doi: 10.2478/v10062-009-0008-8

ANNALES
UNIVERSITATIS MARIAE CURIE-SKŁODOWSKA
LUBLIN – POLONIA

VOL. LXIII, 2009 SECTIO A 83–90

KAZIMIERZ GOEBEL and EWA SĘDŁAK

Nonexpansive retractions in Hilbert spaces

Abstract. Let H be a Hilbert space and $C \subset H$ be closed and convex. The mapping $P : H \to C$ known as *the nearest point projection* is nonexpansive (1-lipschitzian). We observed that, the natural question: "Are there nonexpansive projections $Q : H \to C$ other than P?" is neglected in the literature. Also, the answer is not often present in the "folklore" of the Hilbert space theory. We provide here the answer and discuss some facts connected with the subject.

1. Preliminaries. Let $(H, \|\cdot\|)$ be a Hilbert space and let $E \subset H$ be a closed linear subspace. A well-known fact about E is that there exists a linear, orthogonal projection $P : H \to E$, $P^2 = P$, having the following properties:

- P is of norm one, $\|P\| = 1$.
- The complement $Q = I - P$ is the orthogonal projection onto the orthogonal complement $E^{\perp}$ of E, $Q^2 = Q$, $\|Q\| = 1$.
- P is the nearest point projection which means that for any $x \in H$

$$\|x - Px\| = \min\left[\|x - y\| : y \in E\right].$$

- The reflection with respect to E, $S = 2P - I$, is of norm one, $\|2P - I\| = 1$.

2000 *Mathematics Subject Classification.* 47H10.

Key words and phrases. Hilbert space, convex sets, retractions, nonexpansive mappings.

And *the most important*!

- P is the unique, linear or nonlinear, projection (retraction) on E of norm one.

Similar facts are known when the subspace E is replaced with an arbitrary closed and convex subset $C \subset H$. However in this case, instead of studying linear projections of norm one, we talk about *nonexpansive retractions*.

Definition 1. A mapping $P : H \to C$ is said to be a nonexpansive retraction of H onto C if for all $x \in C$, $Px = x$ and for all x, $y \in H$ we have

$$\|Px - Py\| \leq \|x - y\|.$$

For any nonempty, closed and convex $C \subset H$ there exists at least one such retraction. Indeed, the mapping $P_C : H \to C$ moving each point $x \in H$ to the point $P_C x \in C$ closest to x,

$$\|x - P_C x\| = \min \left[\|x - y\| : y \in C\right]$$

has the following properties:

- P_C is nonexpansive, $\|P_C x - P_C y\| \leq \|x - y\|$ for all $x, y \in C$,
- The complement of P_C, $I - P_C$, is nonexpansive,
- The reflection with respect to P_C, $S_C = 2P_C - I$ is nonexpansive.

The mapping P_C is usually called *nearest point projection* or *metric projection* or *proximity mapping* (*projection*) of H onto C. Proofs of the above facts can be found in many papers and standard books (see e.g. [1]). However the basic question:

Is P_C the unique nonexpansive retraction of H onto C?

is largely ignored.

In general the answer is negative, but examples confirming this fact are hardly found in the literature. The aim of this note is to fill this gap and present ways to construct nonexpansive retractions other than nearest point projections. To illustrate basic facts we provide the reader with some examples in the two dimensional Euclidean plane.

2. Some ways to produce nonexpansive retractions. Fix a closed convex subset C of H and let $\mathcal{R}$ denote the family of all nonexpansive retractions of H onto C. Thus, $\mathcal{R}$ is nonempty and for any two $P_1, P_2 \in \mathcal{R}$ and any $\alpha \in [0,1]$ the averaged mapping $\alpha P_1 + (1-\alpha) P_2$ is also a member of $\mathcal{R}$. So, the family $\mathcal{R}$ is convex. Moreover, since each $P \in \mathcal{R}$ is nonexpansive, $\mathcal{R}$ is closed with respect to pointwise convergence.

Our first observation is the following. Let the set $D \supset C$ be closed and convex. Then the composition $Q = P_C \circ P_D$ is a nonexpansive retraction of H onto C; thus $Q \in \mathcal{R}$. More generally, if sets $D_1, D_2, \ldots, D_n$ are closed, convex and such that $D_1 \supset D_2 \supset \ldots \supset D_n \supset C$, then

$$Q = P_C \circ P_{D_n} \circ P_{D_{n-1}} \circ \ldots \circ P_{D_2} \circ P_{D_1}$$

is a member of $\mathcal{R}$. The picture below illustrates this construction.

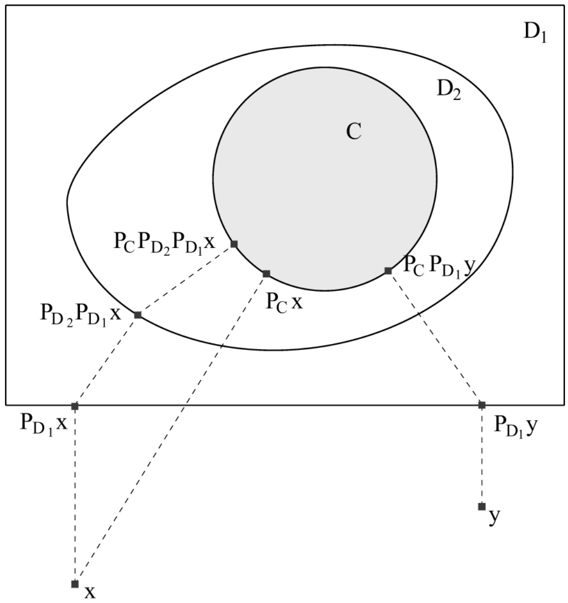

FIGURE 1.

Our second construction is based on the fact that the reflection $S_C = 2P_C - I$ is nonexpansive and equal to the identity I on C. In view of this we can construct the sequence of retractions $Q_n \in \mathcal{R}$, $n = 1, 2, \ldots$ as

$$Q_n = P_C \circ S_C^n = P_C \circ (2P_C - I)^n.$$

Here are some illustrations.

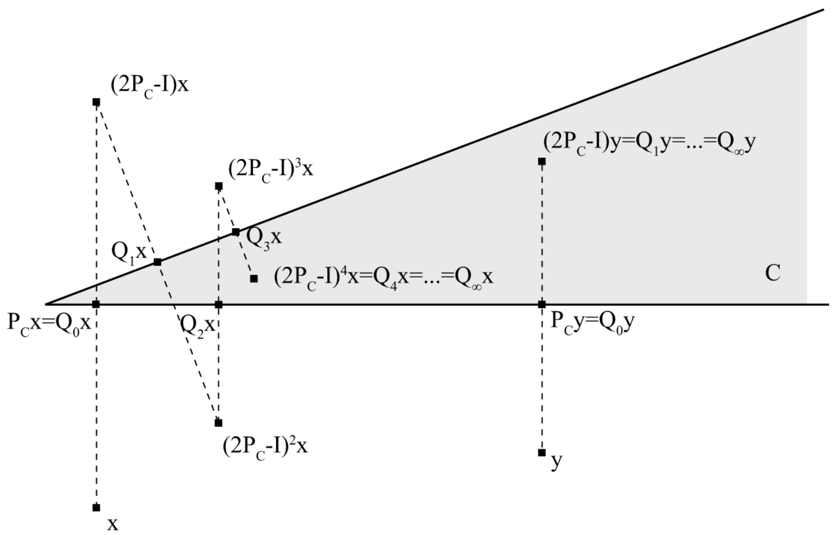

FIGURE 2.

Now, one can produce more retractions using the fact that $\mathcal{R}$ is closed and convex, for example by taking

$$R = \frac{1}{2} P_C \circ P_D + \frac{1}{2} P_C \circ (2P_C - I)$$

(see Figure 3), or when it exists $Q_\infty = \lim_{n\to\infty} Q_n$. One can also mix both observations taking various convex combinations of the retractions constructed above, and passing to the pointwise limit if it exists.

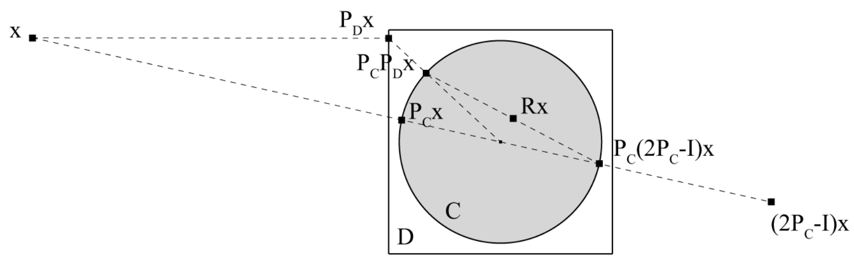

FIGURE 3.

3. Ranges and inverse ranges of points under $\mathcal{R}$. It now seems natural to ask the following questions. For a given $x \in H \setminus C$, which points y of C are the image of x under a nonexpansive retraction of H onto C? For a given point $y \in C$, which points of $H \setminus C$ can be mapped into y by at least one nonexpansive retraction of H onto C? More precisely, let us define *the range of $x \in H \setminus C$ under $\mathcal{R}$* as:

$$\mathcal{R}x = [y \in C : y = Rx \text{ for at least one } R \in \mathcal{R}]\,,$$

and *the inverse range of y under $\mathcal{R}$* as:

$$\mathcal{R}^{-1}y = [x \in H \setminus C : Rx = y \text{ for at least one } R \in \mathcal{R}]\,.$$

Our questions reduce to: What do the above sets look like? For given $x \in H \setminus C$, what criterion determines that $y \in \mathcal{R}x$? For given $y \in C$ what criterion determines that $x \in \mathcal{R}^{-1}y$? The complete answer is given by two facts:

Claim 1. *For $x \in H \setminus C$, $y \in \mathcal{R}x$ if and only if, for any $z \in C$, $\|z - y\| \leq \|z - x\|$.*

Geometrically this means that the hyperplane containing the point $\frac{1}{2}(x + y)$ and orthogonal to the vector $y - x$ leaves the whole set C in the closed half space to which x does not belong. The proof of the "only if" part is obvious. For any $z \in C$, $z = Rz$ for all $R \in \mathcal{R}$. Thus if $y = Rx$, then for all $z \in C$ we have

$$\|z - y\| = \|Rz - Rx\| \leq \|z - x\|\,.$$

The "if" part, can be proved in two ways, abstract and constructive. The first proof uses the fact which distinguish Hilbert spaces from other Banach spaces and deals with an extension property ([2], [3], for the proof see e.g. [1]).

Theorem 1 (Kirzbraun–Valentine [2]). *Let $A \subset H$ be an arbitrary set $A \neq \emptyset$ and let $T : A \to H$ be nonexpansive. Then there exists a nonexpansive extension $\widetilde{T}$ of T, ($\widetilde{T} = T$ on A), such that $T(H) \subset \overline{\text{conv}} T(A)$.*

Now, to prove the "if" part the first way, suppose that $x \notin C$ and $y \in C$ is such that $\|z - y\| \leq \|z - x\|$ for all $z \in C$. Consider the set $A = C \cup \{x\}$ and the mapping $T : A \to C$ defined by

$$Tz = \begin{cases} z & \text{if } z \in C, \\ y & \text{if } z = x. \end{cases}$$

The mapping T is obviously nonexpansive on A and any extension $\widetilde{T}$ of T satisfying Kirzbraun–Valentine condition is a member of $\mathcal{R}$.

To get a constructive proof with the same setting consider the hyperplane $V \subset H$ containing the point $\frac{1}{2}(x + y)$ and orthogonal to $x - y$. Let D be the one of two closed half spaces of H generated by V, the one which contains C. Now for any $P \in \mathcal{R}$ the mapping $P \circ (2P_D - I)$ is also a member of $\mathcal{R}$ and sends x into y. In particular the mapping $R = P_C \circ (2P_D - I)$ represents the nonexpansive retraction on C satisfying $Rx = y$.

In the same way we can prove:

Claim 2. *For any $y \in C$, $x \in \mathcal{R}^{-1}y$ if and only if, for any $z \in C$, $\|z - y\| \leq \|z - x\|$.*

The above claims justify the following procedure to describe the sets $\mathcal{R}x$ and $\mathcal{R}^{-1}y$. Suppose for now that C is bounded. Consider the family $\mathcal{E}$ of all hyperplanes supporting C. Let $x \in H \setminus C$ be fixed. For any $E \in \mathcal{E}$ let P_E and $S_E = 2P_E - I$ be the orthogonal projection on E and the reflection with respect to E. We leave it to the reader to justify that for any E the intersection of the segment $[P_E x, S_E x]$ with C is contained in $\mathcal{R}x$ and

$$\mathcal{R}x = \bigcup_{E \in \mathcal{E}} (C \cap [P_E x, S_E x]) = C \cap \left(\bigcup_{E \in \mathcal{E}} [P_E x, S_E x] \right). \tag{1}$$

If C is unbounded the situation is similar but we have to enlarge the family $\mathcal{E}$ taking into account also hyperplanes E (if they exist) which do not support C at any point, but which nevertheless leave the set C in one of the two half spaces defined by E and such that

$$\inf [\|z - P_E z\| : z \in C] = 0$$

(*asymptotically supporting hyperplanes*).

A characterization of $\mathcal{R}^{-1}y$ for $y \in C$ similar to (1) reads as follows. For any $E \in \mathcal{E}$ (or E asymptotically supporting C) consider points y, $P_E y$, $S_E y$. Either all three differ or all three are equal. If $y \neq P_E y \neq S_E y$, then the half line

$$[S_E y, \infty] = [z = y + t\,(S_E y - y) : t \geq 1]$$

is contained in $\mathcal{R}^{-1}y$.

The other case is the equality $y = P_E y = S_E y$. This means that the hyperplane E supports C at y. In this case we also get one or two half lines contained in $\mathcal{R}^{-1}y$. Let E_0 be the subspace of H of codimension one, parallel to E, $E_0 = E - y$. Let u be a normal vector to E_0. Then at least one of the half lines $[z = y + tu : t \geq 0]$ or $[z = y - tu : t \geq 0]$ is contained in $\mathcal{R}^{-1}y$. The whole line $[z = y + tu : -\infty < t < +\infty]$ is contained in $\mathcal{R}^{-1}y$ if and only if the whole set C is contained in E, $C \subset E$.

The whole set $\mathcal{R}^{-1}y$ is the union of all half lines defined for y in both the ways described above.

4. Two dimensional examples. In the geometry of plane curves the above construction is known. Going "around the set" C with the supporting (tangent) line (and asymptotics, if they exist) and taking the symmetric point to the given point x we get the curve called the *orthotomic* of x with respect to C. Below, we present a number of figures illustrating these notions. The shaded areas are the ranges and inverse ranges of the given points, the orthotomics are marked by dotted lines, and segments and half lines appearing in the asymptotic cases are drawn with thick lines. Other elements of our pictures are left to the reader to interpret.

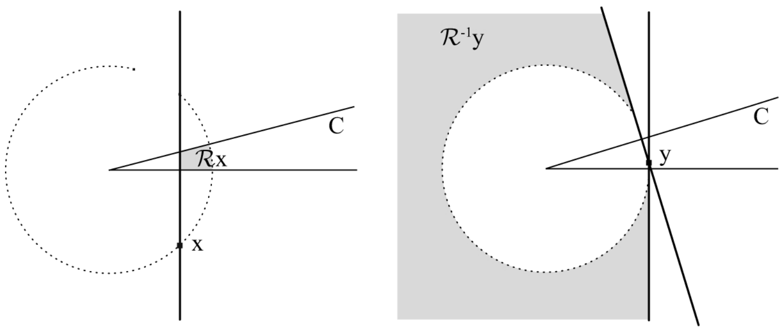

Figure 4. Angle.

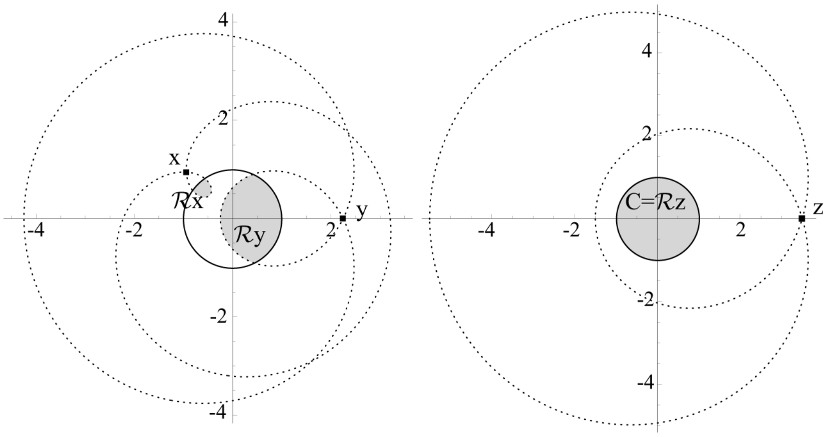

FIGURE 5. Disk – ranges.

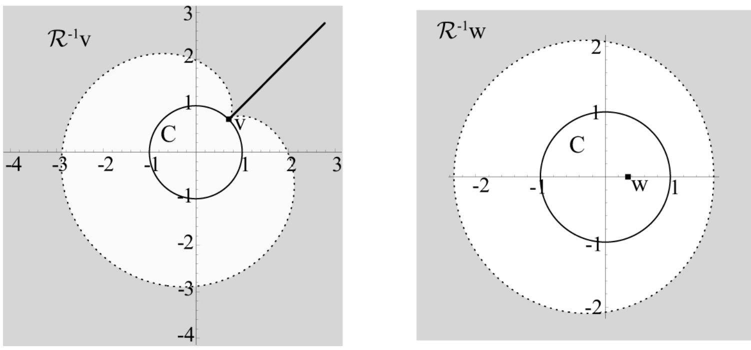

FIGURE 6. Disk – inverse ranges.

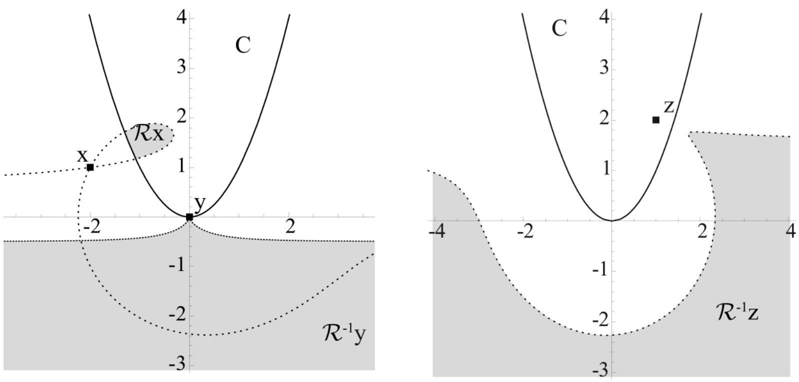

FIGURE 7. Parabola.

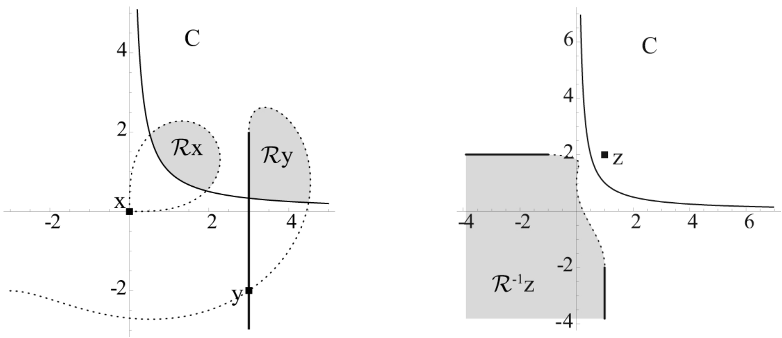

FIGURE 8. Hyperbola.

References

[1] Goebel, K., Kirk, W. A., *Topics in Metric Fixed Point Theory*, Cambridge Univ. Press, Cambridge, 1990.
[2] Kirzbraun, M. D., *Über die Zussamenziehende und Lipschistsche Transformationen*, Fund. Math. **22** (1934), 77–108.
[3] Valentine, F. A., *A Lipschitz condition preserving extension for a vectpr function*, Amer. J. Math. **67** (1945), 83–93.

Kazimierz Goebel
Institute of Mathematics
Maria Curie-Sklodowska University
pl. M. Curie-Sklodowskiej 1
20-031 Lublin
Poland
e-mail: `goebel@hektor.umcs.lublin.pl`

Ewa Sędłak
Institute of Mathematics
Maria Curie-Sklodowska University
pl. M. Curie-Sklodowskiej 1
20-031 Lublin
Poland
e-mail: `esedlak@hektor.umcs.lublin.pl`

Received February 2, 2009