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On certain coefficient bounds for multivalent functions

Abstract. In the present paper, the authors obtain sharp upper bounds for certain coefficient inequalities for linear combination of Mocanu α-convex p-valent functions. Sharp bounds for $\left|a_{p+2} - \mu a_{p+1}^2\right|$ and $|a_{p+3}|$ are derived for multivalent functions.

1. Introduction. Let A_p denote the class of all analytic functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \tag{1.1}$$

defined in the open unit disk

$$\Delta = \{z : z \in \mathbb{C}; |z| < 1\}$$

and let $A_1 := A$. For $f(z)$ given by (1.1) and $g(z)$ given by

$$g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n$$

their convolution (or Hadamard product), denoted by $(f * g)$, is defined as

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n.$$

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The function $f(z)$ is subordinate to the function $g(z)$, written $f(z) \prec g(z)$, provided there is an analytic function $w(z)$ defined on Δ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. In particular, if the function g is univalent in Δ, the above subordination is equivalent to $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$.

Let $\varphi(z)$ be an analytic function with positive real part on Δ with $\varphi(0) = 1$, $\varphi^{'}(0) > 0$ which maps the open unit disk Δ onto a region starlike with respect to 1 and is symmetric with respect to the real axis. R. M. Ali et al. [1] defined and studied the class $S^*_{b,p}(\varphi)$ consisting of functions $f \in A_p$ for which

$$1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf^{'}(z)}{f(z)} - 1\right) \prec \varphi(z) \quad (z \in \Delta,\ b \in \mathbb{C} \setminus \{0\}), \tag{1.2}$$

and the class $C_{b,p}(\varphi)$ of all functions $f \in A_p$ for which

$$1 - \frac{1}{b} + \frac{1}{bp}\left(1 + \frac{zf^{''}(z)}{f^{'}(z)}\right) \prec \varphi(z) \quad (z \in \Delta,\ b \in \mathbb{C} \setminus \{0\}). \tag{1.3}$$

R. M. Ali et al. [1] also defined and studied the class $R_{b,p}(\varphi)$ to be the class of all functions $f \in A_p$ for which

$$1 + \frac{1}{b}\left(\frac{f^{'}(z)}{pz^{p-1}} - 1\right) \prec \varphi(z) \quad (z \in \Delta,\ b \in \mathbb{C} \setminus \{0\}). \tag{1.4}$$

Note that $S^*_{1,1}(\varphi) = S^*(\varphi)$ and $C_{1,1}(\varphi) = C(\varphi)$, the classes introduced and studied by Ma and Minda [3]. The familiar class $S^*(\alpha)$ of starlike functions of order α and the class $C(\alpha)$ of convex functions of order α, $0 \le \alpha < 1$ are the special case of $S^*_{1,1}(\varphi)$ and $C_{1,1}(\varphi)$, respectively, when $\varphi(z) = (1 + (1 - 2\alpha) z) / (1 - z)$.

Owa [4] introduced and studied the class $H_p(A, B, \alpha, \beta)$ of all functions $f \in A_p$ satisfying

$$(1-\beta)\left(\frac{f(z)}{z^p}\right)^{\alpha} + \beta\frac{zf^{'}(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\alpha} \prec \frac{1+Az}{1+Bz} \tag{1.5}$$

where $z \in \Delta$, $-1 \le B < A \le 1$, $0 \le \beta \le 1$, $\alpha \ge 0$.

A function $f \in A_p$ is in the class $R_{b,p,\alpha,\beta}(\varphi)$ if

$$1 + \frac{1}{b}\left\{(1-\beta)\left(\frac{f(z)}{z^p}\right)^{\alpha} + \beta\frac{zf^{'}(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\alpha} - 1\right\} \prec \varphi(z) \tag{1.6}$$

$(0 \le \beta \le 1,\ \alpha \ge 0)$. This class is defined and studied by Ramachandran et al. [6].

The class of functions which unifies the classes $S^*_{b,p}(\varphi)$ and $C_{b,p}(\varphi)$ introduced by T. N. Shanmugam, S. Owa, C. Ramachandran, S. Sivasubramanian and Y. Nakamura [7]. They gave the definition in the following way:

Let $\varphi(z)$ be a univalent starlike function with respect to 1 which maps the open unit disk Δ onto a region in the right half plane and is symmetric with respect to real axis, $\varphi(0) = 1$ and $\varphi'(0) > 0$. A function $f \in A_p$ is in the class $M_{b,p,\alpha,\lambda}(\varphi)$ if

$$1 + \frac{1}{b}\left[\frac{1}{p}\left((1-\alpha)\frac{zF'(z)}{F(z)} + \alpha\left(1 + \frac{zF''(z)}{F'(z)}\right)\right) - 1\right] \prec \varphi(z) \tag{1.7}$$

$(0 \le \alpha \le 1)$, where

$$F(z) := (1-\lambda)\, f(z) + \lambda z f'(z).$$

T. N. Shanmugam et al. [7] obtained certain coefficient inequalities for function $f \in A_p$ in the class $M_{b,p,\alpha,\lambda}(\varphi)$.

In the present paper, we define a class of functions $f \in A_p$ in the following way: Using the Sălăgean operator [8], we can write the following equalities for the functions $f(z)$ belonging to the class A_p:

$$D^0 f(z) = f(z),$$

$$D^1\left(f(z)\right) = Df(z) = \frac{z}{p} f'(z) = z^p + \sum_{n=p+1}^{\infty} \frac{n}{p} a_n z^n,$$

$$D^2\left(f(z)\right) = D(Df(z)) = \frac{z}{p}\left(z^p + \sum_{n=p+1}^{\infty} \frac{n}{p} a_n z^n\right)' = z^p + \sum_{n=p+1}^{\infty} \frac{n^2}{p^2} a_n z^n,$$

$$\vdots$$

$$D^m\left(f(z)\right) = D(D^{m-1} f(z)) = z^p + \sum_{n=p+1}^{\infty} \frac{n^m}{p^m} a_n z^n.$$

Definition 1.1. Let $\varphi(z)$ be a univalent starlike function with respect to 1 which maps the open unit disk Δ onto a region in the right half plane and is symmetric with respect to real axis, $\varphi(0) = 1$ and $\varphi'(0) > 0$. A function $f \in A_p$ is in the class $M_{p,b,\alpha,\lambda,m}(\varphi)$ if

$$1 + \frac{1}{b}\left[\frac{1}{p}\left((1-\alpha)\frac{zF'_{\lambda,m}(z)}{F_{\lambda,m}(z)} + \alpha\left(1 + \frac{zF''_{\lambda,m}(z)}{F'_{\lambda,m}(z)}\right)\right) - 1\right] \prec \varphi(z) \tag{1.8}$$

$(0 \le \alpha \le 1,\ m \in \mathbb{N}_0 = \{0, 1, 2, \dots\})$, where

$$F_{\lambda,m}(z) = (1-\lambda)\, D^m f(z) + \lambda D^{m+1} f(z) \quad (0 \le \lambda \le 1)\,.$$

Also, $M_{b,p,\alpha,\lambda,m,g}(\varphi)$ is the class of all functions $f \in A_p$ for which $f * g \in M_{b,p,\alpha,\lambda,m}(\varphi)$.

The classes $M_{b,p,\alpha,\lambda,m}(\varphi)$ reduce to the following classes.

The classes $M_{1,1,1,0,0}(\varphi) \equiv C(\varphi)$, $M_{1,1,0,0,0}(\varphi) \equiv S^*(\varphi)$ were introduced and studied by Ma and Minda [3]. Also, the classes $M_{p,1,0,0,0}(\varphi) \equiv S_p^*(\varphi)$,

$M_{p,1,1,0,0}(\varphi) \equiv C_p(\varphi)$, $M_{p,b,0,0,0}(\varphi) \equiv S^*_{b,p}(\varphi)$ and $M_{p,b,1,0,0}(\varphi) \equiv C_{b,p}(\varphi)$ were introduced and studied by Ali et al. [1].

In the present paper, we prove the sharp coefficient inequality for a more general class of analytic functions which we have defined above in Definition 1.1. The results obtained in this paper generalizes the results obtained by Ali et al. [1].

2. Coefficient bounds. Let Ω be the class of analytic functions of the form

$$w(z) = w_1 z + w_2 z^2 + \dots \tag{2.1}$$

in the open unit disk Δ satisfying $|w(z)| < 1$.

To prove our main result, we need the following lemmas:

Lemma 2.1 ([1]). *If $w \in \Omega$, then*

$$\left|w_2 - t w_1^2\right| \le \begin{cases} -t & \text{if } t \le -1, \\ 1 & \text{if } -1 \le t \le 1, \\ t & \text{if } t \ge 1. \end{cases} \tag{2.2}$$

When $t < -1$ or $t > 1$, the equality holds if and only if $w(z) = z$ or one of its rotations.

If $-1 < t < 1$, then equality holds if and only if $w(z) = z^2$ or one of its rotations.

Equality holds for $t = -1$ if and only if

$$w(z) = \frac{z(z+\lambda)}{1+\lambda z} \quad (0 \le \lambda \le 1)$$

or one of its rotations, while for $t = 1$ the equality holds if and only if

$$w(z) = -\frac{z(z+\lambda)}{1+\lambda z} \quad (0 \le \lambda \le 1)$$

or one of its rotations.

Although the above upper bound is sharp, it can be improved as follows when $-1 < t < 1$:

$$\left|w_2 - t w_1^2\right| + (t+1)\left|w_1\right|^2 \le 1 \quad (-1 < t \le 0) \tag{2.3}$$

and

$$\left|w_2 - t w_1^2\right| + (1-t)\left|w_1\right|^2 \le 1 \quad (0 < t < 1). \tag{2.4}$$

Lemma 2.2 ([2]). *If $w \in \Omega$, then for any complex number t*

$$\left|w_2 - t w_1^2\right| \le \max\{1; |t|\}. \tag{2.5}$$

The result is sharp for the functions $w(z) = z$ or $w(z) = z^2$.

Lemma 2.3 ([5]). *If $w \in \Omega$, then for any real numbers q_1 and q_2 the following sharp estimate holds:*

$$\left|w_3 + q_1 w_1 w_2 + q_2 w_1^3\right| \le H(q_1, q_2) \tag{2.6}$$

where

$$H(q_1,q_2) = \begin{cases} 1 & \text{for } (q_1,q_2) \in D_1 \cup D_2, \\ |q_2| & \text{for } (q_1,q_2) \in \bigcup_{k=3}^{7} D_k, \\ \frac{2}{3}\left(|q_1|+1\right)\left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{\frac{1}{2}} & \text{for } (q_1,q_2) \in D_8 \cup D_9, \\ \frac{q_2}{3}\left(\frac{q_1^2-4}{q_1^2-4q_2}\right)\left(\frac{q_1^2-4}{3(q_2-1)}\right)^{\frac{1}{2}} & \text{for } (q_1,q_2) \in D_{10} \cup D_{11} \setminus \{\pm 2, 1\}, \\ \frac{2}{3}\left(|q_1|-1\right)\left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{\frac{1}{2}} & \text{for } (q_1,q_2) \in D_{12}. \end{cases}$$

The extremal functions, up to rotations, are of the form

$$w(z) = z^3, \quad w(z) = z, \quad w(z) = w_0(z) = \frac{\left(z\left[(1-\lambda)\,\varepsilon_2 + \lambda\varepsilon_1\right] - \varepsilon_1\varepsilon_2 z\right)}{1 - \left[(1-\lambda)\,\varepsilon_1 + \lambda\varepsilon_2\right] z},$$

$$w(z) = w_1(z) = \frac{z\left(t_1 - z\right)}{1 - t_1 z}, \quad w(z) = w_2(z) = \frac{z\left(t_2 + z\right)}{1 + t_2 z},$$

$$|\varepsilon_1| = |\varepsilon_2| = 1, \quad \varepsilon_1 = t_0 - e^{\frac{-i\theta_0}{2}}\left(a \mp b\right), \quad \varepsilon_2 = -e^{\frac{-i\theta_0}{2}}\left(ia \pm b\right),$$

$$a = t_0 \cos\frac{\theta_0}{2}, \quad b = \sqrt{1 - t_0^2 \sin^2\frac{\theta_0}{2}}, \quad \lambda = \frac{b \pm a}{2b},$$

$$t_0 = \left[\frac{2q_2\left(q_1^2+2\right) - 3q_1^2}{3\left(q_2-1\right)\left(q_1^2+4q_2\right)}\right]^{\frac{1}{2}}, \quad t_1 = \left(\frac{|q_1|+1}{3\left(|q_1|+1+q_2\right)}\right)^{\frac{1}{2}},$$

$$t_2 = \left(\frac{|q_1|-1}{3\left(|q_1|-1-q_2\right)}\right)^{\frac{1}{2}}, \quad \cos\frac{\theta_0}{2} = \frac{q_1}{2}\left[\frac{q_2\left(q_1^2+8\right) - 2\left(q_1^2+2\right)}{2q_2\left(q_1^2+2\right) - 3q_1^2}\right].$$

The sets D_k, $k = 1, 2, \ldots, 12$, are defined as follows:

$$D_1 = \left\{(q_1,q_2) : |q_1| \le \frac{1}{2},\ |q_2| \le 1\right\},$$

$$D_2 = \left\{(q_1,q_2) : \frac{1}{2} \le |q_1| \le 2,\ \frac{4}{27}\left(|q_1|+1\right)^3 - \left(|q_1|+1\right) \le q_2 \le 1\right\},$$

$$D_3 = \left\{(q_1,q_2) : |q_1| \le \frac{1}{2},\ q_2 \le -1\right\},$$

$$D_4 = \left\{(q_1,q_2) : |q_1| \ge \frac{1}{2},\ q_2 \le -\frac{2}{3}\left(|q_1|+1\right)\right\},$$

$$D_5 = \{(q_1,q_2) : |q_1| \le 2,\ q_2 \ge 1\},$$

$$D_6 = \left\{(q_1,q_2) : 2 \le |q_1| \le 4,\ q_2 \ge \frac{1}{12}\left(q_1^2+8\right)\right\},$$

$$D_7 = \left\{(q_1,q_2) : |q_1| \ge 4, q_2 \ge \frac{2}{3}\left(|q_1|-1\right)\right\},$$

$$D_8 = \left\{(q_1,q_2) : \frac{1}{2} \le |q_1| \le 2,\right.$$
$$\left. -\frac{2}{3}\left(|q_1|+1\right) \le q_2 \le \frac{4}{27}\left(|q_1|+1\right)^3 - \left(|q_1|+1\right)\right\},$$

$$D_9 = \left\{(q_1,q_2) : |q_1| \ge 2,\ -\frac{2}{3}\left(|q_1|+1\right) \le q_2 \le \frac{2|q_1|\left(|q_1|+1\right)}{q_1^2+2|q_1|+4}\right\},$$

$$D_{10} = \left\{(q_1,q_2) : 2 \le |q_1| \le 4,\ \frac{2|q_1|\left(|q_1|+1\right)}{q_1^2+2|q_1|+4} \le q_2 \le \frac{1}{12}\left(q_1^2+8\right)\right\},$$

$$D_{11} = \left\{(q_1,q_2) : |q_1| \ge 4,\ \frac{2|q_1|\left(|q_1|+1\right)}{q_1^2+2|q_1|+4} \le q_2 \le \frac{2|q_1|\left(|q_1|-1\right)}{q_1^2-2|q_1|+4}\right\},$$

$$D_{12} = \left\{(q_1,q_2) : |q_1| \ge 4,\ \frac{2|q_1|\left(|q_1|-1\right)}{q_1^2-2|q_1|+4} \le q_2 \le \frac{2}{3}\left(|q_1|-1\right)\right\}.$$

By making use of the Lemmas 2.1–2.3, we prove the following bounds for the class $M_{p,1,\alpha,\lambda,m}(\varphi)$.

Theorem 2.4. *Let $\varphi(z) = 1+B_1z+B_2z^2+B_3z^3+\dots$, where B_n's are real with $B_1>0$ and $B_2 \ge 0$. Let $m \in \mathbb{N}_0$, $0 \le \alpha \le 1$, $0 \le \mu \le 1$, $0 \le \lambda \le 1$, and*

$$\sigma_1 := \frac{(p+\alpha)^2(p+\lambda)^2(p+1)^{2m}}{2p^3B_1^2p^m(p+2\alpha)(p+2\lambda)(p+2)^m}\left\{B_2-B_1+pB_1^2\left(\frac{p^2+2\alpha p+\alpha}{(p+\alpha)^2}\right)\right\},$$

$$\sigma_2 := \frac{(p+\alpha)^2(p+\lambda)^2(p+1)^{2m}}{2p^3B_1^2p^m(p+2\alpha)(p+2\lambda)(p+2)^m}\left\{B_2+B_1+pB_1^2\left(\frac{p^2+2\alpha p+\alpha}{(p+\alpha)^2}\right)\right\},$$

$$\sigma_3 := \frac{(p+\alpha)^2(p+\lambda)^2(p+1)^{2m}}{2p^3B_1^2p^m(p+2\alpha)(p+2\lambda)(p+2)^m}\left\{B_2+pB_1^2\left(\frac{p^2+2\alpha p+\alpha}{(p+\alpha)^2}\right)\right\},$$

$$\xi(p,\alpha,\lambda,\mu,m) = \frac{p^2+2\alpha p+\alpha}{(p+\alpha)^2} - 2\mu p^2\frac{p^m(p+2)^m(p+2\alpha)(p+2\lambda)}{(p+\alpha)^2(p+\lambda)^2(p+1)^{2m}}.$$

If $f(z)$ given by (1.1) belongs to $M_{p,1,\alpha,\lambda,m}(\varphi)$, then

$$|a_{p+2} - \mu a_{p+1}^2| \leq \tag{2.7}$$
$$\begin{cases} \frac{p^3}{2(p+2\alpha)(p+2\lambda)} \left(\frac{p}{p+2}\right)^m \left\{B_2 + pB_1^2 \xi\left(p,\alpha,\lambda,\mu,m\right)\right\} & \textit{if } \mu \leq \sigma_1, \\ \frac{p^3 B_1}{2(p+2\alpha)(p+2\lambda)} \left(\frac{p}{p+2}\right)^m & \textit{if } \sigma_1 \leq \mu \leq \sigma_2, \\ -\frac{p^3}{2(p+2\alpha)(p+2\lambda)} \left(\frac{p}{p+2}\right)^m \left\{B_2 + pB_1^2 \xi\left(p,\alpha,\lambda,\mu,m\right)\right\} & \textit{if } \mu \geq \sigma_2. \end{cases}$$

For any complex number μ, there is the following inequality

$$\left|a_{p+2} - \mu a_{p+2}^2\right| \leq \tag{2.8}$$
$$\frac{p^3 B_1}{2\left(p+2\alpha\right)\left(p+2\lambda\right)} \left(\frac{p}{p+2}\right)^m \max\left\{1, \left|\frac{B_2}{B_1} + pB_1 \xi\left(p,\alpha,\lambda,\mu,m\right)\right|\right\}.$$

Further,

$$|a_{p+3}| \leq \frac{p^3 B_1}{3\left(p+3\alpha\right)\left(p+3\lambda\right)} \left(\frac{p}{p+3}\right)^m H\left(q_1,q_2\right), \tag{2.9}$$

where $H\left(q_1,q_2\right)$ is as defined in Lemma 2.3,

$$q_1 = 2\left(\frac{B_2}{B_1}\right) + \frac{3\left(p^2+3\alpha p+2\alpha\right)}{2\left(p+\alpha\right)\left(p+2\alpha\right)} pB_1,$$

$$q_2 = \frac{B_3}{B_1} + \frac{3\left(p^2+3\alpha p+2\alpha\right)}{2\left(p+\alpha\right)\left(p+2\alpha\right)} pB_2$$
$$+ \left[\frac{3\left(p^2+3\alpha p+2\alpha\right)}{2\left(p+\alpha\right)\left(p+2\alpha\right)} \frac{p^2+2\alpha p+\alpha}{\left(p+\alpha\right)^2} - \frac{p^3 3\alpha^2 p + 3\alpha p + \alpha}{\left(p+\alpha\right)^3}\right] p^2 B_1^2.$$

These results are sharp.

Proof. If $f(z) \in M_{p,1,\alpha,\lambda,m}(\varphi)$, then there is a Schwarz function

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \ldots \in \Omega$$

such that

$$\frac{1}{p}\left\{(1-\alpha)\frac{zF'_{\lambda,m}(z)}{F_{\lambda,m}(z)} + \alpha\left(1 + \frac{zF''_{\lambda,m}(z)}{F'_{\lambda,m}(z)}\right)\right\} = \varphi(w(z)) \tag{2.10}$$

where $0 \leq \alpha \leq 1$, $m \in \mathbb{N}_0 = \{0,1,2,\ldots\}$ and $F_{\lambda,m}(z) = (1-\lambda)\, D^m f(z) + \lambda D^{m+1} f(z)$.

By definition of $D^m f(z)$ and $F_{\lambda,m}(z)$, we can write

$$\begin{aligned} F_{\lambda,m}(z) &= z^p + \frac{1}{p}\sum_{n=p+1}^{\infty}\left(\frac{n}{p}\right)^m [p+(n-p)\lambda]\, a_n z^n \\ &= z^p + \frac{1}{p}\left(\frac{p+1}{p}\right)^m (p+\lambda)\, a_{p+1} z^{p+1} \\ &\quad + \frac{1}{p}\left(\frac{p+2}{p}\right)^m (p+2\lambda)\, a_{p+2} z^{p+2} \\ &\quad + \frac{1}{p}\left(\frac{p+3}{p}\right)^m (p+3\lambda)\, a_{p+3} z^{p+3} + \dots . \end{aligned} \tag{2.11}$$

Let

$$A_{p+c} = \frac{1}{p}\left(\frac{p+c}{p}\right)^m (p+c\lambda)\, a_{p+c}\ ; \quad c \in \mathbb{N} = \{1,2,3,\dots\}.$$

Then, we have

$$F_{\lambda,m}(z) = z^p + A_{p+1}z^{p+1} + A_{p+2}z^{p+2} + A_{p+3}z^{p+3} + \dots \tag{2.12}$$

and, differentiating both sides of the (2.12), we obtain the following equality

$$\begin{aligned} F'_{\lambda,m}(z) &= pz^{p-1} + (p+1)\, A_{p+1}z^p + (p+2)\, A_{p+2}z^{p+1} \\ &\quad + (p+3)\, A_{p+3}z^{p+2} + \dots . \end{aligned} \tag{2.13}$$

From (2.12) and (2.13), we deduce

$$\begin{aligned} \frac{zF'_{\lambda,m}(z)}{F_{\lambda,m}(z)} &= p + A_{p+1}z + \left(2A_{p+2} - A_{p+1}^2\right) z^2 \\ &\quad + \left(3A_{p+3} - 3A_{p+2}A_{p+1} + A_{p+1}^3\right) z^3 + \dots . \end{aligned} \tag{2.14}$$

Similarly, we can write

$$F''_{\lambda,m} = p\,(p-1)\, z^{p-2} + p\,(p+1)\, A_{p+1}z^{p-1} + (p+1)\,(p+2)\, A_{p+2}z^p + \dots$$

and

$$\frac{zF''_{\lambda,m}}{F'_{\lambda,m}} = \frac{p\,(p-1) + p\,(p+1)\, A_{p+1}z + (p+1)\,(p+2)\, A_{p+2}z^2 + \dots}{p + (p+1)\, A_{p+1}z + (p+2)\, A_{p+2}z^2 + \dots}.$$

If we take $B_{p+c} = (p+c)\, A_{p+c}$, we have

$$\begin{aligned} \frac{zF''_{\lambda,m}}{F'_{\lambda,m}} &= \frac{p\,(p-1) + pB_{p+1}z + (p+1)B_{p+2}z^2 + (p+2)\, B_{p+3}z^3 + \dots}{p + B_{p+1}z + B_{p+2}z^2 + B_{p+3}z^3 + \dots} \\ &= p - 1 + \frac{1}{p}B_{p+1}z + \frac{1}{p}\left(2B_{p+2} - \frac{1}{p}B_{p+1}^2\right) z^2 \\ &\quad + \frac{1}{p}\left(3B_{p+3} - \frac{3}{p}B_{p+2}B_{p+1} + \frac{1}{p^2}B_{p+1}^3\right) z^3 + \dots . \end{aligned} \tag{2.15}$$

Since

$$\begin{aligned}(2.16)\quad &\frac{1}{p}\left\{(1-\alpha)\frac{zF'_{\lambda,m}}{F_{\lambda,m}}+\alpha\left(1+\frac{zF''_{\lambda,m}}{F'_{\lambda,m}}\right)\right\}=\frac{1}{p}\Big\{(1-\alpha)\left[p+A_{p+1}z\right.\\
&+\left(2A_{p+2}-A_{p+1}^2\right)z^2+\left(3A_{p+3}-3A_{p+2}A_{p+1}+A_{p+1}^3\right)z^3+\ldots\left.\right]\\
&+\alpha\left[1+p-1+\frac{1}{p}B_{p+1}z+\frac{1}{p}\left(2B_{p+2}-\frac{1}{p}B_{p+1}^2\right)z^2\right.\\
&\left.+\frac{1}{p}\left(3B_{p+3}-\frac{3}{p}B_{p+2}B_{p+1}+\frac{1}{p^2}B_{p+1}^3\right)z^3+\ldots\right]\Big\}\\
&=1+\frac{1}{p}\left(\frac{p+\alpha}{p}\right)A_{p+1}z\\
&+\frac{1}{p}\left[\frac{2(p+2\alpha)}{p}A_{p+2}-\frac{p^2+2\alpha p+\alpha}{p^2}A_{p+1}^2\right]z^2\\
&+\frac{1}{p}\left\{\frac{3}{p}(p+3\alpha)A_{p+3}-\frac{3}{p^2}\left(p^2+3\alpha p+2\alpha\right)A_{p+2}A_{p+1}\right.\\
&\left.+\frac{1}{p^3}\left(p^3+3\alpha p^2+3\alpha p+\alpha\right)A_{p+1}^3\right\}z^3+\ldots\end{aligned}$$

and we can write

$$\begin{aligned}(2.17)\quad \varphi(w(z))=1&+B_1w_1z+\left(B_1w_2+B_2w_1^2\right)z^2\\
&+\left(B_1w_3+2B_2w_1w_2+B_3w_1^3\right)z^3+\ldots,\end{aligned}$$

by using equality (2.10), we have the following equalities:

Firstly, from

$$B_1w_1=\frac{1}{p}\left(\frac{p+\alpha}{p}\right)\frac{1}{p}\left(\frac{p+1}{p}\right)^m(p+\lambda)\,a_{p+1}$$

we can write

$$(2.18)\qquad a_{p+1}=\frac{p^3B_1w_1}{(p+\alpha)(p+\lambda)}\left(\frac{p}{p+1}\right)^m.$$

Secondly, from

$$\begin{aligned}B_1w_2+B_2w_1^2=&\frac{2}{p}\left(\frac{p+2\alpha}{p}\right)\frac{1}{p}\left(\frac{p+2}{p}\right)^m(p+2\lambda)\,a_{p+2}\\
&-\frac{p^2+2\alpha p+\alpha}{p^3}\left[\frac{1}{p}\left(\frac{p+1}{p}\right)^m(p+\lambda)\frac{p^3B_1w_1}{(p+\alpha)(p+\lambda)}\left(\frac{p}{p+1}\right)^m\right]^2\end{aligned}$$

we can write

$$a_{p+2} = \frac{p^3 B_1}{2(p+2\alpha)(p+2\lambda)} \left(\frac{p}{p+2}\right)^m \times \left\{ w_2 - w_1^2 \left[-\frac{B_2}{B_1} - pB_1 \frac{p^2+2\alpha p+\alpha}{(p+\alpha)^2} \right] \right\}. \tag{2.19}$$

Thus, by using (2.18) and (2.19), we can write

$$\begin{aligned} a_{p+2} - \mu a_{p+1}^2 &= \frac{p^3 B_1}{2(p+2\alpha)(p+2\lambda)} \left(\frac{p}{p+2}\right)^m \\ &\quad \times \left\{ w_2 - w_1^2 \left[-\frac{B_2}{B_1} - pB_1 \frac{p^2+2\alpha p+\alpha}{(p+\alpha)^2} \right] \right\} \\ &\quad - \mu \left\{ \frac{p^6 B_1^2 w_1^2}{(p+\alpha)^2 (p+\lambda)^2} \left(\frac{p}{p+1}\right)^{2m} \right\} \\ &= \frac{p^3 B_1}{2(p+2\alpha)(p+2\lambda)} \left(\frac{p}{p+2}\right)^m \\ &\quad \times \left\{ w_2 - w_1^2 \left[-\frac{B_2}{B_1} - pB_1 \frac{p^2+2\alpha p+\alpha}{(p+\alpha)^2} \right.\right. \\ &\qquad \left.\left. + \ 2\mu p^3 B_1 \frac{p^m (p+2)^m (p+2\alpha)(p+2\lambda)}{(p+\alpha)^2 (p+\lambda)^2 (p+1)^{2m}} \right] \right\}. \end{aligned}$$

Let

$$\nu = -\frac{B_2}{B_1} - pB_1 \frac{p^2+2\alpha p+\alpha}{(p+\alpha)^2} + 2\mu p^3 B_1 \frac{p^m (p+2)^m (p+2\alpha)(p+2\lambda)}{(p+\alpha)^2 (p+\lambda)^2 (p+1)^{2m}}.$$

Therefore, we have

$$a_{p+2} - \mu a_{p+1}^2 = \frac{p^3 B_1}{2(p+2\alpha)(p+2\lambda)} \left(\frac{p}{p+2}\right)^m \left\{ w_2 - \nu w_1^2 \right\}. \tag{2.20}$$

By using Lemma 2.1, we can write for $\mu \leq \sigma_1$

$$\begin{aligned} \left| a_{p+2} - \mu a_{p+2}^2 \right| &\leq \frac{p^3 B_1}{2(p+2\alpha)(p+2\lambda)} \left(\frac{p}{p+2}\right)^m \\ &\quad \times \left\{ \frac{B_2}{B_1} + pB_1 \frac{p^2+2\alpha p+\alpha}{(p+\alpha)^2} \right. \\ &\qquad \left. - 2\mu p^3 B_1 \frac{p^m (p+2)^m (p+2\alpha)(p+2\lambda)}{(p+\alpha)^2 (p+\lambda)^2 (p+1)^{2m}} \right\} \end{aligned}$$

$$
\begin{aligned}
&= \frac{p^3}{2(p+2\alpha)(p+2\lambda)}\left(\frac{p}{p+2}\right)^m \\
&\quad\times\left\{B_2 + pB_1^2\left[\frac{p^2+2\alpha p+\alpha}{(p+\alpha)^2}\right.\right. \\
&\qquad\left.\left. -2\mu p^2\frac{p^m(p+2)^m(p+2\alpha)(p+2\lambda)}{(p+\alpha)^2(p+\lambda)^2(p+1)^{2m}}\right]\right\} \\
&= \frac{p^3}{2(p+2\alpha)(p+2\lambda)}\left(\frac{p}{p+2}\right)^m \\
&\quad\times\left\{B_2 + pB_1^2\xi(p,\alpha,\lambda,\mu,m)\right\}
\end{aligned}
$$

and for $\mu \geq \sigma_2$

$$
\begin{aligned}
\left|a_{p+2}-\mu a_{p+2}^2\right| &\leq -\frac{p^3B_1}{2(p+2\alpha)(p+2\lambda)}\left(\frac{p}{p+2}\right)^m \\
&\quad\times\left\{\frac{B_2}{B_1} + pB_1\frac{p^2+2\alpha p+\alpha}{(p+\alpha)^2}\right. \\
&\qquad\left. -2\mu p^3B_1\frac{p^m(p+2)^m(p+2\alpha)(p+2\lambda)}{(p+\alpha)^2(p+\lambda)^2(p+1)^{2m}}\right\} \\
&= -\frac{p^3}{2(p+2\alpha)(p+2\lambda)}\left(\frac{p}{p+2}\right)^m \\
&\quad\times\left\{B_2 + pB_1^2\left[\frac{p^2+2\alpha p+\alpha}{(p+\alpha)^2}\right.\right. \\
&\qquad\left.\left. -2\mu p^2\frac{p^m(p+2)^m(p+2\alpha)(p+2\lambda)}{(p+\alpha)^2(p+\lambda)^2(p+1)^{2m}}\right]\right\} \\
&= -\frac{p^3}{2(p+2\alpha)(p+2\lambda)}\left(\frac{p}{p+2}\right)^m \\
&\quad\times\left\{B_2 + pB_1^2\xi(p,\alpha,\lambda,\mu,m)\right\}
\end{aligned}
\tag{2.21}
$$

where

$$
\sigma_1 := \frac{(p+\alpha)^2(p+\lambda)^2(p+1)^{2m}}{2p^3B_1^2p^m(p+2\alpha)(p+2\lambda)(p+2)^m}\left\{B_2 - B_1 + pB_1^2\left(\frac{p^2+2\alpha p+\alpha}{(p+\alpha)^2}\right)\right\},
$$

$$
\sigma_2 := \frac{(p+\alpha)^2(p+\lambda)^2(p+1)^{2m}}{2p^3B_1^2p^m(p+2\alpha)(p+2\lambda)(p+2)^m}\left\{B_2 + B_1 + pB_1^2\left(\frac{p^2+2\alpha p+\alpha}{(p+\alpha)^2}\right)\right\}
$$

and

$$
\xi(p,\alpha,\lambda,\mu,m) = \frac{p^2+2\alpha p+\alpha}{(p+\alpha)^2} - 2\mu p^2\frac{p^m(p+2)^m(p+2\alpha)(p+2\lambda)}{(p+\alpha)^2(p+\lambda)^2(p+1)^{2m}}.
$$

Further, if $\sigma_1 \leq \mu \leq \sigma_3$, then

$$\begin{aligned} |a_{p+2} - \mu a_{p+2}^2| &+ \frac{1}{(p+2)^m} \frac{(p+\alpha)^2 (p+\lambda)^2 (p+1)^{2m}}{2p^3 B_1^2 p^m (p+2\alpha)(p+2\lambda)} \\ &\times \left\{ B_1 - B_2 - pB_1^2 \xi(p,\alpha,\lambda,\mu,m) \right\} |a_{p+1}|^2 \\ \leq& \frac{p^3 B_1}{2(p+2\alpha)(p+2\lambda)} \left(\frac{p}{p+2} \right)^m |w_2 - \nu w_1^2| \\ &+ \frac{1}{(p+2)^m} \frac{(p+\alpha)^2 (p+\lambda)^2 (p+1)^{2m}}{2p^3 B_1^2 p^m (p+2\alpha)(p+2\lambda)} \\ &\times \left\{ B_1 - B_2 - pB_1^2 \xi(p,\alpha,\lambda,\mu,m) \right\} \\ &\times \frac{p^6 B_1^2}{(p+\alpha)^2 (p+\lambda)^2} \frac{p^{2m}}{(p+1)^{2m}} |w_1|^2 \\ \leq& \frac{p^3 B_1}{2(p+2\alpha)(p+2\lambda)} \left(\frac{p}{p+2} \right)^m \\ &\times \left\{ |w_2 - \nu w_1^2| + (1+\nu)|w_1|^2 \right\} \\ \leq& \frac{p^3 B_1}{2(p+2\alpha)(p+2\lambda)} \left(\frac{p}{p+2} \right)^m . \end{aligned} \tag{2.22}$$

If $\sigma_3 \leq \mu \leq \sigma_2$, then

$$\begin{aligned} |a_{p+2} - \mu a_{p+2}^2| &+ \frac{1}{(p+2)^m} \frac{(p+\alpha)^2 (p+\lambda)^2 (p+1)^{2m}}{2p^3 B_1^2 p^m (p+2\alpha)(p+2\lambda)} \\ &\times \left\{ B_1 + B_2 + pB_1^2 \xi(p,\alpha,\lambda,\mu,m \right\} |a_{p+1}|^2 \\ \leq& \frac{p^3 B_1}{2(p+2\alpha)(p+2\lambda)} \left(\frac{p}{p+2} \right)^m |w_2 - \nu w_1^2| \\ &+ \frac{1}{(p+2)^m} \frac{(p+\alpha)^2 (p+\lambda)^2 (p+1)^{2m}}{2p^3 B_1^2 p^m (p+2\alpha)(p+2\lambda)} \\ &\times \left\{ B_1 + B_2 + pB_1^2 \xi(p,\alpha,\lambda,\mu,m) \right\} \\ &\times \frac{p^6 B_1^2}{(p+\alpha)^2 (p+\lambda)^2} \frac{p^{2m}}{(p+1)^{2m}} |w_1|^2 \\ \leq& \frac{p^3 B_1}{2(p+2\alpha)(p+2\lambda)} \left(\frac{p}{p+2} \right)^m \\ &\times \left\{ |w_2 - \nu w_1^2| + (1-\nu)|w_1|^2 \right\} \\ \leq& \frac{p^3 B_1}{2(p+2\alpha)(p+2\lambda)} \left(\frac{p}{p+2} \right)^m . \end{aligned} \tag{2.23}$$

By using Lemma 2.2, we can write

$$|a_{p+2}-\mu a_{p+2}^{2}| \leq \frac{p^{3}B_{1}}{2\left(p+2\alpha\right)\left(p+2\lambda\right)}\left(\frac{p}{p+2}\right)^{m}\max\left\{1,\left|\frac{B_{2}}{B_{1}}+pB_{1}\xi\left(p,\alpha,\lambda,\mu,m\right)\right|\right\}$$

for any complex number μ.

By using (2.16) and (2.17) equalities, we have

$$\begin{aligned}
&\frac{3}{p^{2}}\left(p+3\alpha\right)\frac{1}{p}\left(\frac{p+3}{p}\right)^{m}\left(p+3\lambda\right)a_{p+3}=B_{1}w_{3}+2B_{2}w_{1}w_{2}+B_{3}w_{1}^{3}\\
&+\frac{3}{p^{3}}\left(p^{2}+3\alpha p+2\alpha\right)\frac{1}{p}\left(p+\lambda\right)\frac{1}{p}\left(p+2\lambda\right)\frac{p^{3}B_{1}w_{1}}{\left(p+\alpha\right)\left(p+\lambda\right)}\\
&\times\frac{p^{3}B_{1}}{2\left(p+2\alpha\right)\left(p+2\lambda\right)}\left\{w_{2}+w_{1}^{2}\left[\frac{B_{2}}{B_{1}}+pB_{1}\frac{p^{2}+2\alpha p+\alpha}{\left(p+\alpha\right)^{2}}\right]\right\}\\
&-\frac{1}{p^{4}}\left(p^{3}+3\alpha p^{2}+3\alpha p+\alpha\right)\\
&\times\left[\frac{1}{p}\left(\frac{p+1}{p}\right)^{m}\left(p+\lambda\right)\frac{p^{3}B_{1}w_{1}}{\left(p+\alpha\right)\left(p+\lambda\right)}\left(\frac{p}{p+1}\right)^{m}\right]^{3}.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
a_{p+3}=&\frac{p^{3}B_{1}}{3\left(p+3\alpha\right)\left(p+3\lambda\right)}\left(\frac{p}{p+3}\right)^{m}\\
&\times\left\{w_{3}+\left[2\frac{B_{2}}{B_{1}}+\frac{3\left(p^{2}+3\alpha p+2\alpha\right)}{2\left(p+\alpha\right)\left(p+2\alpha\right)}pB_{1}\right]w_{1}w_{2}\right.\\
&+\left[\frac{B_{3}}{B_{1}}+\frac{3\left(p^{2}+3\alpha p+2\alpha\right)}{2\left(p+\alpha\right)\left(p+2\alpha\right)}pB_{2}\right.\\
&+\left(\frac{3\left(p^{2}+3\alpha p+2\alpha\right)}{2\left(p+\alpha\right)\left(p+2\alpha\right)}\frac{p^{2}+2\alpha p+\alpha}{\left(p+\alpha\right)^{2}}\right.\\
&\left.\left.\left.-\frac{p^{3}+3\alpha p^{2}+3\alpha p+\alpha}{\left(p+\alpha\right)^{3}}\right)p^{2}B_{1}^{2}\right]w_{1}^{3}\right\}.
\end{aligned} \tag{2.24}$$

Let

$$q_{1}=2\left(\frac{B_{2}}{B_{1}}\right)+\frac{3\left(p^{2}+3\alpha p+2\alpha\right)}{2\left(p+\alpha\right)\left(p+2\alpha\right)}pB_{1},$$

$$\begin{aligned}
q_{2}=&\frac{B_{3}}{B_{1}}+\frac{3\left(p^{2}+3\alpha p+2\alpha\right)}{2\left(p+\alpha\right)\left(p+2\alpha\right)}pB_{2}\\
&+\left[\frac{3\left(p^{2}+3\alpha p+2\alpha\right)}{2\left(p+\alpha\right)\left(p+2\alpha\right)}\frac{p^{2}+2\alpha p+\alpha}{\left(p+\alpha\right)^{2}}-\frac{p^{3}3\alpha^{2}p+3\alpha p+\alpha}{\left(p+\alpha\right)^{3}}\right]p^{2}B_{1}^{2}.
\end{aligned}$$

Then, from equality (2.24), we obtain

$$a_{p+3} = \frac{p^3 B_1}{3(p+3\alpha)(p+3\lambda)} \left(\frac{p}{p+3}\right)^m \left\{w_3 + q_1 w_1 w_2 + q_2 w_1^3\right\}.$$

Thus, we can write

$$|a_{p+3}| \leq \frac{p^3 B_1}{3(p+3\alpha)(p+3\lambda)} \left(\frac{p}{p+3}\right)^m H(q_1, q_2) \tag{2.25}$$

where $H(q_1, q_2)$ is defined as in Lemma 2.3.

To show that the bounds in (2.7), (2.16) and (2.17) are sharp, we define the functions $K_{\varphi n}$ $(n = 2, 3, \dots)$ by

$$\frac{1}{p}\left\{(1-\alpha)\frac{z(K_{\varphi n})'(z)}{(K_{\varphi n})(z)} + \alpha\left(1 + \frac{z(K_{\varphi n})''(z)}{(K_{\varphi n})'(z)}\right)\right\} = \varphi\left(z^{n-1}\right),$$

$$(K_{\varphi n})(0) = 0 = [K_{\varphi n}]'(0) - 1$$

and the functions $F_{\lambda,m}$ and $G_{\lambda,m}$ $(0 \leq \lambda \leq 1,\ m \in \mathbb{N}_0)$ by

$$\frac{1}{p}\left\{(1-\alpha)\frac{z(F_{\lambda,m})'(z)}{(F_{\lambda,m})(z)} + \alpha\left(1 + \frac{z(F_{\lambda,m})''(z)}{(F_{\lambda,m})'(z)}\right)\right\} = \varphi\left(z\frac{z+\lambda}{1+\lambda z}\right),$$

$$(F_{\lambda,m})(0) = 0 = [F_{\lambda,m}]'(0) - 1$$

and

$$\frac{1}{p}\left\{(1-\alpha)\frac{z(G_{\lambda,m})'(z)}{(G_{\lambda,m})(z)} + \alpha\left(1 + \frac{z(G_{\lambda,m})''(z)}{(G_{\lambda,m})'(z)}\right)\right\} = \varphi\left(-z\frac{z+\lambda}{1+\lambda z}\right),$$

$$(G_{\lambda,m})(0) = 0 = [G_{\lambda,m}]'(0) - 1.$$

Clearly the functions $K_{\varphi n}, F_{\lambda,m}, G_{\lambda,m} \in M_{p,1,\alpha,\lambda,m}(\varphi)$. Also we write $K_\varphi = K_{\varphi 2}$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if f is K_φ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds if and only if f is $K_{\varphi 3}$ or one of its rotations. If $\mu = \sigma_1$, then the equality holds if and only if f is $F_{\lambda,m}$ or one of its rotations. If $\mu = \sigma_2$, then the equality holds if and only if f is $G_{\lambda,m}$ or one of its rotations. □

Taking $\lambda = 0$, $\alpha = 0$, $m = 0$ in Theorem 2.4, we can write the following Theorem 2.5 obtained for the class $S^*_{b,p}(\varphi)$ introduced by Ali et al. [1].

Theorem 2.5. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots,$ *and*

$$\sigma_1 := \frac{B_2 - B_1 + pB_1^2}{2pB_1^2}, \quad \sigma_2 := \frac{B_2 + B_1 + pB_1^2}{2pB_1^2}, \quad \sigma_3 := \frac{B_2 + pB_1^2}{2pB_1^2}.$$

If $f(z)$ given by (1.1) belongs to $S^*_{b,p}(\varphi)$, then

$$\left|a_{p+2}-\mu a_{p+1}^2\right| \leq \begin{cases} \frac{p}{2}\left\{B_2+(1-2\mu)\,pB_1^2\right\} & \text{if } \mu \leq \sigma_1, \\ \frac{pB_1}{2} & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\ -\frac{p}{2}\left\{B_2+(1-2\mu)\,pB_1^2\right\} & \text{if } \mu \geq \sigma_2. \end{cases}$$

Further, if $\sigma_1 \leq \mu \leq \sigma_3$, then

$$\left|a_{p+2}-\mu a_{p+1}^2\right| + \frac{1}{2pB_1}\left(1-\frac{B_2}{B_1}+(2\mu-1)pB_1\right)\left|a_{p+1}\right|^2 \leq \frac{pB_1}{2}.$$

If $\sigma_3 \leq \mu \leq \sigma_2$, then

$$\left|a_{p+2}-\mu a_{p+1}^2\right| + \frac{1}{2pB_1}\left(1-\frac{B_2}{B_1}-(2\mu-1)pB_1\right)\left|a_{p+1}\right|^2 \leq \frac{pB_1}{2}.$$

For any complex number μ,

$$\left|a_{p+2}-\mu a_{p+1}^2\right| \leq \frac{pB_1}{2}\max\left\{1,\left|\frac{B_2}{B_1}+(1-2\mu)pB_1\right|\right\}.$$

Further,

$$\left|a_{p+3}\right| \leq \frac{pB_1}{3}H\left(q_1,q_2\right),$$

where $H\left(q_1,q_2\right)$ is as defined in Lemma 2.3 with

$$q_1 := \frac{4B_2+3pB_1^2}{2B_1} \text{ and } q_1 := \frac{2B_3+3pB_1B_2+p^2B_1^3}{2B_1}.$$

The results are sharp.

Remark 2.6. The results which are obtained by taking $\lambda=0$, $\alpha=0$, $m=0$, $p=1$ in Theorem 2.4 coincide with the results obtained for the class $S^*(\varphi)$ by Ma and Minda [3].

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