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On lifts of projectable-projectable classical linear connections to the cotangent bundle

Abstract. We describe all $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators $D\colon Q^{\tau}_{proj\text{-}proj} \rightsquigarrow QT^*$ transforming projectable-projectable classical torsion-free linear connections ∇ on fibred-fibred manifolds Y into classical linear connections $D(\nabla)$ on cotangent bundles T^*Y of Y. We show that this problem can be reduced to finding $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators $D\colon Q^{\tau}_{proj\text{-}proj} \rightsquigarrow (T^*, \otimes^p T^* \otimes \otimes^q T)$ for $p = 2$, $q = 1$ and $p = 3$, $q = 0$.

1. Basic definitions and examples. A fibred-fibred manifold Y is any commutative diagram

$$\begin{array}{ccc} Y = Y_1 & \xrightarrow{p_{12}} & Y_2 \\ {\scriptstyle p_{13}}\downarrow & & \downarrow{\scriptstyle p_{24}} \\ Y_3 & \xrightarrow{p_{34}} & Y_4 \end{array}$$

where maps $p_{12}, p_{13}, p_{24}, p_{34}$ are surjective submersions and an induced map $Y_1 \to Y_2 \times_{Y_4} Y_3$, $y \mapsto (p_{12}(y), p_{13}(y))$ is a surjective submersion. A fibred-fibred manifold has dimension (m_1, m_2, n_1, n_2) if $\dim Y_1 = m_1+m_2+n_1+n_2$, $\dim Y_2 = m_1 + m_2$, $\dim Y_3 = m_1 + n_1$, $\dim Y_4 = m_1$. For two fibred-fibred manifolds $Y, \widetilde{Y}$ of the same dimension (m_1, m_2, n_1, n_2), a morphism $f\colon Y \to \widetilde{Y}$ is a quadruple of local diffeomorphisms $f_1\colon Y_1 \to \widetilde{Y}_1$, $f_2\colon Y_2 \to$

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$\widetilde{Y}_2$, $f_3\colon Y_3 \to \widetilde{Y}_3$, $f_4\colon Y_4 \to \widetilde{Y}_4$ such that all squares of the cube in question are commutative, [2], [7].

All fibred-fibred manifolds of the given dimension (m_1, m_2, n_1, n_2) and all their morphisms form the category which we denote by $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$.

Every object from the category $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$ is locally isomorphic to the standard fibred-fibred manifold

$$\begin{array}{ccc} \mathbb{R}^{m_1,m_2,n_1,n_2} = \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} & \longrightarrow & \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \\ \downarrow & & \downarrow \\ \mathbb{R}^{m_1} \times \mathbb{R}^{n_1} & \longrightarrow & \mathbb{R}^{m_1} \end{array}$$

where arrows are obvious projections.

A classical linear connection ∇ on a fibred-fibred manifold Y is a tangent bundle homothety invariant section $\nabla\colon TY \to J^1TY$ of the 1-jet prolongation $J^1TY \to TY$ of the tangent bundle TY. Recall that a classical linear connection ∇ on Y is called a projectable-projectable linear connection on a fibred-fibred manifold Y if there exist classical linear connections ∇_2, ∇_3, ∇_4 on Y_2, Y_3, Y_4, respectively, such that the connection ∇ projects into ∇_2 and ∇_3 by maps p_{12} and p_{13}, respectively, and connections ∇_2 and ∇_3 project into ∇_4 by maps p_{24} and p_{34}, respectively, [4], [1].

A classical linear connection $\nabla\colon TY \to J^1TY$ on Y determines the corresponding covariant derivative $\nabla\colon \mathfrak{X}(Y) \times \mathfrak{X}(Y) \to \mathfrak{X}(Y)$ of vector fields on Y satisfying the additional projectability-projectability condition.

We say that a classical linear connection ∇ on a fibred-fibred manifold Y is torsion-free if the torsion tensor $T(X,Y)$ of ∇ vanishes, i.e. $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = 0$.

In the present paper we consider a problem of constructing of a classical linear connection $D(\nabla)$ on the cotangent bundle T^*Y of Y by means of a projectable-projectable classical torsion-free linear connection ∇ on an (m_1, m_2, n_1, n_2)-dimensional fibred-fibred manifold Y. To this aim we will consider a characterization of $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators $D\colon Q^{\tau}_{\textit{proj-proj}} \rightsquigarrow QT^*$ corresponding to above constructions.

A similar problem in the case of usual n-dimensional manifolds M and classical linear connections ∇ (not necessarily torsion-free) was studied by M. Kureš [6] and it was extended to $\otimes^k T^*M$ in [5].

We will formulate definitions of natural operators which can be treated as special cases of the general concept of natural operators from [3].

Definition 1. An $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator $D\colon Q^{\tau}_{\textit{proj-proj}} \rightsquigarrow QT^*$ transforming projectable-projectable classical torsion-free linear connections ∇ on (m_1, m_2, n_1, n_2)-dimensional fibred-fibred manifolds Y into classical linear connections $D(\nabla)$ on T^*Y is a family $D = (D_Y)$ of $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-invariant regular operators

$$D_Y\colon Q^{\tau}_{\textit{proj-proj}}(Y) \to Q(T^*Y)$$

for any fibred-fibred manifold Y of the dimension (m_1, m_2, n_1, n_2), where $Q^{\tau}_{\textit{proj-proj}}(Y)$ is the set of all projectable-projectable classical torsion-free linear connections on the fibred-fibred manifold Y and $Q(T^*Y)$ is the set of all classical linear connections (not necessarily torsion-free) on T^*Y. The $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-invariance of (the operator) D means that if any projectable-projectable classical torsion-free linear connections $\nabla \in Q^{\tau}_{\textit{proj-proj}}(Y)$, $\widetilde{\nabla} \in Q^{\tau}_{\textit{proj-proj}}(\widetilde{Y})$ are φ-related by an $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-invariant map $\varphi\colon Y \to \widetilde{Y}$ (i.e. $J^1T\varphi \circ \nabla = \widetilde{\nabla} \circ T\varphi$) then induced classical linear connections $D(\nabla) \in Q(T^*Y)$ and $D(\widetilde{\nabla}) \in Q(T^*\widetilde{Y})$ are $T^*\varphi$-related by $T^*\varphi\colon T^*Y \to T^*\widetilde{Y}$ (i.e. $J^1T(T^*\varphi) \circ D(\nabla) = D(\widetilde{\nabla}) \circ T(T^*\varphi)$), where $T^*\varphi$ is a cotangent map to φ.

The regularity of D means that D transforms smoothly parameterized families of projectable-projectable classical torsion-free linear connections into smoothly parameterized families of classical linear connections.

Example 1. An example of $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator

$$D\colon Q^{\tau}_{\textit{proj-proj}} \rightsquigarrow QT^*$$

is a family $D^{T^*} = (D^{T^*}_Y)$ of operators

$$D^{T^*}_Y : Q^{\tau}_{\textit{proj-proj}}(Y) \to Q(T^*Y)$$

given by the formula $D^{T^*}_Y(\nabla) = \nabla^{T^*Y}$, where $Y \in Obj(\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2})$, $\nabla \in Q^{\tau}_{\textit{proj-proj}}(Y)$ and ∇^{T^*Y} is a horizontal lift of ∇ on Y to the cotangent bundle T^*Y.

We define a horizontal lift ∇^{T^*Y} of a projectable-projectable classical torsion-free linear connection ∇ to the cotangent bundle T^*Y as

$$\nabla^{T^*Y} = \nabla^C - R^V_Y(\nabla),$$

where ∇^C is the complete lift of ∇ and $R^V_Y(\nabla)$ means the vertical lift of the curvature tensor $R_Y(\nabla)$ of ∇, [8].

Definition 2. An $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator

$$D\colon Q^{\tau}_{\textit{proj-proj}} \rightsquigarrow (T^*, \otimes^p T^* \otimes \otimes^q T)$$

transforming projectable-projectable classical torsion-free linear connections ∇ on (m_1, m_2, n_1, n_2)-dimensional fibred-fibred manifolds Y into fibred maps $D(\nabla)\colon T^*Y \to \otimes^p T^*Y \otimes \otimes^q TY$ covering the identity id_Y is a family of $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-invariant regular operators

$$D = (D_Y)\colon Q^{\tau}_{\textit{proj-proj}}(Y) \to C^{\infty}_Y(T^*Y, \otimes^p T^*Y \otimes \otimes^q TY)$$

defined for any (m_1, m_2, n_1, n_2)-dimensional fibred-fibred manifold Y.

The $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-invariance of D means that if two projectable-projectable classical torsion-free linear connections $\nabla \in Q^{\tau}_{\textit{proj-proj}}(Y)$ and $\widetilde{\nabla} \in Q^{\tau}_{\textit{proj-proj}}(\widetilde{Y})$ are φ-related by an $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-map $\varphi\colon Y \to \widetilde{Y}$

then induced maps $D(\nabla)\colon T^*Y \to \otimes^p T^*Y \otimes \otimes^q TY$ and $D(\widetilde{\nabla})\colon T^*\widetilde{Y} \to \otimes^p T^*\widetilde{Y} \otimes \otimes^q T\widetilde{Y}$ are φ-related, i.e. the following diagram is commutative

$$\begin{array}{ccc} T^*Y & \xrightarrow{D(\nabla)} & \otimes^p T^*Y \otimes \otimes^q TY \\ {\scriptstyle T^*\varphi}\big\downarrow & & \big\downarrow{\scriptstyle \otimes^p T^*\varphi \otimes \otimes^q T\varphi} \\ T^*\widetilde{Y} & \xrightarrow{D(\widetilde{\nabla})} & \otimes^p T^*\widetilde{Y} \otimes \otimes^q T\widetilde{Y} \end{array}$$

where $T\varphi\colon TY \to T\widetilde{Y}$ is a tangent map to $\varphi\colon Y \to \widetilde{Y}$ and $T^*\varphi\colon T^*Y \to T^*\widetilde{Y}$ is a cotangent map to φ.

Example 2. An example of $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator

$$D\colon Q^\tau_{\textit{proj-proj}} \rightsquigarrow (T^*, \otimes^3 T^*)$$

is a family of operators $D^1 = (D^1_Y)$,

$$D^1_Y\colon Q^\tau_{\textit{proj-proj}}(Y) \to C^\infty_Y(T^*Y, \otimes^3 T^*Y),$$

$D^1(\nabla)\colon T^*Y \to \otimes^3 T^*Y$ given by $D^1_Y(\nabla)(\omega) = \omega \otimes \omega \otimes \omega$, where $\omega \in T^*_y Y$, $y \in Y$, $Y \in Obj(\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2})$, $\nabla \in Q^\tau_{\textit{proj-proj}}(Y)$.

Another example of $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator

$$D\colon Q^\tau_{\textit{proj-proj}} \rightsquigarrow (T^*, \otimes^3 T^*)$$

is a family of operators $D^2 = (D^2_Y)$,

$$D^2_Y\colon Q^\tau_{\textit{proj-proj}}(Y) \to C^\infty_Y(T^*Y, \otimes^3 T^*Y),$$

$D^2(\nabla)\colon T^*Y \to \otimes^3 T^*Y$ given by $D^2_Y(\nabla)(\omega) = \langle R_y(\nabla), \omega\rangle$, where $R(\nabla)$ is the curvature tensor of ∇, $\omega \in T^*_y Y$, $y \in Y$, $Y \in Obj(\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2})$, $\nabla \in Q^\tau_{\textit{proj-proj}}(Y)$.

Example 3. An example of $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator

$$D\colon Q^\tau_{\textit{proj-proj}} \rightsquigarrow (T^*, \otimes^2 T^* \otimes T)$$

is a family of operators $D^3 = (D^3_Y)$,

$$D^3_Y\colon Q^\tau_{\textit{proj-proj}}(Y) \to C^\infty_Y(T^*Y, \otimes^2 T^*Y \otimes TY),$$

$D^3(\nabla)\colon T^*Y \to \otimes^2 T^*Y \otimes TY$ given by $\langle D^3_Y(\nabla)(\omega), v_1 \otimes v_2\rangle = \langle \omega, v_1\rangle v_2$, where $\omega \in T^*_y Y$, $v_1, v_2 \in T_y Y$, $y \in Y$, $Y \in Obj(\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2})$, $\nabla \in Q^\tau_{\textit{proj-proj}}(Y)$.

2. Some lemmas. The following lemma shows that the description of $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators $D\colon Q^{\tau}_{\textit{proj-proj}} \rightsquigarrow QT^*$ can be replaced by the description of $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators $D\colon Q^{\tau}_{\textit{proj-proj}} \rightsquigarrow (T^*, \otimes^p T^* \otimes \otimes^q T)$.

Lemma 1. *There exists a bijection between the set of $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators $D\colon Q^{\tau}_{\textit{proj-proj}} \rightsquigarrow QT^*$ and the set of sequences $(D_i)_{i=1,\dots,8}$ consisting of $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators of the following forms:*

$$
\begin{aligned}
&D_1\colon Q^{\tau}_{\textit{proj-proj}} \rightsquigarrow (T^*, T^*\otimes T^*\otimes T)\\
&D_2\colon Q^{\tau}_{\textit{proj-proj}} \rightsquigarrow (T^*, T^*\otimes T^*\otimes T^*)\\
&D_3\colon Q^{\tau}_{\textit{proj-proj}} \rightsquigarrow (T^*, T\otimes T^*\otimes T)\\
&D_4\colon Q^{\tau}_{\textit{proj-proj}} \rightsquigarrow (T^*, T\otimes T^*\otimes T^*)\\
&D_5\colon Q^{\tau}_{\textit{proj-proj}} \rightsquigarrow (T^*, T^*\otimes T\otimes T)\\
&D_6\colon Q^{\tau}_{\textit{proj-proj}} \rightsquigarrow (T^*, T^*\otimes T\otimes T^*)\\
&D_7\colon Q^{\tau}_{\textit{proj-proj}} \rightsquigarrow (T^*, T\otimes T\otimes T)\\
&D_8\colon Q^{\tau}_{\textit{proj-proj}} \rightsquigarrow (T^*, T\otimes T\otimes T^*).
\end{aligned}
$$

Proof. Let $\nabla \in Q^{\tau}_{\textit{proj-proj}}(Y)$ be a projectable-projectable classical torsion-free linear connection on an (m_1, m_2, n_1, n_2)-dimensional fibred-fibred manifold Y. Let $v \in T^*_y Y$, $y \in Y$.

The connection ∇ yields a decomposition of the tangent space $T_v T^*Y$ of T^*Y at v of the form

$$T_v T^*Y = H^{\nabla}_v \oplus V_v T^*Y,$$

where H^{∇}_v is a ∇-horizontal subspace and $V_v T^*Y$ is a vertical subspace.

We have an isomorphism $H^{\nabla}_v \cong T_y Y$ by the restriction of the differential $T_v\pi\colon T_vT^*Y \to T_yY$ of the cotangent bundle projection $\pi\colon T^*Y \to Y$ to H^{∇}_v. Moreover, we have an isomorphism $V_vT^*Y \cong T^*_yY$ by the standard isomorphism

$$T^*_yY \ni \omega \to \left.\frac{d}{dt}\right|_0 (v + t\omega) \in T_vT^*Y = V_vT^*Y.$$

Thus we have a decomposition

$$T_vT^*Y \cong T_yY \oplus T^*_yY$$

canonically depending on ∇.

Consequently, we have a linear isomorphism

$$T^*_vT^*Y \otimes T^*_vT^*Y \otimes T_vT^*Y \cong (T_yY \oplus T^*_yY)^* \otimes (T_yY \oplus T^*_yY)^* \otimes (T_yY \oplus T^*_yY)$$

canonically depending on ∇.

We have an isomorphism

$$(T_yY \oplus T^*_yY)^* \cong T^*_yY \oplus T_yY$$

by standard identifications

$$(V \oplus W)^* = V^* \oplus W^* \quad \text{and} \quad V^{**} = V,$$

from linear algebra.

Thus we have the following linear isomorphism

$$\begin{aligned} T_v^*T^*Y \otimes T_v^*T^*Y \otimes T_vT^*Y \cong (T_y^*Y \otimes T_y^*Y \otimes T_yY) \oplus (T_y^*Y \otimes T_y^*Y \otimes T_y^*Y) \\ \oplus (T_yY \otimes T_y^*Y \otimes T_yY) \oplus (T_yY \otimes T_y^*Y \otimes T_y^*Y) \oplus (T_y^*Y \otimes T_yY \otimes T_yY) \\ \oplus (T_y^*Y \otimes T_yY \otimes T_y^*Y) \oplus (T_yY \otimes T_yY \otimes T_yY) \oplus (T_yY \otimes T_yY \otimes T_y^*Y) \end{aligned}$$

canonically depending on ∇.

Using the above isomorphism, for any $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator $D\colon Q^\tau_{\textit{proj-proj}} \rightsquigarrow QT^*$, we can define a sequence of eight $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators $D_1, \ldots, D_8$ such as in Lemma 1, taking

$$(1) \qquad (D_1(\nabla)(v), \ldots, D_8(\nabla)(v)) := (D(\nabla) - \nabla^{T^*})(v)$$

for any $\nabla \in Q^\tau_{\textit{proj-proj}}(Y)$, $Y \in \textit{Obj}(\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2})$, $v \in T_y^*Y$, $y \in Y$, where ∇^{T^*} is the horizontal lift of ∇ to T^*Y.

The difference $D(\nabla) - \nabla^{T^*}$ of linear connections $D(\nabla)$ and ∇^{T^*} means a tensor field of type $T^* \otimes T^* \otimes T$ on T^*Y.

Above relation (1) makes sense because it holds $(D(\nabla) - \nabla^{T^*})(v) \in T_v^*T^*Y \otimes T_v^*T^*Y \otimes T_vT^*Y$ and $(D_1(\nabla)(v), \ldots, D_8(\nabla)(v)) \in ((T_y^*Y \otimes T_y^*Y \otimes T_yY) \oplus \ldots \oplus (T_yY \otimes T_yY \otimes T_y^*Y)) \cong T_v^*T^*Y \otimes T_v^*T^*Y \otimes T_vT^*Y$, where $\cong$ is a linear isomorphism canonically depending on ∇ describing above.

It is obvious that an assignment $D \mapsto (D_i)_{i=1,\ldots,8}$ yields the bijection from Lemma 1. □

Note that the description of natural operators D_1, D_4 and D_6 from Lemma 1 can be reduced to the description of operators of type D_1 since by obviously linear isomorphisms obtaining by permutations of factors

$$T_y^*Y \otimes T_y^*Y \otimes T_yY \cong T_yY \otimes T_y^*Y \otimes T_y^*Y \cong T_y^*Y \otimes T_yY \otimes T_y^*Y$$

for any $Y \in \textit{Obj}(\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2})$ and $y \in Y$ we have

Lemma 2. *There exists the bijection between the set of $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators $D_1\colon Q^\tau_{\textit{proj-proj}} \rightsquigarrow (T^*, T^* \otimes T^* \otimes T)$ and the set of $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$ natural operators $D_4\colon Q^\tau_{\textit{proj-proj}} \rightsquigarrow (T^*, T \otimes T^* \otimes T^*)$.*

Similarly, there exists the bijection between the set of $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators $D_1\colon Q^\tau_{\textit{proj-proj}} \rightsquigarrow (T^*, T^* \otimes T^* \otimes T)$ and the set of $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$ natural operators $D_6\colon Q^\tau_{\textit{proj-proj}} \rightsquigarrow (T^*, T^* \otimes T \otimes T^*)$.

Proof. The first bijection is of the form $D_1 \mapsto D_4$, where $D_4(\nabla)(v) := D_1(\nabla)(v)$, $v \in T_y^*Y$, $y \in Y$, $\nabla \in Q^\tau_{\textit{proj-proj}}(Y)$ modulo the identification $T_y^*Y \otimes T_y^*Y \otimes T_yY \cong T_yY \otimes T_y^*Y \otimes T_y^*Y$ of the form $\omega_1 \otimes \omega_2 \otimes \omega \mapsto \omega \otimes \omega_1 \otimes \omega_2$.

The second bijection is analogous. □

Moreover, we show that $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators D_3, D_5, D_7 and D_8 from Lemma 1 are zero. It holds the following general fact.

Lemma 3. *Let $p < q$. Then every $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator*

$$D\colon Q^\tau_{\textit{proj-proj}} \rightsquigarrow (T^*, \otimes^p T^* \otimes \otimes^q T)$$

is zero.

Proof. Let $\nabla \in Q^\tau_{\textit{proj-proj}}(Y)$, $Y \in Obj(\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2})$, $v \in T^*_y Y$, $y \in Y$. We have to show that $D(\nabla)(v) = 0 \in \otimes^p T^*_y Y \otimes \otimes^q T_y Y$. By the $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-invariance of D with respect to $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-charts we can assume $Y = \mathbb{R}^{m_1,m_2,n_1,n_2}$, $y = (0,0,0,0) \in \mathbb{R}^{m_1+m_2+n_1+n_2}$.

Then using the invariance of D with respect to $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-maps (homotheties)

$$\frac{1}{t} id\colon \mathbb{R}^{m_1,m_2,n_1,n_2} \to \mathbb{R}^{m_1,m_2,n_1,n_2} \qquad \text{for } t \neq 0,$$

we get the condition

$$D(\nabla)(v) = \left(\frac{1}{t}\right)^{q-p} D\left(\left(\frac{1}{t} id\right)_* \nabla\right)(tv), \qquad t \neq 0.$$

But the family (∇_t) of projectable-projectable classical torsion-free linear connections given by

$$\nabla_t := \begin{cases} \left(\frac{1}{t} id\right)_* \nabla, & t \neq 0 \\ \nabla_0, & t = 0, \end{cases}$$

where ∇_0 is the flat torsion-free linear connection (i.e. with zero Christoffel symbols), is smoothly parameterized because of the fact that ∇_t has Christoffel symbols of the form $t \cdot \Gamma^a_{bc}(tx)$ at the chart $id_{\mathbb{R}^{m_1,m_2,n_1,n_2}}$, where $\Gamma^a_{bc}(x)$ are the Christoffel symbols for ∇.

Thus using the regularity of D and taking $t \to \infty$, we get $D(\nabla)(v) = 0$ since $(\frac{1}{t})^{q-p} = t^{p-q} \to 0$ for $p < q$. □

3. The main results. As the summary of Lemmas 1–3 we get the following main theorem.

Theorem 1. *There exists the bijection between the set of $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators $D\colon Q^\tau_{\textit{proj-proj}} \rightsquigarrow QT^*$ and the set of sequences $(\widetilde{D}_i)_{i=1,2,3,4}$ consisting of $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators $\widetilde{D}_1, \widetilde{D}_2, \widetilde{D}_3\colon Q^\tau_{\textit{proj-proj}} \rightsquigarrow (T^*, T^* \otimes T^* \otimes T)$ and $\widetilde{D}_4\colon Q^\tau_{\textit{proj-proj}} \rightsquigarrow (T^*, T^* \otimes T^* \otimes T^*)$.*

More precisely, the system of operators $(\widetilde{D}_i)_{i=1,2,3,4}$ defines a new sequence of operators $(D_i)_{i=1,\dots,8}$ (of the type from Lemma 1) such as

$$D_1 := \widetilde{D}_1,\ D_4 := \widetilde{D}_2,\ D_6 := \widetilde{D}_3 \ \textit{(modulo the bijection from Lemma 2)}$$
$$D_2 := \widetilde{D}_4,\ D_3 = 0,\ D_5 = 0,\ D_7 = 0, D_8 = 0.$$

This system of operators $(D_i)_{i=1,\dots,8}$ defines the $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator D (by Lemma 1).

Lemma 3 shows that the above assignment $(\widetilde{D}_i)_{i=1,2,3,4} \mapsto D$ is a bijection.

Theorem 1 reduces the classification of $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators $D\colon Q^{\tau}_{\textit{proj-proj}} \rightsquigarrow QT^*$ to the classification of $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators $D\colon Q^{\tau}_{\textit{proj-proj}} \rightsquigarrow (T^*, \otimes^p T^* \otimes \otimes^q T)$ for $p = 2$, $q = 1$ and $p = 3$, $q = 0$.

Definition 3. An $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator

$$D\colon Q^{\tau}_{\textit{proj-proj}} \rightsquigarrow \otimes^p T^* \otimes \otimes^q T$$

is an $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-invariant family of regular operators

$$D = (D_Y)\colon Q^{\tau}_{\textit{proj-proj}}(Y) \to C^{\infty}_Y(\otimes^p T^*Y \otimes \otimes^q TY)$$

defined for every $Y \in Obj(\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2})$, where $Q^{\tau}_{\textit{proj-proj}}(Y)$ is defined in Definition 1 and $C^{\infty}_Y(\otimes^p T^*Y \otimes \otimes^q TY)$ means the set of smooth tensor fields on Y.

The $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-invariance of D means almost the same as in Definition 1, i.e. φ-related connections are transformed into φ-related tensor fields.

Example 4. An example of an $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator

$$D\colon Q^{\tau}_{\textit{proj-proj}} \rightsquigarrow \otimes^3 T^* \otimes T$$

is a family $D = (R_Y)$ of operators

$$R_Y\colon Q^{\tau}_{\textit{proj-proj}}(Y) \to C^{\infty}_Y(\otimes^3 T^*Y \otimes TY)$$

for any $Y \in Obj(\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2})$, where $R_Y(\nabla)$ is the curvature tensor of ∇.

Theorem 2. *Let $p \geq q$, $r := p - q$. There exists the bijection between the set of all $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators $D\colon Q^{\tau}_{\textit{proj-proj}} \rightsquigarrow (T^*, \otimes^p T^* \otimes \otimes^q T)$ and the set of $(r+1)$-elements sequences $(D_i)_{i=0,1,\dots,r}$ consisting of $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators $D_i\colon Q^{\tau}_{\textit{proj-proj}} \rightsquigarrow \otimes^p T^* \otimes \otimes^q T \otimes S^i T$, i.e. $D_i\colon Q^{\tau}_{\textit{proj-proj}} \rightsquigarrow \otimes^p T^* \otimes \otimes^{q+i} T$ and $D_i(\nabla)(w_1, \dots, w_p, v_1, \dots, v_{q+i})$ is symmetric with respect to $v_{q+1}, \dots, v_{q+i}$.*

Schema of the proof. Consider any $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator

$$D\colon Q^{\tau}_{\textit{proj-proj}} \rightsquigarrow (T^*, \otimes^p T^* \otimes \otimes^q T).$$

Let $\nabla \in Q^{\tau}_{\textit{proj-proj}}(\mathbb{R}^{m_1,m_2,n_1,n_2})$ and $v \in T^*_{(0,0,0,0)}\mathbb{R}^{m_1,m_2,n_1,n_2}$. We are going to study $D(\nabla)(v)$.

By the non-linear Petree theorem (see [3]) we have

$$D(\nabla)(v) = D(\widetilde{\nabla})(v),$$

where $\widetilde{\nabla}$ is some projectable-projectable classical torsion-free linear connection on $\mathbb{R}^{m_1,m_2,n_1,n_2}$ with Christoffel symbols $\widetilde{\nabla}^a_{bc}$ being polynomials of degree k. Thus we have

$$\widetilde{\nabla}^a_{bc} = \sum_{|\alpha|\le k} \nabla^a_{bc;\alpha} x^\alpha,$$

where $\nabla^a_{bc;\alpha} \in \mathbb{R}$ and $x^1, \dots, x^{m_1+m_2+n_1+n_2}$ is the usual fibred-fibred coordinate system on $\mathbb{R}^{m_1,m_2,n_1,n_2}$.

In short, we write $D(\nabla)(v) = D(\nabla^a_{bc;\alpha})(v)$.

Using the invariance of D with respect to homotheties $\frac{1}{t}id$, $t \neq 0$, we get the homogeneity condition

$$t^r D(\nabla^a_{bc;\alpha})(v) = D(t^{|\alpha|+1}\nabla^a_{bc;\alpha})(tv).$$

By the homogeneous function theorem (see [3]) and by the invariance of D with respect to $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-charts we get that $D(\nabla)(v)$ is a polynomial of degree not higher than $r := p - q$ with respect to $v \in T^*_yY$, $y \in Y$, for every $Y \in Obj(\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2})$ and $\nabla \in Q^\tau_{\textit{proj-proj}}(Y)$.

Thus we have

$$D(\nabla)(tv) = \sum_{i=0}^{r} D_i(\nabla)(v)t^i$$

for some uniquely determined coefficients $D_i(\nabla)(v) \in \otimes^p T^*_yY \otimes \otimes^q T_yY$.

For every $a \in \mathbb{R}$ we have

$$D(\nabla)(tav) = \sum_{i=0}^{r} D_i(\nabla)(av)t^i$$

and

$$D(\nabla)(tav) = \sum_{i=0}^{r} D_i(\nabla)(v)a^i t^i,$$

hence we get

$$D(\nabla)(av) = a^i D_i(\nabla)(v).$$

It means that $D_i(\nabla)(v)$ is a polynomial of degree i with respect to v and it can be identified with the corresponding element

$$D_i(\nabla)(v) \in \otimes^p T^*_yY \otimes \otimes^q T_yY \otimes S^i T_yY.$$

Summarizing, for every $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator

$$D\colon Q^\tau_{\textit{proj-proj}} \rightsquigarrow (T^*, \otimes^p T^* \otimes \otimes^q T)$$

we defined the sequence $(D_i)_{i=0,1,\dots,r}$ consisting of $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operators

$$D_i\colon Q^\tau_{\textit{proj-proj}} \rightsquigarrow \otimes^p T^* \otimes \otimes^q T \otimes S^i T.$$

Conversely, analysing the above reasoning, one can see that every $\mathcal{F}^2\mathcal{M}_{m_1,m_2,n_1,n_2}$-natural operator $D\colon Q^{\tau}_{proj\text{-}proj} \rightsquigarrow (T^*, \otimes^p T^* \otimes \otimes^q T)$ can be reconstructed from the sequence $(D_i)_{i=0,1,\dots,r}$ of operators

$$D_i\colon Q^{\tau}_{proj\text{-}proj} \rightsquigarrow \otimes^p T^* \otimes \otimes^q T \otimes S^i T. \qquad \square$$

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