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Inclusion properties of certain subclass of analytic functions defined by multiplier transformations

Abstract. Let A denote the class of analytic functions with normalization $f(0) = f^{'}(0) - 1 = 0$ in the open unit disk $U = \{z : |z| < 1\}$. Set

$$f^m_{\lambda,\ell}(z) = z + \sum_{k=2}^{\infty} \left[\frac{\ell + 1 + \lambda(k-1)}{\ell + 1}\right]^m z^k \quad (z \in U;\, m \in N_0;\, \lambda \geq 0;\, \ell \geq 0),$$

and define $f^m_{\lambda\ell,\mu}$ in terms of the Hadamard product

$$f^m_{\lambda,\ell}(z) * f^m_{\lambda,\ell,\mu}(z) = \frac{z}{(1-z)^\mu} \quad (z \in U; \mu > 0).$$

In this paper, we introduce several new subclasses of analytic functions defined by means of the operator $I^m_{\lambda,\ell,\mu} f(z) = f^m_{\lambda,\ell,\mu}(z) * f(z)$ ($f \in A$; $m \in N_0$; $\lambda \geq 0$; $\ell \geq 0$; $\mu > 0$).

Inclusion properties of these classes and the classes involving the generalized Libera integral operator are also considered.

1. Introduction. Let A denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. If $f$and g are analytic in U, we say that f is subordinate to g, written $f(z) \prec g(z)$, if

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there exists an analytic function w in U with $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$ such that $f(z) = g(w(z))$. For $0 \leq \eta < 1$, we denote by $S^*(\eta)$, $K(\eta)$ and C the subclasses of A consisting of all analytic functions which are, respectively, starlike of order η, convex of order η and close-to-convex in U (see, e.g. Srivastava and Owa [18]).

For $m \in N_0 = N \cup \{0\}$, where $N = \{1, 2, \dots\}$, $\lambda \geq 0$ and $\ell \geq 0$, Cătaş [3] defined the multiplier transformations $I^m(\lambda, \ell)$ on A by the following infinite series

$$I^m(\lambda,\ell)f(z) = z + \sum_{k=2}^{\infty} \left[\frac{\ell + 1 + \lambda(k-1)}{\ell + 1}\right]^m a_k z^k. \tag{1.2}$$

It follows from (1.2) that

$$I^0(\lambda,\ell) = f(z), \tag{1.3}$$

$$(\ell+1)I^2(\lambda,\ell)f(z) = (\ell+1-\lambda)I^1(\lambda,\ell)f(z) + \lambda z(I^1(\lambda,\ell)f(z))', \tag{1.4}$$

$\lambda > 0$, and

$$I^{m_1}(\lambda,\ell)(I^{m_2}(\lambda,\ell)f(z)) = I^{m_2}(\lambda,\ell)(I^{m_1}(\lambda,\ell)f(z)) \tag{1.5}$$

for all integers m_1 and m_2.

We note that:

(i) $I^m(1,\ell) = I_\ell^m$ (see Cho and Srivastava [4] and Cho and Kim [5]);
(ii) $I^m(\lambda, 0) = D_\lambda^m$ ($m \in N_0$; $\lambda \geq 0$) (see Al-Oboudi [1]);
(iii) $I^m(1, 0) = D^m$ ($m \in N_0$) (see Sălăgean [17]);
(iv) $I^m(1, 1) = I_m$ (see Uralegaddi and Somanatha [19]).

Let S be the class of all functions φ which are analytic and univalent in U and for which $\varphi(U)$ is convex and $\varphi(0) = 1$ and $\operatorname{Re}\{\varphi(z)\} > 0$ ($z \in U$).

Making use of the principle of subordination between analytic functions, we introduce the subclasses $S^*(\eta;\varphi)$, $K(\eta;\varphi)$ and $C(\eta,\delta;\varphi,\psi)$ of the class A for $0 \leq \eta, \delta < 1$ and $\varphi, \psi \in S$ (cf., e.g., [6], [8] and [12]), which are defined as follows:

$$S^*(\eta;\varphi) = \left\{ f \in A : \frac{1}{1-\eta}\left(\frac{zf'(z)}{f(z)} - \eta\right) \prec \varphi(z),\, z \in U \right\},$$

$$K(\eta;\varphi) = \left\{ f \in A : \frac{1}{1-\eta}\left(1 + \frac{zf''(z)}{f'(z)} - \eta\right) \prec \varphi(z),\, z \in U \right\},$$

and

$$C(\eta,\delta;\varphi,\psi) = \left\{ f \in A : \exists g \in S^*(\eta,\varphi) \text{ s.t. } \frac{1}{1-\delta}\left(\frac{zf'(z)}{g(z)} - \delta\right) \prec \psi(z),\, z \in U \right\}.$$

We note that, for special choices for the functions φ and ψ in the above definitions we obtain the well-known subclasses of A. For examples, we have

(i) $S^*\left(\eta; \frac{1+z}{1-z}\right) = S^*(\eta) \quad (0 \leq \eta < 1)$,

(ii) $K\left(\eta; \frac{1+z}{1-z}\right) = K(\eta) \quad (0 \leq \eta < 1)$

and

(iii) $C\left(0, 0; \frac{1+z}{1-z}; \frac{1+z}{1-z}\right) = C$.

Setting

$$f^m_{\lambda,\ell}(z) = z + \sum_{k=2}^{\infty} \left[\frac{\ell + 1 + \lambda(k-1)}{\ell + 1}\right]^m z^k \quad (m \in N_0,\ \lambda \geq 0,\ \ell \geq 0),$$

we define a new function $f^m_{\lambda,\ell,\mu}(z)$ in terms of the Hadamard product (or convolution) by:

$$f^m_{\lambda,\ell}(z) * f^m_{\lambda,\ell,\mu}(z) = \frac{z}{(1-z)^\mu} \quad (\mu > 0;\ z \in U). \tag{1.6}$$

Then, motivated essentially by the Choi–Saigo–Srivastava operator [6] (see also [10], [11], [14], and [15]), we now introduce the operators $f^m_{\lambda,\ell,\mu} : A \to A$, which are defined here by

$$I^m_{\lambda,\ell,\mu} f(z) = f^m_{\lambda,\ell,\mu} * f(z) \tag{1.7}$$

$(f \in A;\ m \in N_0;\ \lambda \geq 0;\ \ell \geq 0;\ \mu > 0)$. For a function $f(z) \in A$, given by (1.1), it is easily seen from (1.7) that

$$I^m_{\lambda,\ell,\mu} f(z) = z + \sum_{k=2}^{\infty} \left[\frac{\ell + 1}{\ell + 1 + \lambda(k-1)}\right]^m \frac{(\mu)_{k-1}}{(1)_{k-1}} a_k z^k \tag{1.8}$$

$(m \in N_0;\ \lambda \geq 0;\ \ell \geq 0;\ z \in U)$.

We note that:

(i) $I^1_{1,0,2} f(z) = f(z)$ and $I^0_{1,0,2} f(z) = z f'(z)$,

and

(ii) $I^s_{1,\ell,\mu} f(z) = I^s_{\ell,\mu} f(z)$ ($s \in R$; see Cho and Kim [5]).

In view of (1.8), we obtain the following relations:

$$\lambda z (I^{m+1}_{\lambda,\ell,\mu} f(z))' = (\ell + 1) I^m_{\lambda,\ell,\mu} f(z) - [\lambda - (\ell + 1)]\, I^{m+1}_{\lambda,\ell,\mu} f(z) \tag{1.9}$$

$(f \in A;\ m \in N_0;\ \lambda > 0;\ \ell \geq 0;\ \mu > 0)$ and

$$z (I^m_{\lambda,\ell,\mu} f(z))' = \mu I^m_{\lambda,\ell,\mu+1} f(z) - (\mu - 1) I^m_{\lambda,\ell,\mu} f(z) \tag{1.10}$$

($f \in A$; $m \in N_0$; $\lambda \geq 0$; $\ell \geq 0$; $\mu > 0$). Next, by using the operator $I^m_{\lambda,\ell,\mu}$, we introduce the following classes of analytic functions for $\varphi, \psi \in S$, $m \in N_0$, $\lambda \geq 0$, $\ell \geq 0$, $\mu > 0$ and $0 \leq \eta$, $\delta < 1$:

$$S^m_{\lambda,\ell,\mu}(\eta;\varphi) = \left\{f \in A : I^m_{\lambda,\ell,\mu} f(z) \in S^*(\eta;\varphi)\right\}, \tag{1.11}$$

$$K^m_{\lambda,\ell,\mu}(\eta;\varphi) = \left\{f \in A : I^m_{\lambda,\ell,\mu} f(z) \in K(\eta;\varphi)\right\} \tag{1.12}$$

and

$$C^m_{\lambda,\ell,\mu}(\eta,\delta;\varphi,\psi) = \left\{f \in A : I^m_{\lambda,\ell,\mu} f(z) \in C(\eta,\delta;\varphi,\psi)\right\}. \tag{1.13}$$

We also have

$$f(z) \in K^m_{\lambda,\ell,\mu}(\eta;\varphi) \Leftrightarrow zf'(z) \in S^m_{\lambda,\ell,\mu}(\eta;\varphi). \tag{1.14}$$

In particular, we set

$$S^m_{\lambda,\ell,\mu}\left(\eta;\left(\frac{1+Az}{1+Bz}\right)^{\alpha}\right) = S^m_{\lambda,\ell,\mu}(\eta;A,B,\alpha)$$

($0 < \alpha \leq 1$; $-1 \leq B < A \leq 1$) and

$$K^m_{\lambda,\ell,\mu}\left(\eta;\left(\frac{1+Az}{1+Bz}\right)^{\alpha}\right) = K^m_{\lambda,\ell,\mu}(\eta;A,B,\alpha)$$

($0 < \alpha \leq 1$; $-1 \leq B < A \leq 1$).

In this paper, we investigate several inclusion properties of the classes $S^m_{\lambda,\ell,\mu}(\eta;\varphi)$, $K^m_{\lambda,\ell,\mu}(\eta;\varphi)$ and $C^m_{\lambda,\ell,\mu}(\eta,\delta;\varphi,\psi)$ associated with the operator $I^m_{\lambda,\ell,\mu}$. Some applications involving these and other classes of integral operators are also considered.

2. Inclusion properties involving the operator $I^m_{\lambda,\ell,\mu}$. The following lemmas will be required in our investigation.

Lemma 1 ([7]). *Let φ be convex, univalent in U with $\varphi(0) = 1$ and* $\mathrm{Re}\{\beta\varphi(z) + \nu\} > 0$ *($\beta, \nu \in C$). If p is analytic in U with $p(0) = 1$, then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \nu} \prec \varphi(z) \quad (z \in U)$$

implies that

$$p(z) \prec \varphi(z) \quad (z \in U).$$

Lemma 2 ([13]). *Let φ be convex, univalent in U and w be analytic in U with* $\mathrm{Re}\{w(z)\} \geq 0$. *If $p(z)$ is analytic in U and $p(0) = \varphi(0)$, then*

$$p(z) + w(z)zp'(z) \prec \varphi(z) \quad (z \in U)$$

implies that

$$p(z) \prec \varphi(z) \quad (z \in U).$$

At first, with the help of Lemma 1, we prove the following theorem.

Theorem 1. *Let $m \in N_0$, $\lambda > 0$, $\ell \geq 0$, $\ell + 1 > \lambda$ and $\mu \geq 1$. Then*

$$S^m_{\lambda,\ell,\mu+1}(\eta;\varphi) \subset S^m_{\lambda,\ell,\mu}(\eta;\varphi) \subset S^{m+1}_{\lambda,\ell,\mu}(\eta;\varphi)$$

$(0 \leq \eta < 1;\ \phi \in S)$.

Proof. First of all, we will show that

$$S^m_{\lambda,\ell,\mu+1}(\eta;\varphi) \subset S^m_{\lambda,\ell,\mu}(\eta;\varphi).$$

Let $f \in S^m_{\lambda,\ell,\mu+1}(\eta;\varphi)$ and put

$$p(z) = \frac{1}{1-\eta}\left(\frac{z(I^m_{\lambda,\ell,\mu}f(z))'}{I^m_{\lambda,\ell,\mu}f(z)} - \eta\right), \tag{2.1}$$

where $p(z)$ is analytic in U with $p(0) = 1$. Using (1.10) and (2.1), we obtain

$$\mu\frac{I^m_{\lambda,\ell,\mu+1}f(z)}{I^m_{\lambda,\ell,\mu}f(z)} = (1-\eta)p(z) + \eta + (\mu - 1). \tag{2.2}$$

Differentiating (2.2) logarithmically with respect to z, we obtain

$$\frac{1}{1-\eta}\left(\frac{z(I^m_{\lambda,\ell,\mu+1}f(z))'}{I^m_{\lambda,\ell,\mu+1}f(z)} - \eta\right) = p(z) + \frac{zp'(z)}{(1-\eta)p(z) + \eta + (\mu-1)} \tag{2.3}$$

$(z \in U)$. Applying Lemma 1 to (2.3), it follows that $p \prec \varphi$, that is $f \in S^m_{\lambda,\ell,\mu}(\eta;\varphi)$.

To prove the second part, let $f \in S^m_{\lambda,\ell,\mu}(\eta;\varphi)$ and put

$$h(z) = \frac{1}{1-\eta}\left(\frac{z(I^{m+1}_{\lambda,\ell,\mu}f(z))'}{I^{m+1}_{\lambda,\ell,\mu}f(z)} - \eta\right),$$

where h is analytic in U with $h(0) = 1$. Then, by using the arguments similar to those detailed above with (1.9), it follows that $h \prec \varphi$. This completes the proof of Theorem 1. □

Theorem 2. *Let $m \in N_0$, $\lambda > 0$, $\ell \geq 0$, $\ell + 1 > \lambda$ and $\mu \geq 1$. Then*

$$K^m_{\lambda,\ell,\mu+1}(\eta;\varphi) \subset K^m_{\lambda,\ell,\mu}(\eta;\varphi) \subset K^{m+1}_{\lambda,\ell,\mu}(\eta;\varphi)$$

$(0 \leq \eta < 1;\ \phi \in S)$.

Proof. Applying (1.11) and Theorem 1, we observe that

$$\begin{aligned} f \in K^m_{\lambda,\ell,\mu+1}(\eta;\varphi) &\Leftrightarrow I^m_{\lambda,\ell,\mu+1}f(z) \in K(\eta;\varphi) \Leftrightarrow z(I^m_{\lambda,\ell,\mu+1}f(z))' \in S^*(\eta;\varphi) \\ &\Leftrightarrow I^m_{\lambda,\ell,\mu+1}(zf'(z)) \in S^*(\eta;\varphi) \\ &\Leftrightarrow zf'(z) \in S^m_{\lambda,\ell,\mu+1}(\eta;\varphi) \\ &\Rightarrow zf'(z) \in S^m_{\lambda,\ell,\mu}(\eta;\varphi) \\ &\Leftrightarrow I^m_{\lambda,\ell,\mu}(zf'(z)) \in S^m(\eta;\varphi) \end{aligned}$$

$$\Leftrightarrow z(I^m_{\lambda,\ell,\mu}(zf(z))' \in S^m(\eta;\varphi)$$
$$\Leftrightarrow I^m_{\lambda,\ell,\mu}f(z) \in K(\eta;\varphi)$$
$$\Leftrightarrow f(z) \in K^m_{\lambda,\ell,\mu}(\eta;\varphi)$$

and

$$\begin{aligned} f(z) \in K^m_{\lambda,\ell,\mu}(\eta;\varphi) &\Leftrightarrow zf'(z) \in S^*(\eta;\varphi) \\ &\Rightarrow zf'(z) \in S^{m+1}_{\lambda,\ell,\mu}(\eta;\varphi) \\ &\Leftrightarrow z(I^{m+1}_{\lambda,\ell,\mu}f(z))' \in S^*(\eta;\varphi) \\ &\Leftrightarrow I^{m+1}_{\lambda,\ell,\mu}f(z) \in K(\eta;\varphi) \\ &\Leftrightarrow f(z) \in K^{m+1}_{\lambda,\ell,\mu}(\eta;\varphi), \end{aligned}$$

which evidently proves Theorem 2. □

Taking

$$\varphi(z) = \left(\frac{1+Az}{1+Bz}\right)^{\alpha}$$

$(-1 \leq B < A \leq 1;\ 0 < \alpha \leq 1;\ z \in U)$ in Theorem 1 and Theorem 2, we obtain the following corollary.

Corollary 1. *Let $m \in N_0$, $\lambda > 0$, $\ell \geq 0$, $\ell + 1 > \lambda$ and $\mu \geq 1$. Then*

$$S^m_{\lambda,\ell,\mu+1}(\eta;A,B;\alpha) \subset S^m_{\lambda,\ell,\mu}(\eta;A,B;\alpha) \subset S^{m+1}_{\lambda,\ell,\mu}(\eta;A,B;\alpha)$$

$(0 \leq \mu < 1;\ -1 \leq B < A \leq 1;\ 0 < \alpha \leq 1)$, and

$$K^m_{\lambda,\ell,\mu+1}(\eta;A,B;\alpha) \subset K^m_{\lambda,\ell,\mu}(\eta;A,B;\alpha) \subset K^{m+1}_{\lambda,\ell,\mu}(\eta;A,B;\alpha)$$

$(0 \leq \mu < 1;\ -1 \leq B < A \leq 1;\ 0 < \alpha \leq 1)$.

By using Lemma 2, we obtain the following inclusion relation of the class $C^m_{\lambda,\ell,\mu}(\eta,\delta;\phi,\psi)$.

Theorem 3. *Let $m \in N_0$, $\lambda > 0$, $\ell \geq 0, \ell + 1 > \lambda$ and $\mu \geq 1$. Then*

$$C^m_{\lambda,\ell,\mu+1}(\eta,\delta;\varphi,\psi) \subset C^m_{\lambda,\ell,\mu}(\eta,\delta;\varphi,\psi) \subset C^{m+1}_{\lambda,\ell,\mu}(\eta,\delta;\varphi,\psi)$$

$(0 \leq \eta;\ \delta < 1;\ \varphi, \psi \in S)$.

Proof. We begin by proving that

$$C^m_{\lambda,\ell,\mu+1}(\eta,\delta;\varphi,\psi) \subset C^m_{\lambda,\ell,\mu}(\eta,\delta;\varphi,\psi).$$

Let $f \in C^m_{\lambda,\ell,\mu+1}(\eta,\delta;\varphi,\psi)$. Then, in view of the definition of the class $C^m_{\lambda,\ell,\mu+1}(\eta,\delta;\varphi,\psi)$, there exists a function $g \in S^m_{\lambda,\ell,\mu+1}(\eta;\varphi)$ such that

$$\frac{1}{1-\delta}\left(\frac{z(I^m_{\lambda,\ell,\mu+1}f(z))'}{I^m_{\lambda,\ell,\mu+1}g(z)} - \delta\right) \prec \psi(z) \quad (z \in U).$$

Now let

$$p(z) = \frac{1}{1-\delta}\left(\frac{z(I^m_{\lambda,\ell,\mu}f(z))'}{I^m_{\lambda,\ell,\mu}g(z)} - \delta\right),$$

where p is analytic in U with $p(0) = 1$. Using the identity (1.10), we obtain

$$[(1-\delta)p(z) + \delta]\, I^m_{\lambda,\ell,\mu}g(z) + (\mu - 1)I^m_{\lambda,\ell,\mu}f(z) = \mu I^m_{\lambda,\ell,\mu+1}f(z). \tag{2.4}$$

Differentiating (2.4) with respect to z and multiplying by z, we have

$$\begin{aligned}(1-\delta)zp'(z)I^m_{\lambda,\ell,\mu}g(z) + [(1-\delta)p(z)+\delta]\, z(I^m_{\lambda,\ell,\mu}g(z))' \\ = \mu z(I^m_{\lambda,\ell,\mu+1}f(z))' - (\mu-1)z(I^m_{\lambda,\ell,\mu}f(z))'.\end{aligned} \tag{2.5}$$

Since $g \in S^m_{\lambda,\ell,\mu+1}(\eta;\varphi)$, by Theorem 1, we know that $g \in S^m_{\lambda,\ell,\mu}(\eta;\varphi)$. Let

$$q(z) = \frac{1}{1-\eta}\left(\frac{z(I^m_{\lambda,\ell,\mu}g(z))'}{I^m_{\lambda,\ell,\mu}g(z)} - \eta\right).$$

Then, using the identity (1.10) once again, we obtain

$$\mu\frac{I^m_{\lambda,\ell,\mu+1}g(z)}{I^m_{\lambda,\ell,\mu}g(z)} = (1-\eta)q(z) + \eta + (\mu - 1). \tag{2.6}$$

From (2.5) and (2.6), we have

$$\frac{1}{1-\delta}\left(\frac{z(I^m_{\lambda,\ell,\mu+1}f(z))'}{I^m_{\lambda,\ell,\mu+1}g(z)} - \delta\right) = p(z) + \frac{zp'(z)}{(1-\eta)q(z) + \eta + (\mu-1)}.$$

Since $0 \le \eta < 1$, $\mu \ge 1$ and $q \prec \varphi$ in U,

$$\mathrm{Re}\,\{(1-\eta)q(z) + \eta + \mu - 1\} > 0$$

($z \in U$). Hence applying Lemma 2, we can show that $p \prec \psi$, so that $f \in C^m_{\lambda,\ell,\mu}(\eta;\delta;\varphi,\psi)$.

For the second part, by using the arguments similar to those detailed above with (1.9), we obtain

$$C^m_{\lambda,\ell,\mu}(\eta,\delta;\varphi,\psi) \subset C^{m+1}_{\lambda,\ell,\mu}(\eta,\delta;\varphi,\psi).$$

This completes the proof of Theorem 3. □

3. Inclusion properties involving the integral operator F_c. In this section, we consider the generalized Libera integral operator F_c (see [16], [2] and [9]) defined by

$$F_c(f) = F_c(f)(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \tag{3.1}$$

($c > -1$; $f \in A$). We first prove the following theorem.

Theorem 4. *Let $c, \lambda \ge 0$, $m \in N_0$, $\ell \ge 0$ and $\mu > 0$. If $f \in S^m_{\lambda,\ell,\mu}(\eta;\varphi)$ ($0 \le \eta < 1$; $\varphi \in S$), then $F_c(f) \in S^m_{\lambda,\ell,\mu}(\eta;\varphi)$ ($0 \le \eta < 1$; $\varphi \in S$).*

Proof. Let $f \in S^m_{\lambda,\ell,\mu}(\eta;\varphi)$ and put

$$p(z) = \frac{1}{1-\eta}\left(\frac{z(I^m_{\lambda,\ell,\mu}F_c(f)(z))'}{I^m_{\lambda,\ell,\mu}F_c(f)(z)} - \eta\right), \tag{3.2}$$

where p is analytic in U with $p(0) = 1$. From (3.1), we have

$$z(I^m_{\lambda,\ell,\mu}F_c(f)(z))' = (c+1)I^m_{\lambda,\ell,\mu}f(z) - cI^m_{\lambda,\ell,\mu}F_c(f)(z). \tag{3.3}$$

Then, by using (3.2) and (3.3), we have

$$(c+1)\frac{I^m_{\lambda,\ell,\mu}f(z)}{I^m_{\lambda,\ell,\mu}F_c(f)(z)} = (1-\eta)p(z) + \eta + c. \tag{3.4}$$

Differentiating (3.4) logarithmically with respect to z and multiplying by z, we have

$$p(z) + \frac{zp'(z)}{(1-\eta)p(z)+\eta+c} = \frac{1}{1-\eta}\left(\frac{z(I^m_{\lambda,\ell,\mu}f(z))'}{I^m_{\lambda,\ell,\mu}f(z)} - \eta\right) \quad (z \in U).$$

Hence, by virtue of Lemma 1, we conclude that $p \prec \varphi$ $(z \in U)$, which implies that $F_c(f) \in S^m_{\lambda,\ell,\mu}(\eta;\varphi)$. □

Next, we derive an inclusion property involving F_c, which is given by the following theorem.

Theorem 5. *Let $c, \ell \geq 0$, $m \in N_0$, $\lambda \geq 0$ and $\mu > 0$. If $f \in K^m_{\lambda,\ell,\mu}(\eta;\varphi)$ $(0 \leq \eta < 1;\ \varphi \in S)$, then $F_c(f) \in K^m_{\lambda,\ell,\mu}(\eta;\varphi)$ $(0 \leq \eta < 1;\ \varphi \in S)$.*

Proof. By applying Theorem 4, it follows that

$$\begin{aligned} f(z) \in K^m_{\lambda,\ell,\mu}(\eta;\varphi) &\Leftrightarrow zf'(z) \in S^m_{\lambda,\ell,\mu}(\eta;\varphi) \\ &\Rightarrow F_c(zf'(z)) \in S^m_{\lambda,\ell,\mu}(\eta;\varphi) \\ &\Leftrightarrow z(F_c(f)(z))' \in S^m_{\lambda,\ell,\mu}(\eta;\varphi) \\ &\Leftrightarrow F_c(f)(z) \in K^m_{\lambda,\ell,\mu}(\eta;\varphi), \end{aligned}$$

which proves Theorem 5. □

From Theorem 4 and Theorem 5, we have the following corollary.

Corollary 2. *Let $c, \ell \geq 0$, $m \in N_0$, $\lambda > 0$ and $\mu > 0$. If $f \in S^m_{\lambda,\ell,\mu}(\eta;A,B;\alpha)$ (or $K^m_{\lambda,\ell,\mu}(\eta;A,B;\alpha)$) $(0 \leq \eta < 1;\ -1 \leq B < A \leq 1;\ 0 < \alpha \leq 1)$, then $F_c(f)$ belongs to the class $S^m_{\lambda,\ell,\mu}(\eta;A,B;\alpha)$ (or $K^m_{\lambda,\ell,\mu}(\eta;A,B;\alpha)$) $(0 \leq \eta < 1;$ $-1 \leq B < A \leq 1;\ 0 < \alpha \leq 1)$.*

Finally, we prove the following theorem.

Theorem 6. *Let $c, \ell \geq 0$, $m \in N_0$, $\lambda > 0$ and $\mu > 0$. If $f \in C^m_{\lambda,\ell,\mu}(\eta;\delta,\varphi;\psi)$ $(0 \leq \eta;\ \delta < 1;\ \varphi,\psi \in S)$, then $F_c(f) \in C^m_{\lambda,\ell,\mu}(\eta;\delta,\varphi;\psi)$ $(0 \leq \eta;\ \delta < 1;$ $\varphi,\psi \in S)$.*

Proof. Let $f \in C^m_{\lambda,\ell,\mu}(\eta;\delta,\varphi;\psi)$. Then, in view of the definition of the class $C^m_{\lambda,\ell,\mu}(\eta;\delta,\varphi;\psi)$, there exists a function $g \in S^m_{\lambda,\ell,\mu}(\eta;\varphi)$ such that

$$\frac{1}{1-\delta}\left(\frac{z(I^m_{\lambda,\ell,\mu}f(z))'}{I^m_{\lambda,\ell,\mu}g(z)}-\delta\right) \prec \psi(z) \quad (z \in U).$$

Thus, we put

$$p(z)=\frac{1}{1-\delta}\left(\frac{z(I^m_{\lambda,\ell,\mu}F_c(f)(z))'}{I^m_{\lambda,\ell,\mu}F_c(g)(z)}-\delta\right),$$

where p is analytic in U with $p(0)=1$. Since $g \in S^m_{\lambda,\ell,\mu}(\eta;\varphi)$, we see from Theorem 4 that $F_c(g) \in S^m_{\lambda,\ell,\mu}(\eta;\varphi)$. Using (3.3), we have

$$[(1-\delta)p(z)+\delta]\, I^m_{\lambda,\ell,\mu}F_c(g)(z)+cI^m_{\lambda,\ell,\mu}F_c(f)(z)=(c+1)I^m_{\lambda,\ell,\mu}f(z).$$

Then, by a simple calculations, we get

$$(c+1)\frac{z(I^m_{\lambda,\ell,\mu}f(z))'}{I^m_{\lambda,\ell,\mu}F_c(g)(z)}=[(1-\delta)p(z)+\delta]\,[(1-\eta)q(z)+\eta+c]+(1-\delta)zp'(z),$$

where

$$q(z)=\frac{1}{1-\eta}\left(\frac{z(I^m_{\lambda,\ell,\mu}F_c(g)(z))'}{I^m_{\lambda,\ell,\mu}F_c(g)(z)}-\eta\right).$$

Hence, we have

$$\frac{1}{1-\delta}\left(\frac{z(I^m_{\lambda,\ell,\mu}f(z))'}{I^m_{\lambda,\ell,\mu}g(z)}-\delta\right)=p(z)+\frac{zp'(z)}{(1-\eta)q(z)+\eta+c}.$$

The remaining part of the proof of Theorem 6 is similar to that of Theorem 3 and so we omit it. □

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