{\prime}(z ;-1 / 3,2)=e^{-i \theta}\left(e^{i \theta}\left(f^{\prime}(z)\right)^{-\frac{1}{3}}\right)(g(z))^{2} z^{-2} .
$$

Hence $h(z ;-1 / 3,2) \in L$. This ends the first part of the proof and gives a sufficient condition for univalence of $h(z ; \alpha, \beta)$.

We now prove the second part of the theorem.

1. Proceeding analogously to the proof of Theorem 1 we obtain that $h(z ; \alpha, \beta)$ is not univalent for $(\beta<-1) \vee(\beta>3) \vee(\alpha<-1 / 2)$ and these cases were handled in [4].
2. Cases $(\beta<-2 \alpha-1) \vee(\beta>-2 \alpha+3)$ are discussed in the proof of Theorem 1.
3. Setting $f(z)=z(1-z)^{-2}$ and $g(z)=z(1-z)^{-1}$ we have

$$
h^{\prime \prime}(z ; \alpha, \beta)=\alpha(1+z)^{\alpha-1}(1-z)^{-3 \alpha-\beta}+(3 \alpha+\beta)(1+z)^{\alpha}(1-z)^{-3 \alpha-\beta-1} .
$$

By the well-known Bieberbach theorem: if $h(z)=z+\sum_{n=2}^{\infty} a_{n} z^{n}$ is univalent in $\Delta$, then $\left|a_{2}\right| \leq 2$, which shows that

$$
\left|h^{\prime \prime}(0) / 2\right| \leq 2 \Leftrightarrow\left|h^{\prime \prime}(0)\right| \leq 4
$$

Let us observe that in our case we have $h^{\prime \prime}(0 ; \alpha, \beta)=4 \alpha+\beta$, hence, if $4 \alpha+\beta>4$ that is $\beta>4-4 \alpha$, then $\left|h^{\prime \prime}(0 ; \alpha, \beta)\right|>4$. In other words $h(z ; \alpha, \beta)=\int_{0}^{z}\left(\left(t(1-t)^{-2}\right)^{\prime}\right)^{\alpha}(1-t)^{-\beta} d t$ is not univalent in $\Delta$.
4. Setting $f(z)=z(1+z)^{-2}$ and $g(z)=z /(1-z)$ we have

$$
h^{\prime \prime}(z ; \alpha, \beta)=(\beta-\alpha)(1+z)^{-3 \alpha}(1-z)^{\alpha-\beta-1}-3 \alpha(1+z)^{-3 \alpha-1}(1-z)^{\alpha-\beta}
$$

Again, from the Bieberbach theorem it follows, that $\left|h^{\prime \prime}(0)\right| \leq 4$.
In our case we have $h^{\prime \prime}(0 ; \alpha, \beta)=-4 \alpha+\beta$, hence, if $-4 \alpha+\beta>4$ that is $\beta>4+4 \alpha$ or $-4 \alpha+\beta<-4$ that is $\beta<-4+4 \alpha$, then $\left|h^{\prime \prime}(0 ; \alpha, \beta)\right|>4$. In other words $h(z ; \alpha, \beta)=\int_{0}^{z}\left(\left(t(1+t)^{-2}\right)^{\prime}\right)^{\alpha}(1-t)^{-\beta} d t$ is not univalent in $\Delta$.

Cases 1, 2, 3, 4 complete this part of the proof and determine the set $B_{3}$. This ends the proof.

For $(\alpha, \beta) \in C_{3}=\mathbb{R}^{2} \backslash\left(A_{3} \cup B_{3}\right)=T_{5} \cup T_{6} \cup T_{7}$ the problem of univalence of $h(z ; \alpha, \beta)$ is open (see Figure 5).


Figure 5.

## Theorem 5.

1. Assume that

$$
A_{4}=\operatorname{conv}(\{(0,3 / 2),(0,-1 / 2),(1,0),(-1 / 3,0),(-1 / 3,1)\})
$$

and $f(z), g(z) \in S^{*}$. Then for each pair $(\alpha, \beta) \in A_{4}$ the operator $h(z ; \alpha, \beta)$ is in $L$.
2. Assume that

$$
\begin{gathered}
B_{4}=\left\{(x, y) \in \mathbb{R}^{2}:(y<-1 / 2) \vee(y>3 / 2) \vee(x<-1 / 2) \vee(y<-x-1 / 2)\right. \\
\vee(y>2 x+2) \vee(y>-x+3 / 2) \vee(y>-2 x+2) \vee(y<2 x-2)\} .
\end{gathered}
$$

Then for each pair $(\alpha, \beta) \in B_{4}$ there exist functions $f(z), g(z) \in S^{*}$ such that the function $h(z ; \alpha, \beta)=\int_{0}^{z}\left(f^{\prime}(t)\right)^{\alpha}(g(t))^{\beta} t^{-\beta} d t$ is not univalent in $\Delta$.

Proof. The set $A_{4}$ is a convex polygon. In view of Lemma 1 it is sufficient to establish close-to-convexity of $h(z ; \alpha, \beta)$ at the vertices of $A_{4}$. The points $(0,3 / 2),(0,-1 / 2)$ were considered in [4] and the corresponding functions $h(z ; \alpha, \beta)$ were shown to be in $L$. For the vertex $(1,0)$ we have

$$
h(z ; 1,0)=\int_{0}^{z} f^{\prime}(t) d t=f(z) \in S^{*} \subset L
$$

The vertex $(-1 / 3,0)$ we get directly from Lemma 3. For the vertex $(-1 / 3,1)$ we have

$$
h^{\prime}(z ;-1 / 3,1)=\left(f^{\prime}(z)\right)^{-\frac{1}{3}} g(z) z^{-1}
$$

and from Lemma 3 we know that there exists $\theta \in(-\pi / 2, \pi / 2)$, such that $\operatorname{Re}\left(e^{i \theta}\left(f^{\prime}(z)\right)^{-\frac{1}{3}}>0\right.$ and $g(z) \in S^{*}$. We have

$$
h^{\prime}(z ;-1 / 3,1)=e^{-i \theta}\left(e^{i \theta}\left(f^{\prime}(z)\right)^{-\frac{1}{3}}\right) g(z) z^{-1}
$$

Hence $h(z ;-1 / 3,1) \in L$. This ends the first part of the proof and gives a sufficient condition for univalence of $h(z ; \alpha, \beta)$.

We now prove the second part of the theorem.

1. Proceeding analogously to the proof of Theorem 1 we obtain that $h(z ; \alpha, \beta)$ is not univalent for $(\beta<-1 / 2) \vee(\beta>3 / 2) \vee(\alpha<-1 / 2)$ and these cases were handled in [4].
2. Cases $(\beta<-\alpha-1 / 2) \vee(\beta>-\alpha+3 / 2)$ were discussed in the proof of Theorem 3.
3. Setting $f(z)=z(1-z)^{-2}$ and $g(z)=z(1-z)^{-2}$ we have
$h^{\prime \prime}(z ; \alpha, \beta)=\alpha(1+z)^{\alpha-1}(1-z)^{-3 \alpha-2 \beta}+(3 \alpha+2 \beta)(1-z)^{-3 \alpha-2 \beta-1}(1+z)^{\alpha}$.
From the Bieberbach theorem: for univalence of $h(z)=z+\sum_{n=2}^{\infty} a_{n} z^{n}$ there must be $\left|h^{\prime \prime}(0)\right| \leq 4$.

But in our case we have $h^{\prime \prime}(0 ; \alpha, \beta)=4 \alpha+2 \beta$. Hence, if $4 \alpha+2 \beta>4$ that is $\beta>2-2 \alpha$, then $\left|h^{\prime \prime}(0 ; \alpha, \beta)\right|>4$. It means that $h(z ; \alpha, \beta)=$ $\int_{0}^{z}\left(\left(t(1-t)^{-2}\right)^{\prime}\right)^{\alpha}(1-t)^{-2 \beta} d t$ is not univalent in $\Delta$.
4. Setting $f(z)=z(1+z)^{-2}$ and $g(z)=z(1-z)^{-2}$ we have
$h^{\prime \prime}(z ; \alpha, \beta)=-3 \alpha(1+z)^{-3 \alpha-1}(1-z)^{\alpha-2 \beta}+(2 \beta-\alpha)(1-z)^{-2 \beta+\alpha-1}(1+z)^{-3 \alpha}$.

The condition $\left|h^{\prime \prime}(0)\right| \leq 4$ is satisfied. But in our case we have $h^{\prime \prime}(0 ; \alpha, \beta)=$ $-4 \alpha+2 \beta$. Hence, if $-4 \alpha+2 \beta>4$ that is $\beta>2+2 \alpha$ or $-4 \alpha+2 \beta<-4$ that is $\beta<-2+2 \alpha$, then $\left|h^{\prime \prime}(0 ; \alpha, \beta)\right|>4$, which means that $h(z ; \alpha, \beta)=$ $\int_{0}^{z}\left(\left(t(1+t)^{-2}\right)^{\prime}\right)^{\alpha}(1-t)^{-2 \beta} d t$ is not univalent in $\Delta$.

Cases 1, 2, 3 and 4 complete this part of proof and define the set $B_{4}$. This ends the proof.

For $(\alpha, \beta) \in C_{4}=\mathbb{R}^{2} \backslash\left(A_{4} \cup B_{4}\right)=T_{8} \cup T_{9} \cup T_{10}$ the problem of univalence of $h(z ; \alpha, \beta)$ is open (see Figure 6).


Figure 6.

Proofs of the remaining theorems are very similar to those presented above. So we have decided to omit them, but nevertheless we provide the corresponding geometric interpretations.

## Theorem 6.

1. Assume that

$$
A_{5}=\operatorname{conv}(\{(0,3),(0,-1),(1,0),(-1 / 3,0),(-1 / 3,2)\})
$$

and $f(z) \in L, g(z) \in K$, then for each pair $(\alpha, \beta) \in A_{5}$ the operator $h(z ; \alpha, \beta)$ is in $L$.
2. Assume that

$$
\begin{aligned}
B_{5}=\left\{(x, y) \in \mathbb{R}^{2}:\right. & (y<-1) \vee(y>3) \vee(x<-1 / 3) \vee(y<-3 x-1) \\
& \vee(y>-3 x+3) \vee(y>4 x+4) \vee(y<4 x-4)\}
\end{aligned}
$$

Then for each pair $(\alpha, \beta) \in B_{5}$ there exist functions $f(z) \in L, g(z) \in K$ such that the function $h(z ; \alpha, \beta)=\int_{0}^{z}\left(f^{\prime}(t)\right)^{\alpha}(g(t))^{\beta} t^{-\beta} d t$ is not univalent in $\Delta$.

For $(\alpha, \beta) \in C_{5}=\mathbb{R}^{2} \backslash\left(A_{5} \cup B_{5}\right)=T_{11} \cup T_{12}$ the problem of univalence of $h(z ; \alpha, \beta)$ is open (see Figure 7).


Figure 7.

## Theorem 7.

1. Assume that

$$
A_{6}=\operatorname{conv}(\{(0,3 / 2),(0,-1 / 2),(1,0),(-1 / 3,0),(-1 / 3,1)\})
$$

and $f(z) \in L, g(z) \in S^{*}$. Then for each pair $(\alpha, \beta) \in A_{6}$ the operator $h(z ; \alpha, \beta)$ is in $L$.
2. Assume that

$$
\begin{aligned}
B_{6}=\left\{(x, y) \in \mathbb{R}^{2}:\right. & (y<-1 / 2) \vee(y>3 / 2) \vee(x<-1 / 3) \\
& \vee(y<-3 / 2 x-1 / 2) \vee(y>-3 / 2 x+3 / 2) \\
& \vee(y>2 x+2) \vee(y<2 x-2)\} .
\end{aligned}
$$

Then for each pair $(\alpha, \beta) \in B_{6}$ there exist functions $f(z) \in L, g(z) \in S^{*}$ such that the function $h(z ; \alpha, \beta)=\int_{0}^{z}\left(f^{\prime}(t)\right)^{\alpha}(g(t))^{\beta} t^{-\beta} d t$ is not univalent in $\Delta$.

For $(\alpha, \beta) \in C_{6}=\mathbb{R}^{2} \backslash\left(A_{6} \cup B_{6}\right)=T_{13} \cup T_{14}$ the problem of univalence of $h(z ; \alpha, \beta)$ is open (see Figure 8).


Figure 8.

## Theorem 8.

1. If $f(z) \in K, g(z)$ is a function of positive real part in $\Delta,(\alpha, \beta) \in D$, where

$$
\begin{equation*}
D=\operatorname{conv}(\{(-1 / 2,0),(0,1),(3 / 2,0),(0,-1),(1,1),(1,-1)\}) \tag{4}
\end{equation*}
$$

then

$$
h(z ; \alpha, \beta)=\int_{0}^{z}\left(f^{\prime}(t)\right)^{\alpha}(g(t))^{\beta} d t \in L
$$

2. For $(\alpha, \beta) \notin D$, where set $D$ is given by (4) there exist functions $f(z) \in$ $K$ and $g(z)$ of positive real part in $\Delta$ such that the function $h(z ; \alpha, \beta)=$ $\int_{0}^{z}\left(f^{\prime}(t)\right)^{\alpha}(g(t))^{\beta} d t$ is not univalent in $\Delta$.


Figure 9.

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