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The almost sure central limit theorems for certain order statistics of some stationary Gaussian sequences

Abstract. Suppose that $X_1, X_2, \ldots$ is some stationary zero mean Gaussian sequence with unit variance. Let $\{k_n\}$ be a certain nondecreasing sequence of positive integers, $M_n^{(k_n)}$ denote the k_nth largest maximum of $X_1, \ldots, X_n$. We aim at proving the almost sure central limit theorems for the suitably normalized sequence $\left\{M_n^{(k_n)}\right\}$ under certain additional assumptions on $\{k_n\}$ and the covariance function $r(t) := Cov\,(X_1, X_{1+t})$.

1. Introduction. The almost sure central limit theorem (ASCLT) has become an intensively studied subject in recent time. In the research concerning the ASCLT the following property is investigated. Let $X_1, X_2, \ldots$ be some r.v.'s, $f_1, f_2, \ldots, f_k, \ldots$ denote some real-valued measurable functions, defined on $\mathbb{R}, \mathbb{R}^2, \ldots, \mathbb{R}^k, \ldots$, respectively. We seek conditions under which, for some nondegenerate d.f. G,

$$\lim_{N\to\infty} \frac{1}{D_N} \sum_{n=1}^{N} d_n I(f_n(X_1, \ldots, X_n) \le x) = G(x) \quad \text{a.s.}$$

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for all $x \in C_G$, where: $\{d_n\}$ is some sequence of weights, $D_N = \sum_{n=1}^{N} d_n$, I stands for the indicator function, and C_G denotes the set of continuity points of G.

In our investigations, we will restrict ourselves to the case, when the relation above holds with: $d_n = 1/n$, $D_N \sim \log N$, $f_n(X_1, \ldots, X_n) = a_n\left(M_n^{(k_n)} - b_n\right)$, where: $\{k_n\}$ is a certain nondecreasing sequence of positive integers, $M_n^{(k_n)}$ denotes the k_nth largest maximum of $X_1, \ldots, X_n$, and $a_n > 0$, b_n are certain normalizing constants.

Let Φ be the standard normal d.f. The purpose of this paper is to prove that if $X_1, X_2, \ldots$ is a standardized stationary Gaussian sequence, then, under some assumptions on the numerical sequences $\{k_n\}$, $\{u_n\}$ and the covariance function $r(t) := Cov(X_1, X_{1+t})$, we have for some τ, $0 < \tau < \infty$,

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left(M_n^{(k_n)} \le u_n\right) = \Phi(\tau) \quad \text{a.s.} \tag{1}$$

As a direct consequence, we will also show that if:

$$\begin{aligned} a_n &= \left(\frac{2\log(n/k_n)}{k_n}\right)^{1/2}, \\ b_n &= (2\log(n/k_n))^{1/2} - \frac{\log\log(n/k_n) + \log 4\pi}{2(2\log(n/k_n))^{1/2}}, \end{aligned} \tag{2}$$

then the following strong convergence occurs for all $x \in \mathbb{R}$

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left(a_n\left(M_n^{(k_n)} - b_n\right) \le x\right) = \Phi(x) \quad \text{a.s.} \tag{3}$$

In the case, when $k_n \equiv k$, where k is a fixed positive integer, we will show that, under certain conditions on $\{u_n\}$ and $r(t)$,

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left(M_n^{(k)} \le u_n\right) = e^{-\tau} \sum_{s=0}^{k-1} \frac{\tau^s}{s!} \quad \text{a.s.} \tag{4}$$

for some τ, $0 < \tau < \infty$.

As a direct consequence, we will also show that if:

$$a_n = (2\log n)^{1/2}, \; b_n = (2\log n)^{1/2} - \frac{\log\log n + \log 4\pi}{2(2\log n)^{1/2}}, \tag{5}$$

then the following strong convergence occurs for all $x \in \mathbb{R}$

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left(a_n\left(M_n^{(k)} - b_n\right) \le x\right) = \exp\left(-e^{-x}\right) \sum_{s=0}^{k-1} \frac{(e^{-x})^s}{s!} \quad \text{a.s.} \tag{6}$$

We should mention here that, in the case of i.i.d. r.v.'s, the ASCLT for the order statistics $M_n^{(k_n)}$ has been proved in Stadtmueller [4], under some extra assumptions on $\{k_n\}$. It is also worthwhile to mention that the ASCLT for the ordinary maxima $M_n = M_n^{(1)} := \max(X_1, \ldots, X_n)$ of some dependent stationary Gaussian sequences has been proved in Csaki and Gonchigdanzan [1] and Dudziński [2].

The following notations will be used throughout the paper: $M_n^{(k_n)}$ – the k_nth largest maximum of $X_1, \ldots, X_n$; $M_{m,n}^{(k_n)}$ – the k_nth largest maximum among $X_{m+1}, \ldots, X_n$; $r(t) := Cov(X_1, X_{1+t})$; Φ – the standard normal d.f.; $\#A$ – the cardinality of the set A; $|x|$ – an absolute value of x; $[x]$ – the greatest integer less than or equal to x. Furthermore, $f(n) \ll g(n)$ and $f(n) \sim g(n)$ will stand for $f(n) = \mathcal{O}(g(n))$ and $f(n)/g(n) \to 1$, as $n \to \infty$, respectively.

2. Main results. Our main results are the ASCLTs for certain order statistics of some stationary Gaussian sequences. The first one can be formulated as follows.

Theorem 1. *Let $X_1, X_2, \ldots$ be a stationary zero mean Gaussian sequence with unit variance and $\{k_n\}$ denote a nondecreasing sequence of positive integers, which satisfies:*

$$k_n \to \infty \quad \text{as } n \to \infty, \tag{7}$$

$$\log k_n \ll (\log n)^{1-\alpha} \quad \text{for some } \alpha > 0, \tag{8}$$

(9) *there exists a number $\beta > 1$, such that the sequence $\left\{(\log n)/k_n^{\beta}\right\}$ is nondecreasing for all sufficiently large n.*

Assume in addition that the covariance function $r(t) := Cov(X_1, X_{1+t})$ fulfils the following condition

$$\sum_{t=\left[n^{1/k_n^{\beta}}\right]}^{\infty} |r(t)| \ll \frac{1}{n^{k_n - 1 - 1/k_n^{\beta-1} + 1/k_n^{\beta}}} \quad \text{for some } \beta \text{ satisfying (9)}. \tag{10}$$

Then:

(i) *if the numerical sequence $\{u_n\}$ satisfies:*

$$n(1 - \Phi(u_n))\Phi(u_n) \to \infty \quad \text{as } n \to \infty, \tag{11}$$

$$\frac{k_n - n(1 - \Phi(u_n))}{\{n(1 - \Phi(u_n))\Phi(u_n)\}^{1/2}} \to \tau \quad \text{for some } \tau,\ 0 < \tau < \infty,\ \text{as } n \to \infty, \tag{12}$$

then relation (1) *holds,*

(ii) *if the sequences $\{a_n\}$, $\{b_n\}$ are such as in* (2), *then relation* (3) *holds for all $x \in \mathbb{R}$.*

Our next main result is the ASCLT for the kth largest maxima. Here it is.

Theorem 2. *Let $X_1, X_2, \dots$ be a stationary zero mean Gaussian sequence with unit variance and k denote a fixed positive integer. Assume moreover that the covariance function $r(t) := Cov\,(X_1, X_{1+t})$ fulfils the following condition*

$$\sum_{t=\left[n^{1/k^{\beta}}\right]}^{\infty} |r\,(t)| \ll \frac{1}{n^{k-1-1/k^{\beta-1}+1/k^{\beta}}} \quad \text{for some } \beta > 1. \tag{13}$$

Then:

(i) *if the numerical sequence $\{u_n\}$ satisfies*

$$n\,(1 - \Phi\,(u_n)) \to \tau \quad \text{for some } \tau,\ 0 < \tau < \infty,\ \text{ as } n \to \infty, \tag{14}$$

then relation (4) *holds,*

(ii) *if the sequences $\{a_n\}$, $\{b_n\}$ are such as in* (5)*, then relation* (6) *holds for all $x \in \mathbb{R}$.*

3. Auxiliary results. In this section, we state and prove three lemmas, which will be used in the proofs of Theorems 1, 2.

Lemma 1. *Under the assumptions of Theorem 1, we have that if m, n satisfy $m \le n/k_n - 1$, then*

$$E\left|I\left(M_n^{(k_n)} \le u_n\right) - I\left(M_{m,n}^{(k_n)} \le u_n\right)\right| \ll \frac{1}{n^{\gamma}} + \frac{k_n m}{n - k_n} \tag{15}$$

for some $\gamma > 0$.

Proof. We have

$$\begin{aligned} &E\left|I\left(M_n^{(k_n)} \le u_n\right) - I\left(M_{m,n}^{(k_n)} \le u_n\right)\right| \\ &\qquad = \left|P\left(M_n^{(k_n)} \le u_n\right) - P\left(M_{m,n}^{(k_n)} \le u_n\right)\right|. \end{aligned} \tag{16}$$

It is clear that $P\Big(M_n^{(k_n)} \le u_n\Big) = P(\text{at most } k_n{-}1 \text{ of } X_1, \dots, X_n \text{ exceed } u_n)$. Similarly, $P\Big(M_{m,n}^{(k_n)} \le u_n\Big) = P(\text{at most } k_n{-}1 \text{ of } X_{m+1}, \dots, X_n \text{ exceed } u_n)$.

For the given m, n, we put:

$$\sum_{(A_1,A_2)} := \sum_{\substack{(A_1,A_2):\\ A_1\cup A_2=\{1,\dots,n\},\ \#A_1\le k_n-1,\ A_2=\{1,\dots,n\}\setminus A_1}}, \tag{17}$$

$$\sum_{(B_1,B_2)} := \sum_{\substack{(B_1,B_2):\\ B_1\cup B_2=\{m+1,\dots,n\},\ \#B_1\le k_n-1,\ B_2=\{m+1,\dots,n\}\setminus B_1}}. \tag{18}$$

Hence, by applying (16) and the notations in (17), (18),

$$\begin{aligned}
&E\left|I\left(M_n^{(k_n)} \le u_n\right) - I\left(M_{m,n}^{(k_n)} \le u_n\right)\right| \\
&= \left| \sum_{(A_1,A_2)} P\left(\bigcap_{a_p\in A_1} \{X_{a_p} > u_n\} \cap \bigcap_{a_s\in A_2} \{X_{a_s} \le u_n\} \right) \right. \\
&\qquad \left. - \sum_{(B_1,B_2)} P\left(\bigcap_{b_p\in B_1} \{X_{b_p} > u_n\} \cap \bigcap_{b_s\in B_2} \{X_{b_s} \le u_n\} \right) \right|.
\end{aligned}$$

Thus, we can write that

$$\begin{aligned}
&E\left|I\left(M_n^{(k_n)} \le u_n\right) - I\left(M_{m,n}^{(k_n)} \le u_n\right)\right| \\
&\le \sum_{(A_1,A_2)} \left| P\left(\bigcap_{a_p\in A_1} \{X_{a_p} > u_n\} \cap \bigcap_{a_s\in A_2} \{X_{a_s} \le u_n\} \right) \right. \\
&\qquad \left. - \prod_{a_p\in A_1} P\left(X_{a_p} > u_n\right) \prod_{a_s\in A_2} P\left(X_{a_s} \le u_n\right) \right| \\
&\quad + \sum_{(B_1,B_2)} \left| P\left(\bigcap_{b_p\in B_1} \{X_{b_p} > u_n\} \cap \bigcap_{b_s\in B_2} \{X_{b_s} \le u_n\} \right) \right. \\
&\qquad \left. - \prod_{b_p\in B_1} P\left(X_{b_p} > u_n\right) \prod_{b_s\in B_2} P\left(X_{b_s} \le u_n\right) \right| \\
&\quad + \left\{ \sum_{(B_1,B_2)} \prod_{b_p\in B_1} P\left(X_{b_p} > u_n\right) \prod_{b_s\in B_2} P\left(X_{b_s} \le u_n\right) \right. \\
&\qquad \left. - \sum_{(A_1,A_2)} \prod_{a_p\in A_1} P\left(X_{a_p} > u_n\right) \prod_{a_s\in A_2} P\left(X_{a_s} \le u_n\right) \right\} \\
&=: D_1 + D_2 + D_3.
\end{aligned} \tag{19}$$

By (10) and the fact that $k_n - 1 - 1/k_n^{\beta-1} + 1/k_n^{\beta} \ge 0$ for any $n \ge 1$, we obtain

$$\sum_{t=1}^{\infty} |r(t)| < \infty. \tag{20}$$

It follows from (20) and (7) that there exist positive numbers δ, γ, n_0, such that:

$$\sup_{t\geq 1}|r(t)| = \delta < 1, \tag{21}$$

$$1/k_n^{\beta-1} - 1/k_n^{\beta} < 2/(1+\delta) - 1 - 2\gamma \tag{22}$$

for all $n > n_0$, if β fulfils (9), (10). Let $c(n)$ denote the largest integer, such that

$$c(n)\left[n^{1/k_n^{\beta}}\right] + 1 < n. \tag{23}$$

Thus, we can divide the sequence $X_1, \ldots, X_n$ into the following $c(n)+1$ blocks:

$$\left(X_1, \ldots, X_{\left[n^{1/k_n^{\beta}}\right]}\right), \left(X_{\left[n^{1/k_n^{\beta}}\right]+1}, \ldots, X_{2\left[n^{1/k_n^{\beta}}\right]}\right), \ldots,$$
$$\left(X_{c(n)\left[n^{1/k_n^{\beta}}\right]+1}, \ldots, X_n\right).$$

Since $X_1, \ldots, X_n$ is a standard normal sequence and (21), (23) hold, then, by applying Theorem 4.2.1 in Leadbetter et al. [3] (the so-called Normal Comparison Lemma), and by using the previously described division of $\{X_1, \ldots, X_n\}$, as well as the definitions of D_1 in (19) and $\sum_{(A_1,A_2)}$ in (17), we have

$$\begin{aligned}
D_1 \ll \; & \\
\ll \Bigg\{ & C_1(n) \sum_{d=0}^{c(n)-2} \sum_{i=d\left[n^{1/k_n^{\beta}}\right]+1}^{(d+1)\left[n^{1/k_n^{\beta}}\right]} \sum_{j=i+1}^{(d+2)\left[n^{1/k_n^{\beta}}\right]} |r(j-i)| \exp\left(-\frac{u_n^2}{1+\delta}\right) \\
& + C_1(n) \sum_{i=(c(n)-1)\left[n^{1/k_n^{\beta}}\right]+1}^{c(n)\left[n^{1/k_n^{\beta}}\right]} \sum_{j=i+1}^{n} |r(j-i)| \exp\left(-\frac{u_n^2}{1+\delta}\right) \\
& + C_1(n) \sum_{i=c(n)\left[n^{1/k_n^{\beta}}\right]+1}^{n-1} \sum_{j=i+1}^{n} |r(j-i)| \exp\left(-\frac{u_n^2}{1+\delta}\right) \Bigg\} \\
+ \Bigg\{ & C_2(n) \sum_{d=0}^{c(n)-2} \sum_{i=d\left[n^{1/k_n^{\beta}}\right]+1}^{(d+1)\left[n^{1/k_n^{\beta}}\right]} \sum_{j=(d+2)\left[n^{1/k_n^{\beta}}\right]+1}^{n} |r(j-i)| \exp\left(-\frac{u_n^2}{1+\delta}\right) \Bigg\},
\end{aligned} \tag{24}$$

where $C_1(n)$, $C_2(n)$ satisfy:

$$C_1(n) = \sum_{l=0}^{k_n-1} \binom{\left[n^{1/k_n^\beta}\right]}{l} + \sum_{l=0}^{k_n-1} \binom{\left[n^{1/k_n^\beta}\right]}{l} \binom{\left[n^{1/k_n^\beta}\right]}{k_n-1-l}, \tag{25}$$

$$C_2(n) = \sum_{l=0}^{k_n-1} \binom{\left[n^{1/k_n^\beta}\right]}{l} \binom{n-2\left[n^{1/k_n^\beta}\right]}{k_n-1-l}. \tag{26}$$

Due to the derivation in (24), we have

$$\begin{aligned} D_1 \ll (c(n)+1)\, C_1(n) \left[n^{1/k_n^\beta}\right] &\sum_{t=1}^{2\left[n^{1/k_n^\beta}\right]-1} |r(t)| \exp\left(-\frac{u_n^2}{1+\delta}\right) \\ + (c(n)-1)\, C_2(n) \left[n^{1/k_n^\beta}\right] &\sum_{t=\left[n^{1/k_n^\beta}\right]+1}^{n-1} |r(t)| \exp\left(-\frac{u_n^2}{1+\delta}\right). \end{aligned} \tag{27}$$

Notice that, by the definition of $c(n)$ in (23),

$$(c(n)+1) \left[n^{1/k_n^\beta}\right] \ll n. \tag{28}$$

Consequently, due to (27), (28),

$$\begin{aligned} D_1 \ll nC_1(n) &\sum_{t=1}^{2\left[n^{1/k_n^\beta}\right]-1} |r(t)| \exp\left(-\frac{u_n^2}{1+\delta}\right) \\ + nC_2(n) &\sum_{t=\left[n^{1/k_n^\beta}\right]+1}^{n-1} |r(t)| \exp\left(-\frac{u_n^2}{1+\delta}\right). \end{aligned} \tag{29}$$

In addition, it follows from (25), (26) that:

$$C_1(n) \ll k_n \left(n^{1/k_n^\beta}\right)^{k_n-1} = k_n n^{1/k_n^{\beta-1}-1/k_n^\beta}, \tag{30}$$

$$C_2(n) \ll k_n n^{k_n-1}. \tag{31}$$

Relations (30), (31) together with derivation (29) imply

$$\begin{aligned} D_1 \ll k_n n^{1+1/k_n^{\beta-1}-1/k_n^\beta} &\sum_{t=1}^{2\left[n^{1/k_n^\beta}\right]-1} |r(t)| \exp\left(-\frac{u_n^2}{1+\delta}\right) \\ + k_n n^{k_n} &\sum_{t=\left[n^{1/k_n^\beta}\right]+1}^{n-1} |r(t)| \exp\left(-\frac{u_n^2}{1+\delta}\right). \end{aligned} \tag{32}$$

Recall that, by (10) and (20) (see the reasoning above (20)),

$$\sum_{t=\left[n^{1/k_n^{\beta}}\right]+1}^{n-1} |r(t)| \ll \frac{1}{n^{k_n-1-1/k_n^{\beta-1}+1/k_n^{\beta}}} \quad \text{and} \quad \sum_{t=1}^{\infty} |r(t)| < \infty. \tag{33}$$

The relations in (32), (33) yield

$$D_1 \ll k_n n^{1+1/k_n^{\beta-1}-1/k_n^{\beta}} \exp\left(-\frac{u_n^2}{1+\delta}\right). \tag{34}$$

Since

$$1-\Phi\left(u_n\right) \sim \frac{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{u_n^2}{2}\right)}{u_n}$$

and, by (11), (12), $k_n \sim n\left(1-\Phi\left(u_n\right)\right)$, we get $u_n \sim \left(2\log\left(n/k_n\right)\right)^{1/2}$ and

$$\exp\left(-\frac{u_n^2}{2}\right) \sim \sqrt{2\pi}\frac{k_n}{n}\left(2\log\left(n/k_n\right)\right)^{1/2}.$$

Hence

$$\exp\left(-\frac{u_n^2}{1+\delta}\right) \ll \frac{\left(k_n\right)^{2/(1+\delta)}}{n^{2/(1+\delta)}}\left(\log\left(n/k_n\right)\right)^{1/(1+\delta)}. \tag{35}$$

From (34), (35), we get

$$D_1 \ll \frac{\left(k_n\right)^{1+2/(1+\delta)}\left(\log\left(n/k_n\right)\right)^{1/(1+\delta)}}{n^{2/(1+\delta)-1-1/k_n^{\beta-1}+1/k_n^{\beta}}}.$$

Furthermore, notice that $\log\left(n/k_n\right) \leq \log n$ and, by (8), $k_n \ll n^{\epsilon}$ for any $\epsilon > 0$. Therefore

$$D_1 \ll \frac{\left(n^{\epsilon}\right)^{1+2/(1+\delta)}\left(\log n\right)^{1/(1+\delta)}}{n^{2/(1+\delta)-1-1/k_n^{\beta-1}+1/k_n^{\beta}}} \quad \text{for any } \epsilon > 0. \tag{36}$$

Since in addition, (22) holds, we have $2/\left(1+\delta\right)-1-1/k_n^{\beta-1}+1/k_n^{\beta} > 2\gamma$ for any $n > n_0$ and some $\gamma > 0$. As ϵ in (36) may be arbitrary positive number, we can choose ϵ satisfying the relation $\epsilon\left(1+2/\left(1+\delta\right)\right) < \gamma$. Then

$$\left(n^{\epsilon}\right)^{1+2/(1+\delta)}\left(\log n\right)^{1/(1+\delta)} \ll n^{\gamma},$$

and we can write that

$$D_1 \ll \frac{n^{\gamma}}{n^{2\gamma}} = 1/n^{\gamma} \quad \text{for some } \gamma > 0. \tag{37}$$

In order to estimate the component D_2 in (19), it is sufficient to apply identical methods to those used in the estimation of D_1. Therefore, we obtain that

$$D_2 \ll 1/n^{\gamma} \quad \text{for some } \gamma > 0. \tag{38}$$

Thus, it remains to bound the term D_3 in (19). Let $\tilde{X}_1, \dots, \tilde{X}_n$ be an i.i.d. standard normal sequence. We denote by $\tilde{M}_n^{(k_n)}$ the k_nth largest maximum of $\tilde{X}_1, \dots, \tilde{X}_n$ and by $\tilde{M}_{m,n}^{(k_n)}$ the k_nth largest maximum among $\tilde{X}_{m+1}, \dots, \tilde{X}_n$. By the notations in (17), (18) and the definition of the component D_3 in (19)

$$D_3 = P\left(\tilde{M}_{m,n}^{(k_n)} \le u_n\right) - P\left(\tilde{M}_n^{(k_n)} \le u_n\right) \le P\left(\tilde{M}_n^{(k_n)} \ne \tilde{M}_{m,n}^{(k_n)}\right). \tag{39}$$

As $m \le n/k_n - 1$ and $k_n \ll n^\epsilon$ for any $\epsilon > 0$, it follows from Lemma 1 in Stadtmueller [4] that

$$P\left(\tilde{M}_n^{(k_n)} \ne \tilde{M}_{m,n}^{(k_n)}\right) \ll k_n m / (n - k_n).$$

Thus, due to (39),

$$D_3 \ll k_n m / (n - k_n). \tag{40}$$

Relations (19), (37), (38) and (40) imply the desired result in (15). □

Lemma 2. *Under the assumptions of Theorem 1, we have that if m, n satisfy $m \le n/k_n - 1$, then*

$$\left|Cov\left(I\left(M_m^{(k_m)} \le u_m\right), I\left(M_{m,n}^{(k_n)} \le u_n\right)\right)\right| \ll \frac{1}{n^\gamma} \tag{41}$$

for some $\gamma > 0$.

Proof. Let $X_1, X_2, \dots$ be a standardized stationary Gaussian sequence, $\{k_n\}$, $\{r(t)\}$, $\{u_n\}$ satisfy (7)–(12), respectively, and $m \le n/k_n - 1$. Clearly

$$\begin{aligned} &\left|Cov\left(I\left(M_m^{(k_m)} \le u_m\right), I\left(M_{m,n}^{(k_n)} \le u_n\right)\right)\right| \\ &= \left|P\left(M_m^{(k_m)} \le u_m, M_{m,n}^{(k_n)} \le u_n\right) - P\left(M_m^{(k_m)} \le u_m\right)P\left(M_{m,n}^{(k_n)} \le u_n\right)\right|. \end{aligned} \tag{42}$$

For the given m, n, we set

$$\sum_{(A_1,A_2,B_1,B_2)} := \sum_{\substack{(A_1,A_2,B_1,B_2): \\ A_1 \cup A_2 = \{1,\dots,m\},\ \#A_1 \le k_m - 1,\ A_2 = \{1,\dots,m\} \setminus A_1, \\ B_1 \cup B_2 = \{m+1,\dots,n\},\ \#B_1 \le k_n - 1,\ B_2 = \{m+1,\dots,n\} \setminus B_1}}. \tag{43}$$

Since

$$P\left(M_m^{(k_m)} \le u_m\right) = P\left(\text{at most } k_m - 1 \text{ of } X_1, \dots, X_m \text{ exceed } u_m\right)$$

and

$$P\left(M_{m,n}^{(k_n)} \le u_n\right) = P\left(\text{at most } k_n - 1 \text{ of } X_{m+1}, \dots, X_n \text{ exceed } u_n\right),$$

then, by relation (42) and the notation in (43), we can write that

$$\begin{aligned}\left|Cov\left(I\left(M_m^{(k_m)}\le u_m\right), I\left(M_{m,n}^{(k_n)}\le u_n\right)\right)\right| \\ \le \sum_{(A_1,A_2,B_1,B_2)} F\left(A_1,A_2,B_1,B_2\right),\end{aligned} \tag{44}$$

where

$$\begin{aligned}&F\left(A_1,A_2,B_1,B_2\right)\\ &:= \Bigg|P\Bigg(\bigcap_{a_p\in A_1}\{X_{a_p}>u_m\}\cap\bigcap_{a_s\in A_2}\{X_{a_s}\le u_m\}\cap\bigcap_{b_p\in B_1}\{X_{b_p}>u_n\}\cap\bigcap_{b_s\in B_2}\{X_{b_s}\le u_n\}\Bigg)\\ &- P\Bigg(\bigcap_{a_p\in A_1}\{X_{a_p}>u_m\}\cap\bigcap_{a_s\in A_2}\{X_{a_s}\le u_m\}\Bigg)P\Bigg(\bigcap_{b_p\in B_1}\{X_{b_p}>u_n\}\cap\bigcap_{b_s\in B_2}\{X_{b_s}\le u_n\}\Bigg)\Bigg|.\end{aligned}$$

By (7) and (10) (see the reasoning above (20)), $|r(t)|\to 0$ as $t\to\infty$. Hence, there exist positive numbers δ, γ, n_1, such that:

$$\sup_{t\ge 1}|r(t)| = \delta < 1, \tag{45}$$

$$1/k_n^{\beta-1} - 1/k_n^{\beta} < 1/\left(1+\delta\right) - 1/2 - \gamma \tag{46}$$

for all $n > n_1$, if β fulfils (9), (10).

Let $c(m)$ denote the largest integer, such that

$$c(m)\left[m^{1/k_m^{\beta}}\right]+1<m.$$

Thus, we can divide the sequence $X_1,\dots,X_m,X_{m+1},\dots,X_n$ into the blocks:

$$\begin{aligned}&\left(X_1,\dots,X_{\left[m^{1/k_m^{\beta}}\right]}\right),\ \left(X_{\left[m^{1/k_m^{\beta}}\right]+1},\dots,X_{2\left[m^{1/k_m^{\beta}}\right]}\right),\dots,\\ &\left(X_{c(m)\left[m^{1/k_m^{\beta}}\right]+1},\dots,X_m\right),\ \left(X_{m+1},\dots,X_{m+\left[n^{1/k_n^{\beta}}\right]}\right),\\ &\left(X_{m+\left[n^{1/k_n^{\beta}}\right]+1},\dots,X_n\right).\end{aligned}$$

By using such a division, as well as Theorem 4.2.1 in Leadbetter et al. [3], the relation in (44) and the definition of $\sum_{(A_1,A_2,B_1,B_2)}$ in (43), we obtain

$$\left|Cov\left(I\left(M_m^{(k_m)}\le u_m\right),I\left(M_{m,n}^{(k_n)}\le u_n\right)\right)\right| \tag{47}$$

$$\ll\left\{C_1(m,n)\sum_{d=0}^{c(m)-1}\sum_{i=d\left[m^{1/k_m^\beta}\right]+1}^{(d+1)\left[m^{1/k_m^\beta}\right]}\sum_{j=m+1}^{m+\left[n^{1/k_n^\beta}\right]}|r(j-i)|\exp\left(-\frac{u_m^2+u_n^2}{2(1+\delta)}\right)\right.$$

$$\left.+C_1(m,n)\sum_{i=c(m)\left[m^{1/k_m^\beta}\right]+1}^{m}\sum_{j=m+1}^{m+\left[n^{1/k_n^\beta}\right]}|r(j-i)|\exp\left(-\frac{u_m^2+u_n^2}{2(1+\delta)}\right)\right\}$$

$$+\left\{C_2(m,n)\sum_{d=0}^{c(m)-1}\sum_{i=d\left[m^{1/k_m^\beta}\right]+1}^{(d+1)\left[m^{1/k_m^\beta}\right]}\sum_{j=m+\left[n^{1/k_n^\beta}\right]+1}^{n}|r(j-i)|\exp\left(-\frac{u_m^2+u_n^2}{2(1+\delta)}\right)\right.$$

$$\left.+C_2(m,n)\sum_{i=c(m)\left[m^{1/k_m^\beta}\right]+1}^{m}\sum_{j=m+\left[n^{1/k_n^\beta}\right]+1}^{n}|r(j-i)|\exp\left(-\frac{u_m^2+u_n^2}{2(1+\delta)}\right)\right\},$$

where $C_1(m,n)$, $C_2(m,n)$ satisfy:

$$C_1(m,n)=\sum_{l_1=0}^{k_m-1}\sum_{l_2=0}^{k_n-1}\binom{\left[m^{1/k_m^\beta}\right]}{l_1}\binom{\left[n^{1/k_n^\beta}\right]}{l_2}, \tag{48}$$

$$C_2(m,n)=\sum_{l_1=0}^{k_m-1}\sum_{l_2=0}^{k_n-1}\binom{\left[m^{1/k_m^\beta}\right]}{l_1}\binom{n-m-\left[n^{1/k_n^\beta}\right]}{l_2}. \tag{49}$$

It follows from the derivation in (47) that

$$\left|Cov\left(I\left(M_m^{(k_m)}\le u_m\right),I\left(M_{m,n}^{(k_n)}\le u_n\right)\right)\right|$$

$$\ll(c(m)+1)C_1(m,n)\left[m^{1/k_m^\beta}\right]\sum_{t=1}^{m+\left[n^{1/k_n^\beta}\right]-1}|r(t)|\exp\left(-\frac{u_m^2+u_n^2}{2(1+\delta)}\right) \tag{50}$$

$$+(c(m)+1)C_2(m,n)\left[m^{1/k_m^\beta}\right]\sum_{t=\left[n^{1/k_n^\beta}\right]+1}^{n-1}|r(t)|\exp\left(-\frac{u_m^2+u_n^2}{2(1+\delta)}\right).$$

In addition, by the definition of $c(m)$ (see the relation below (46)),

$$(c(m)+1)\left[m^{1/k_m^\beta}\right] \ll m. \tag{51}$$

Relations (50), (51) yield

$$\begin{aligned} &\left|Cov\left(I\left(M_m^{(k_m)} \le u_m\right), I\left(M_{m,n}^{(k_n)} \le u_n\right)\right)\right| \\ &\ll mC_1(m,n) \sum_{t=1}^{m+\left[n^{1/k_n^\beta}\right]-1} |r(t)| \exp\left(-\frac{u_m^2+u_n^2}{2(1+\delta)}\right) \\ &\quad + mC_2(m,n) \sum_{t=\left[n^{1/k_n^\beta}\right]+1}^{n-1} |r(t)| \exp\left(-\frac{u_m^2+u_n^2}{2(1+\delta)}\right). \end{aligned} \tag{52}$$

Moreover, observe that, due to (48), (49):

$$\begin{aligned} C_1(m,n) &\ll k_m k_n \left(m^{1/k_m^\beta}\right)^{k_m-1} \left(n^{1/k_n^\beta}\right)^{k_n-1} \\ &= k_m k_n m^{1/k_m^{\beta-1}-1/k_m^\beta} n^{1/k_n^{\beta-1}-1/k_n^\beta}, \end{aligned} \tag{53}$$

$$C_2(m,n) \ll k_m k_n \left(m^{1/k_m^\beta}\right)^{k_m-1} n^{k_n-1} = k_m k_n m^{1/k_m^{\beta-1}-1/k_m^\beta} n^{k_n-1}. \tag{54}$$

Relations (53), (54) together with derivation (52) imply

$$\begin{aligned} &\left|Cov\left(I\left(M_m^{(k_m)} \le u_m\right), I\left(M_{m,n}^{(k_n)} \le u_n\right)\right)\right| \\ &\ll k_m k_n m^{1+1/k_m^{\beta-1}-1/k_m^\beta} n^{1/k_n^{\beta-1}-1/k_n^\beta} \sum_{t=1}^{m+\left[n^{1/k_n^\beta}\right]-1} |r(t)| \exp\left(-\frac{u_m^2+u_n^2}{2(1+\delta)}\right) \\ &\quad + k_m k_n m^{1+1/k_m^{\beta-1}-1/k_m^\beta} n^{k_n-1} \sum_{t=\left[n^{1/k_n^\beta}\right]+1}^{n-1} |r(t)| \exp\left(-\frac{u_m^2+u_n^2}{2(1+\delta)}\right). \end{aligned} \tag{55}$$

By assumption (10) and relation (20) (see the reasoning above (20))

$$\sum_{t=\left[n^{1/k_n^\beta}\right]+1}^{n-1} |r(t)| \ll \frac{1}{n^{k_n-1-1/k_n^{\beta-1}+1/k_n^\beta}} \quad \text{and} \quad \sum_{t=1}^{\infty} |r(t)| < \infty.$$

This and (55) yield

$$\begin{aligned} &\left|Cov\left(I\left(M_m^{(k_m)} \le u_m\right), I\left(M_{m,n}^{(k_n)} \le u_n\right)\right)\right| \\ &\qquad \ll k_m k_n m^{1+1/k_m^{\beta-1}-1/k_m^\beta} n^{1/k_n^{\beta-1}-1/k_n^\beta} \exp\left(-\frac{u_m^2+u_n^2}{2(1+\delta)}\right). \end{aligned} \tag{56}$$

Moreover, it follows from (35) that

$$\exp\left(-\frac{u_m^2+u_n^2}{2\,(1+\delta)}\right) \ll \frac{(k_m)^{1/(1+\delta)}}{m^{1/(1+\delta)}}\left(\log\left(m/k_m\right)\right)^{1/2(1+\delta)}\frac{(k_n)^{1/(1+\delta)}}{n^{1/(1+\delta)}}\left(\log\left(n/k_n\right)\right)^{1/2(1+\delta)}. \tag{57}$$

As in addition, the sequence $\{k_n\}$ is nondecreasing, we have $k_m \le k_n$. This, relation (57) and the fact that

$$\begin{aligned}\left(\log\left(m/k_m\right)\right)^{1/2(1+\delta)}\left(\log\left(n/k_n\right)\right)^{1/2(1+\delta)} &\le (\log m)^{1/2(1+\delta)}(\log n)^{1/2(1+\delta)}\\ &\le (\log n)^{1/(1+\delta)}\end{aligned}$$

yield

$$\exp\left(-\frac{u_m^2+u_n^2}{2\,(1+\delta)}\right) \ll \frac{(k_n)^{2/(1+\delta)}(\log n)^{1/(1+\delta)}}{m^{1/(1+\delta)}n^{1/(1+\delta)}}. \tag{58}$$

Relations (56), (58) imply

$$\begin{aligned}&\left|Cov\left(I\left(M_m^{(k_m)}\le u_m\right), I\left(M_{m,n}^{(k_n)}\le u_n\right)\right)\right|\\ &\ll \frac{m^{1+1/k_m^{\beta-1}-1/k_m^{\beta}-1/(1+\delta)}}{n^{1/(1+\delta)-1/k_n^{\beta-1}+1/k_n^{\beta}}}(k_n)^{2+2/(1+\delta)}(\log n)^{1/(1+\delta)}\\ &\ll \frac{(k_n)^{2+2/(1+\delta)}(\log n)^{1/(1+\delta)}}{n^{2/(1+\delta)-1-1/k_n^{\beta-1}+1/k_n^{\beta}-1/k_m^{\beta-1}+1/k_m^{\beta}}}\\ &= \frac{(k_n)^{2+2/(1+\delta)}(\log n)^{1/(1+\delta)}}{n^{1/(1+\delta)-1/2-1/k_n^{\beta-1}+1/k_n^{\beta}+1/(1+\delta)-1/2-1/k_m^{\beta-1}+1/k_m^{\beta}}}.\end{aligned} \tag{59}$$

Notice that, by (46), we obtain

$$1/\left(1+\delta\right)-1/2-1/k_n^{\beta-1}+1/k_n^{\beta}+1/\left(1+\delta\right)-1/2-1/k_m^{\beta-1}+1/k_m^{\beta} > 2\gamma \tag{60}$$

for all m, $n > n_1$ and some $\gamma > 0$.

Due to (59) and (60) and the fact that, by assumption (8), $k_n \ll n^{\epsilon}$ for any $\epsilon > 0$, we have

$$\begin{aligned}&\left|Cov\left(I\left(M_m^{(k_m)}\le u_m\right), I\left(M_{m,n}^{(k_n)}\le u_n\right)\right)\right|\\ &\ll \frac{(k_n)^{2+2/(1+\delta)}(\log n)^{1/(1+\delta)}}{n^{2\gamma}}\\ &\ll \frac{(n^{\epsilon})^{2+2/(1+\delta)}(\log n)^{1/(1+\delta)}}{n^{2\gamma}}\end{aligned} \tag{61}$$

for some $\gamma > 0$ and any $\epsilon > 0$. As ϵ in (61) may be arbitrary positive number, we can choose ϵ satisfying the relation $\epsilon\,(2+2/\left(1+\delta\right)) < \gamma$. Then

$(n^{\epsilon})^{2+2/(1+\delta)}(\log n)^{1/(1+\delta)} \ll n^{\gamma}$ and

$$\left|Cov\left(I\left(M_m^{(k_m)} \le u_m\right), I\left(M_{m,n}^{(k_n)} \le u_n\right)\right)\right| \ll \frac{n^{\gamma}}{n^{2\gamma}} = \frac{1}{n^{\gamma}}$$

for some $\gamma > 0$, which is the result in (41), we wished to prove. □

The following property will be also needed in our further considerations.

Lemma 3. *Under the assumptions of Theorem 1, we have*

$$\lim_{n\to\infty} P\left(M_n^{(k_n)} \le u_n\right) = \Phi(\tau), \tag{62}$$

where τ *satisfies* (12).

Proof. The relation in (62) follows immediately from Theorem 4.2.1 (the Normal Comparison Lemma) and Theorem 2.5.2 in Leadbetter et al. [3]. □

4. Proofs of main results. In this section, we give the proofs of Theorems 1, 2. As we mentioned earlier, the results stated in Lemmas 1–3 are important ingredients of these proofs.

Proof of Theorem 1 (i). First, we will show that

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n}\left\{I\left(M_n^{(k_n)} \le u_n\right) - P\left(M_n^{(k_n)} \le u_n\right)\right\} = 0 \quad \text{a.s.} \tag{63}$$

By Lemma 3.1 in Csaki and Gonchigdanzan [1], in order to prove (63), it is sufficient to show that the following property occurs for some $\varepsilon > 0$

$$Var\left(\sum_{n=1}^{N} \frac{1}{n} I\left(M_n^{(k_n)} \le u_n\right)\right) \ll (\log N)^2 (\log\log N)^{-(1+\varepsilon)}. \tag{64}$$

We have

$$\begin{aligned} Var\left(\sum_{n=1}^{N} \frac{1}{n} I\left(M_n^{(k_n)} \le u_n\right)\right) &\le \sum_{n=1}^{N} \frac{1}{n^2} Var\left(I\left(M_n^{(k_n)} \le u_n\right)\right) \\ &+ 2\sum_{1\le m<n\le N} \frac{1}{mn}\left|Cov\left(I\left(M_m^{(k_m)} \le u_m\right), I\left(M_n^{(k_n)} \le u_n\right)\right)\right| \\ &=: \sum\nolimits_1 + \sum\nolimits_2. \end{aligned} \tag{65}$$

It is clear that

$$\sum\nolimits_1 \le \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty. \tag{66}$$

Our purpose now is to estimate $\sum_2$ in (65). Observe that

$$\begin{aligned}&\left|Cov\left(I\left(M_m^{(k_m)}\le u_m\right),I\left(M_n^{(k_n)}\le u_n\right)\right)\right|\\&\quad\ll E\left|I\left(M_n^{(k_n)}\le u_n\right)-I\left(M_{m,n}^{(k_n)}\le u_n\right)\right|\\&\qquad+\left|Cov\left(I\left(M_m^{(k_m)}\le u_m\right),I\left(M_{m,n}^{(k_n)}\le u_n\right)\right)\right|.\end{aligned}$$

Thus, by (67) and Lemmas 1, 2, we have that if m, n satisfy $m\le\dfrac{n}{k_n}-1$, then

$$\left|Cov\left(I\left(M_m^{(k_m)}\le u_m\right),I\left(M_n^{(k_n)}\le u_n\right)\right)\right|\ll\frac{1}{n^\gamma}+\frac{k_nm}{n-k_n}$$

for some $\gamma>0$. Consequently

$$\begin{aligned}\textstyle\sum_2&\ll\sum_{\substack{1\le m<n\le N,\\ m\le n/k_n-1}}\frac{1}{mn}\frac{1}{n^\gamma}+\sum_{\substack{1\le m<n\le N,\\ m\le n/k_n-1}}\frac{1}{mn}\frac{k_nm}{n-k_n}+\sum_{\substack{1\le m<n\le N,\\ m>n/k_n-1}}\frac{1}{mn}\\&=:G_1+G_2+G_3.\end{aligned}\tag{67}$$

Notice that

$$G_1\le\sum_{1\le m<n\le N}\frac{1}{mn}\frac{1}{n^\gamma}=\sum_{m=1}^{N-1}\frac{1}{m}\sum_{n=m+1}^{N}\frac{1}{n^{1+\gamma}}\le\frac{1}{\gamma}\sum_{m=1}^{N-1}\frac{1}{m^{1+\gamma}}<\infty.\tag{68}$$

In order to estimate G_2 in (67), observe that

$$\begin{aligned}G_2&\ll\sum_{1\le n\le N}\sum_{1\le m\le n/k_n-1}\frac{k_n}{n\left(n-k_n\right)}\ll\sum_{1\le n\le N}\left(n/k_n-1\right)\frac{k_n}{n\left(n-k_n\right)}\\&\ll\sum_{n=1}^{N}\frac{1}{n}\ll\log N.\end{aligned}\tag{69}$$

Thus, it remains to estimate component G_3. From its definition in (67), we get

$$G_3\le\sum_{1\le n\le N}\frac{1}{n}\sum_{m=[n/k_n]}^{n-1}\frac{1}{m}\ll\sum_{1\le n\le N}\frac{1}{n}\log\frac{n}{[n/k_n]}\ll\sum_{1\le n\le N}\frac{1}{n}\log\frac{n}{n/k_n-1}.$$

Therefore

$$G_3\ll\sum_{1\le n\le N}\frac{1}{n}\log\frac{nk_n}{n-k_n}.\tag{70}$$

It follows from (8) that, there exists constant n_0, such that $k_n\le n/2$ for all $n>n_0$. Hence, for any $n>n_0$,

$$\log\frac{nk_n}{n-k_n}\le\log\frac{nk_n}{n-n/2}=\log 2k_n,$$

and

$$\log \frac{nk_n}{n-k_n} \ll \log k_n. \tag{71}$$

Relations (70), (71) together with assumption (8) imply

$$G_3 \ll \sum_{n=1}^{N} \frac{1}{n} \log k_n \ll \sum_{n=1}^{N} \frac{1}{n} (\log n)^{1-\alpha} \ll (\log N)^{2-\alpha} \tag{72}$$

for some $\alpha > 0$. Due to (67)–(69) and (72)

$$\sum\nolimits_2 \ll (\log N)^{2-\alpha} \tag{73}$$

for some $\alpha > 0$. It follows from (65), (66) and (73) that

$$Var\left(\sum_{n=1}^{N} \frac{1}{n} I\left(M_n^{(k_n)} \le u_n\right)\right) \ll (\log N)^{2-\alpha}$$

for some $\alpha > 0$. Thus, (64) holds for any $\varepsilon > 0$.

Consequently, by the already mentioned Lemma 3.1 in Csaki and Gonchigdanzan [1], condition (63) is also satisfied. In turn, as (63) holds, then Lemma 3 and the regularity property of logarithmic means imply (1). Thus, statement (i) of our assertion has been proved. □

Proof of Theorem 1 (ii). Let x be arbitrary real number. It is easy to check that, provided $\{a_n\}$, $\{b_n\}$ are such as in (2), then, under the assumptions of our theorem,

$$\lim_{n\to\infty} P\left(a_n\left(M_n^{(k_n)} - b_n\right) \le x\right) = \Phi(x)$$

(see also the remark on p. 416 in Stadtmueller [4]). This and Theorem 2.5.2 in Leadbetter et al. [3] imply that assumptions (11), (12) are satisfied with: $u_n := x/a_n + b_n$, $\tau := x$. It is easily seen now that statement (ii) of Theorem 1 is a special case of its, the earlier proved, statement (i). □

Remark. Suppose that $k_n = [(\log n)^c]$ for some $0 < c < 1$, and the number $\beta > 1$ satisfies the condition $c\beta < 1$. Let in addition, $X_1, X_2, \ldots$ be a stationary zero mean Gaussian sequence with unit variance and the covariance function $r(t) = e^{-\lambda t}$ for some $\lambda > 0$. Then, the assumptions (7)–(10) of Theorem 1 are fulfilled and the property in (3) holds with $\{a_n\}$, $\{b_n\}$ given by (2).

We now prove our second main result.

Proof of Theorem 2 (i). Let k denote a fixed positive integer, $M_n^{(k)}$ stand for the kth largest maximum of $X_1, \ldots, X_n$. By applying assumption (13) on the covariance function $r(t) = Cov(X_1, X_{1+t})$ and assumption (14) on the sequence $\{u_n\}$, and by using similar methods to that applied in the proofs of Lemmas 1, 2, we can show that if $m \le n/k - 1$, then:

$$E\left|I\left(M_n^{(k)} \le u_n\right) - I\left(M_{m,n}^{(k)} \le u_n\right)\right| \ll \frac{1}{n^\gamma} + \frac{km}{n-k} \quad \text{for some } \gamma > 0,$$

$$\left|Cov\left(I\left(M_m^{(k)} \le u_m\right), I\left(M_{m,n}^{(k)} \le u_n\right)\right)\right| \ll \frac{1}{n^\gamma} \quad \text{for some } \gamma > 0.$$

This and the relation in (67), applied for $k_n \equiv k$, yield

$$\left|Cov\left(I\left(M_m^{(k)} \le u_m\right), I\left(M_n^{(k)} \le u_n\right)\right)\right| \ll \frac{1}{n^\gamma} + \frac{km}{n-k} \tag{74}$$

for some $\gamma > 0$, provided $m \le n/k - 1$. Suppose that $\left\{\tilde{X}_i\right\}$ is an i.i.d. standard normal sequence and $\tilde{M}_n^{(k)}$ denotes the kth largest maximum of $\tilde{X}_1, \ldots, \tilde{X}_n$. It follows from Theorem 2.2.1 in Leadbetter et al. [3] that, under the assumptions of our theorem,

$$\lim_{n\to\infty} P\left(\tilde{M}_n^{(k)} \le u_n\right) = e^{-\tau}\sum_{s=0}^{k-1}\frac{\tau^s}{s!}, \tag{75}$$

where τ satisfies (14). By using similar methods to those applied in the estimation of D_1 in the proof of Lemma 1, it is easy to check that

$$\left|P\left(M_n^{(k)} \le u_n\right) - P\left(\tilde{M}_n^{(k)} \le u_n\right)\right| \ll \frac{1}{n^\gamma} \tag{76}$$

for some $\gamma > 0$. Relations (75) and (76) imply

$$\lim_{n\to\infty} P\left(M_n^{(k)} \le u_n\right) = e^{-\tau}\sum_{s=0}^{k-1}\frac{\tau^s}{s!}, \tag{77}$$

provided τ fulfils (14). Thus, in view of the already mentioned Lemma 3.1 in Csaki and Gonchigdanzan [1], in order to prove (4), it is enough to show that

$$Var\left(\sum_{n=k}^{N}\frac{1}{n}I\left(M_n^{(k)} \le u_n\right)\right) \ll (\log N)^2 (\log\log N)^{-(1+\varepsilon)} \tag{78}$$

for some $\varepsilon > 0$. We have

$$Var\left(\sum_{n=k}^{N}\frac{1}{n}I\left(M_n^{(k)} \le u_n\right)\right) \le \sum\nolimits_3 + \sum\nolimits_4, \tag{79}$$

where $\sum_3$, $\sum_4$ are defined as $\sum_1$, $\sum_2$ in (65), but for the case, when $k_n \equiv k$.

Obviously

$$\sum\nolimits_3 < \infty. \tag{80}$$

In addition, by (74),

$$\sum\nolimits_4 \ll \sum_{\substack{k\le m<n\le N,\\ m\le n/k-1}} \frac{1}{mn}\frac{1}{n^{\gamma}} + \sum_{\substack{k\le m<n\le N,\\ m\le n/k-1}} \frac{1}{mn}\frac{km}{n-k} + \sum_{\substack{k\le m<n\le N,\\ m>n/k-1}} \frac{1}{mn} =: H_1 + H_2 + H_3. \tag{81}$$

By proceeding analogously as in the estimation of G_1, G_2 in the proof of Theorem 1 (i), we immediately get

$$H_1 + H_2 \ll \log N. \tag{82}$$

Furthermore, it is easily seen from the definition of H_3 in (81) that

$$H_3 \ll \sum_{n=k}^{N} \frac{1}{n} \ll \log N. \tag{83}$$

Thus, due to (81)–(83),

$$\sum\nolimits_4 \ll \log N. \tag{84}$$

By (79), (80) and (84), we obtain

$$Var\left(\sum_{n=k}^{N} \frac{1}{n} I\left(M_n^{(k)} \le u_n\right)\right) \ll \log N.$$

Hence, the relation in (78) is satisfied. Consequently, by Lemma 3.1 in Csaki and Gonchigdanzan [1],

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=k}^{N} \frac{1}{n} \left\{ I\left(M_n^{(k)} \le u_n\right) - P\left(M_n^{(k)} \le u_n\right) \right\} = 0 \quad \text{a.s.}$$

This, (77) and the regularity property of logarithmic means yield (4), which is statement (i) of Theorem 2. □

Proof of Theorem 2 (ii). Let x be arbitrary real number. Since (13) holds, it follows from Theorem 4.3.3 in Leadbetter et al. [3] that, provided $\{a_n\}$, $\{b_n\}$ are such as in (5), we have

$$\lim_{n\to\infty} n\left(1 - \Phi\left(x/a_n + b_n\right)\right) = e^{-x}.$$

It is easily seen now that statement (ii) of Theorem 2 is a special case of its, the earlier proved, statement (i), with: $u_n := x/a_n + b_n$, $\tau := e^{-x}$. □

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