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Two-Dimensional Stability of Nonlinear Exponential Schrödinger Waves

We consider the two-dimensional exponential nonlinear Schrödinger equation

$$iu_t + \nabla^2 u + \alpha[1 - e^{-|u|}]u = 0. \quad (1)$$

One-dimensional version of this equation has been introduced recently to nonlinear plasma physics by D'Evelyn and Morales [1], Kaw *et al.* [2] Sheering and Ong [3] and to nonlinear optics by Murawski and Koper [4].

Transverse stability of solutions of the exponential nonlinear Schrödinger equation has been studied by several authors [5–9]. Particularly, Infeld and Ziemkiewicz [5] have shown that all stationary entities are unstable with respect to two dimensional perturbations. The authors consider only a case of the positive value of α. This case corresponds to periodic waves and a soliton as solutions of equation (1). A variation of action method has been applied by Anderson, Bondeson, Lisak [6] to find that the solitons are stable for all transverse wave numbers $k > k_c$, where k_c is the so-called cut-off wavenumber. In contrast to one--dimensional solutions, which are always transversely unstable (existence of the cut-off), there exist completely stable 3-dimensional solutions of the vectorial exponential nonlinear Schrödinger equation if the wave amplitude exceeds a critical threshold. The solutions are also stable with respect to perturbations which depend on the radius only, but for azimuthally dependent perturbations instability exists. These results have been checked by Laedke and Spatschek [7,9] to prove fairly good agreement. A Liapunov functional method has been used by Blaha, Laedke, Spatschek [9] to assert that the solutions of the scalar exponential nonlinear Schrödinger equation are two-dimensionally completely stable. The corresponding formulae are easily obtained by the estimation of those for the one-dimensional case. See Murawski [10] for the corresponding calculations for other equations.

In this paper we will apply Infeld-Rowlands [11] method for the nonlinear exponential Schrödinger equation. Wherever possible the obtained results will be compared with those for the nonlinear Schrödinger equation obtained by Infeld and Rowlands [11] and for the nonlinear exponential Schrödinger equation by Infeld and Ziemkiewicz [5]. The latter authors have studied only a family of solutions which contain the soliton solution.

We now lood for stationary envelope solutions

$$u = u_0(\xi)\, e^{i(cX/2+bt)}, \tag{2}$$

where $\xi = x - ct$. Equation (1) leads to

$$u_{\xi\xi} - pu_0 - \alpha u_0 e^{-u_0^2} = 0, \tag{3}$$

where we defined

$$p \equiv b + \frac{c^2}{4} - \alpha.$$

Upon integration of equation (3) multiplied by $u_{0\xi}$ we get

$$u_{0\xi}^2 = pu_0^2 - \alpha e^{-u_0^2} + pl \equiv Y(u_0), \tag{4}$$

Here l is another integration constant.

The qualitative nature of the solution of the equation may be determined from consideration of the function $Y(u_0)$ which should be bounded for bounded u_0 and must possess double roots. It holds when

$$Y'(u_0) = Y(u_0) = 0.$$

Hence, we find the condition for α and p:

$$p\alpha < 0, \qquad p(\alpha + p) \leq 0 \tag{5}$$

and values of l corresponding to the double roots:

$$l_{min,max} = \min, \max\left\{\frac{\alpha}{p}, \ln\left(-\frac{p}{\alpha}\right) - 1\right\}. \tag{6}$$

We consider now the case of $\alpha > 0$ assuming that the conditions (5) are fullfiled. We call this case the 'soliton' case. For $l = l_{max}$ and $l = l_{min}$ we have linear waves and solitons as solutions of equation (4). There are periodic waves both for $l_{max} > l > l_{min}$ and $l < l_{min}$.

A second case we will take into consideration is for $\alpha > 0$ and

$$p < 0, \qquad p(\alpha + p) > 0. \tag{7}$$

For $1 > \frac{\alpha}{p}$ there is a range of periodic waves as solution of equation (4). For $1 = \frac{\alpha}{p}$ there is a linear wave. If conditions (5) are satisfied we have a range of periodic waves for $l_{min} < l < l_{max}$. For $l = l_{max}$ there are the linear wave and shock-wave, respectively. We call this case the 'shock-wave' case.

A third case we discuss here is for $\alpha > 0$ and $p < 0$. For $1 = \alpha/p$ there is a linear wave and for $1 > \alpha/p$ there are periodic waves only. We call this case the 'periodic-wave' case.

We superimpose a small disturbance of envelope with a long wavelength and small amplitude upon the steady state given by equation (4):

$$u = \left[u_0(\xi) + \delta u_1(\xi)\, e^{i(k_1\xi+k_2y+\omega\tau)} + \delta u_2(\xi) e^{-i(k_1\xi+k_2y+\omega^*\tau)}\right] e^{i(cX/2+bt)}. \tag{8}$$

Here, we introduced coordinates of the moving frame

$$\xi = x - ct, \qquad \tau = t, \tag{9}$$

and

$$k_1 = k\cos\phi, \qquad k_2 = k\sin\phi,$$

where ϕ is an angle between the wavevector k and the x axis. The star denotes a complex conjugate. Substituting (8) into (1) and dropping nonlinear terms, we find

$$\hat{L}\delta u_+ - \omega\delta u_- + 2ik_1\delta u_{+\xi} - k^2\delta u_+ = 0, \tag{10}$$

$$L\delta u_- - \omega\delta u_+ + 2ik_1\delta u_{-\xi} - k^2\delta u_- = 0, \tag{11}$$

where the following notation is used:

$$L = \partial_\xi^2 - p - \alpha e^{-u_0^2},$$

$$\hat{L} = L + 2\alpha u_0^2 e^{-u_0^2},$$

$$\delta u_\pm = \delta u_1 \pm \delta u_2^*,$$

and where the asterisk denotes the complex conjugate. In further calculations we assume k to be small (with respect to a wavelength of the basic wave) and use the following expansion

$$\omega = k\omega_1 + k^2\omega_2 + \cdots, \tag{12}$$

$$\delta u_+ = \delta u_{+0} + k\delta u_{+1} + \cdots, \tag{13}$$

$$\delta u_- = K(\delta u_{-0} + k\delta u_{-1} + \cdots). \tag{14}$$

Here K, is an arbitrary constant. The discussion of expansion (12) is presented by Infeld and Rowlands [11].

From the zeroth-and first-order equations in k, after an elimination of secular terms, we obtain

$$\delta u_{-0} = u_0, \tag{15}$$

$$\delta u_{+0} = u_{0\xi}, \tag{16}$$

$$\delta u_{-1} = u_0 + \frac{2i\cos\phi K - \omega_1}{2\eta K} P_0, \tag{17}$$

$$\delta u_{+0} = u_{0\xi} + \frac{1}{2\beta}(2\cos\phi + i\omega K\kappa)Q_0 + \frac{\omega_1 K}{2} Q_2, \tag{18}$$

where:

$$u_0 \int \frac{d\xi}{u_0^2} = \eta\xi u_0 + P_0(\xi), \tag{19}$$

$$u_{0\xi} \int \frac{d\xi}{u_{0\xi}^2} = \overline{\beta}\xi u_{0\xi} + Q_0(\xi), \tag{20}$$

$$u_{0\xi} \int \frac{u_0^2 d\xi}{u_{0\xi}^2} = \kappa\xi u_{0\tau} + Q_2(\xi). \tag{21}$$

Here, P_0, Q_0 and Q_2 are periodic functions with the same period as the nonlinear wave $\gamma\lambda$.

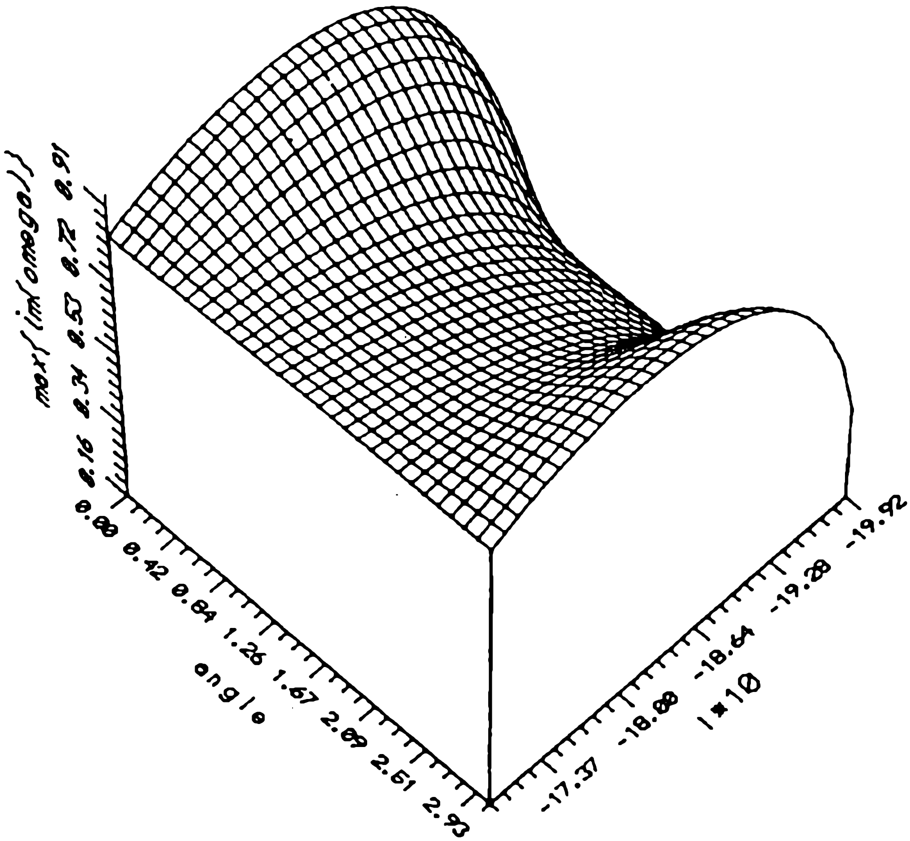

Fig. 1. A maximum value of the imaginary root of the dispersion relation plotted versus the parameter l and an angle ϕ, which changes between 0 and π. Here $a = 1$, $b = 0.25$, $c = 1$ and $l_{max} < l < l_{min}$. The smallest value of l corresponds to the soliton. A case of small amplitude waves

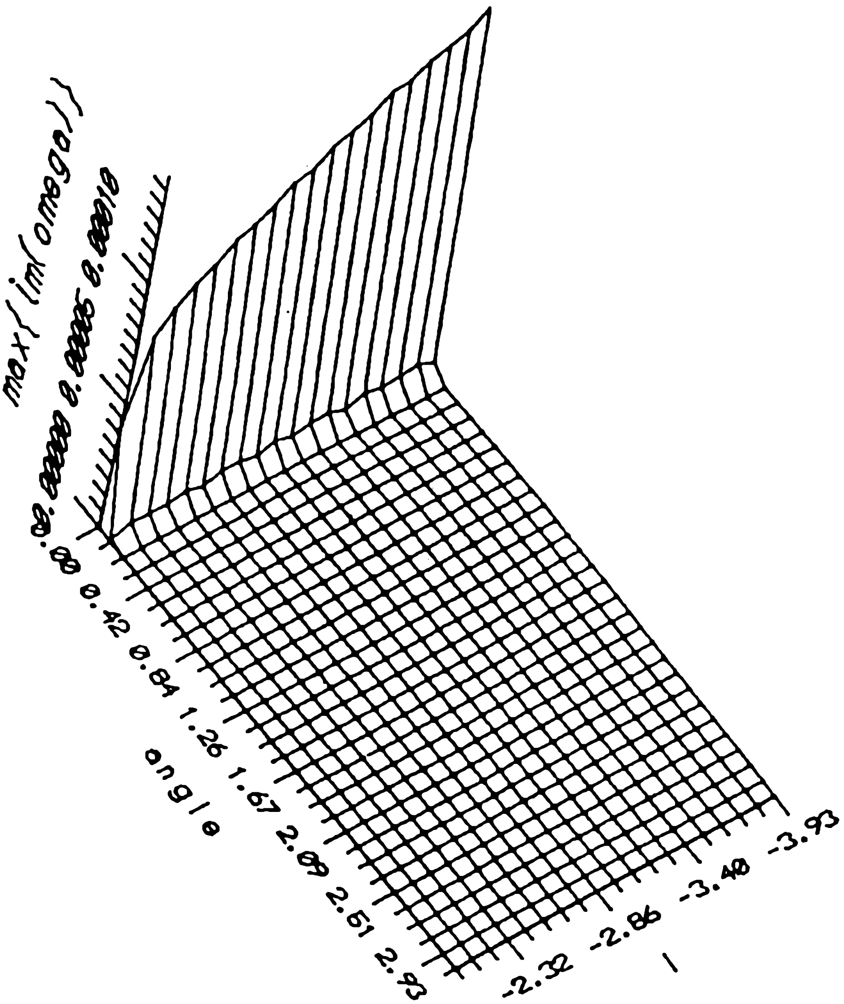

Fig. 2. As for Fig. 1, but $l < l_{max}$. The largest value of l corresponds to the soliton

From the second-order in k, we find

$$2i\cos\phi\langle u_{0\xi}\delta u_{+1}\rangle - \omega_1 K\langle u_{0\xi}\delta u_{-1}\rangle - \langle u_{0\xi}^2\rangle = 0, \tag{22}$$

$$2i\cos\phi\langle u_0\delta u_{-1\xi}\rangle - \frac{\omega_1}{K}\langle u_0\delta u_{+1}\rangle - \langle u_0^2\rangle = 0, \tag{23}$$

where we used the definition

$$\langle f\rangle = \frac{1}{\lambda}\int_0^\lambda f d\xi.$$

After the straightforward but lengthy calculation we find the dispersion

$$AO_1BO_1\omega_1^4 + (AO_2BO_1 + AO_1BO_2 + AO_3BO_3)\omega_1^2 + AO_2BO_2 = 0, \tag{24}$$

where:

$$\begin{aligned}
AO_1 &= \overline{\beta}\langle u_{0\xi}P_0\rangle,\\
AO_2 &= -2\eta(2\cos^2\phi\langle u_{0\xi}Q_{0\xi}\rangle\overline{\beta}\langle u_{0\xi}^2\rangle),\\
AO_3 &= 2\cos\phi(\overline{\beta}\langle u_{0\xi}P_0\rangle + \kappa\eta\langle u_{0\xi}Q_{0\xi}\rangle - \eta\overline{\beta}\langle u_{0\xi}Q_{2\xi}\rangle),\\
BO_1 &= \eta(\kappa\langle u_0Q_0\rangle - \overline{\beta}\langle u_0Q_2\rangle),\\
BO_2 &= 2\overline{\beta}(2\cos^2\phi\langle u_{0\xi}P_0\rangle - \eta\langle u_0^2\rangle),\\
BO_3 &= 2\cos\phi(\eta\langle u_0Q_0\rangle - \overline{\beta}\langle u_{0\xi}P_0\rangle).
\end{aligned}$$

Firstly, we discuss the 'soliton' case for relatively small amplitude waves. We should recover those results for the cubic nonlinear Schrödinger equation obtained by Infeld and Rowlands [11] and for the nonlinear exponential Schrödinger equation by Infeld and Ziemkiewicz [5]. One-dimensional case has already been discussed by Murawski and Koper [4].

THE 'SOLITON' CASE

We have a range of periodic waves for $l_{min} < l < l_{max}$ and $l < l_{min}$. For $l = l_{min}$ and $l = l_{max}$ there are the linear wave and the soliton, respectively. We consider both small and large amplitude waves cases (larger amplitude waves correspond to smaller values of p):

a) small amplitude case

Maximum value of the imaginary part of ω is plotted versus an angle of the disturbances and the parameter l corresponding to a wave amplitude. The cases $l_{min} < l < l_{max}$ and $l < l_{min}$ are shown in Figs. 1 and 2, respectively. We learn that for $l_{min} < l < l_{max}$ all periodic waves are unstable to the two-dimensional disturbances. For $l < l_{min}$ a maximum value of the imaginary part of ω is very small and this practically means that waves are stable for the perturbations because of the numerical accuracy of the calculations of the roots (10^{-6}) and integrals (10^{-4}).

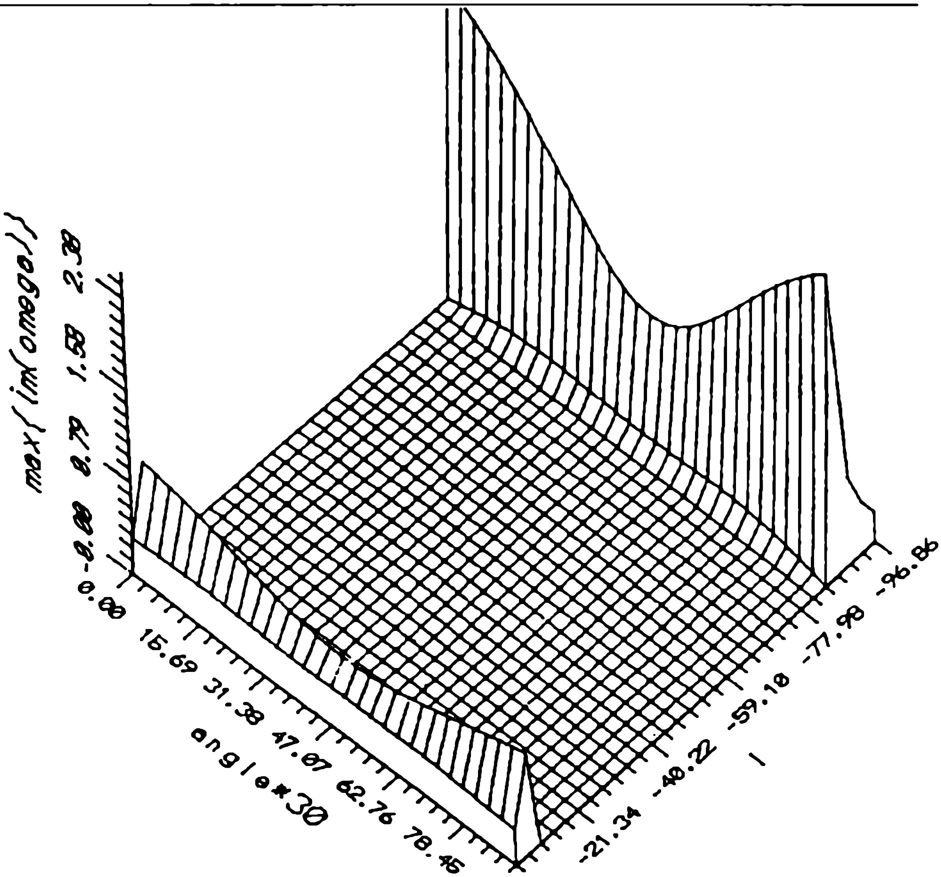

Fig. 3. As for Fig. 1, but $b = 0.99$ and $U = 0$. A case of larger amplitudes

These two figures also show that the soliton corresponding to $l = l_{max}$ is one--dimensional stable entity but unstable to perpendicular perturbances.

b) large amplitude waves

For large amplitude waves we notice that periodic waves for $l_{min} < l < l_{max}$ are only unstable both at the linear wave and soliton limits and this instability exists for all angles of the perturbations. For other values of l periodic waves are stable with respect to perturbations propagating at arbitrary angles. See Fig. 3 for details.

We observe in the case of $l < l_{min}$ (Fig. 4) that the qualitative nature of the stability does not change for large amplitude waves.

THE 'PERIODIC WAVE' CASE

In this case we have got the range of periodic waves for $1 > \alpha/p$. All these waves are practically stable with respect to two-dimensional perturbances. The waves are most unstable with respect to parallel perturbances and the growth-rate becomes larger for smaller values of l. Although very small values of the growth-rate are

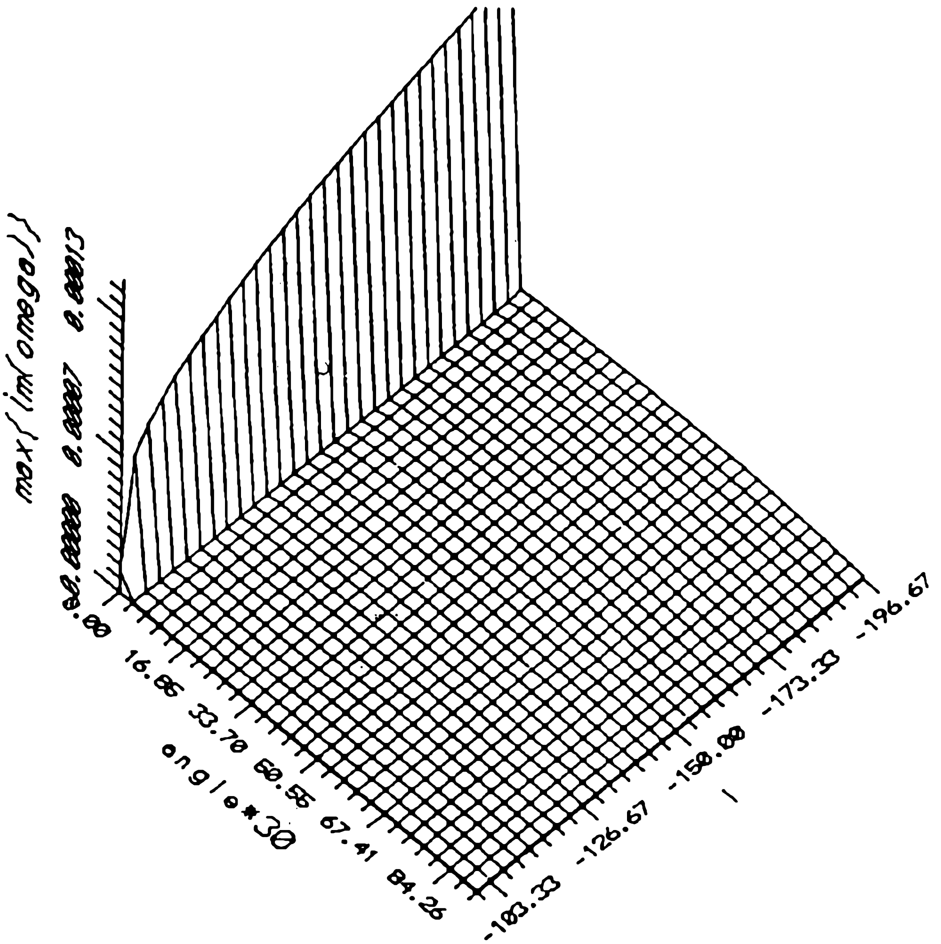

Fig. 4. As for Fig. 3, but for $l < l_{max}$

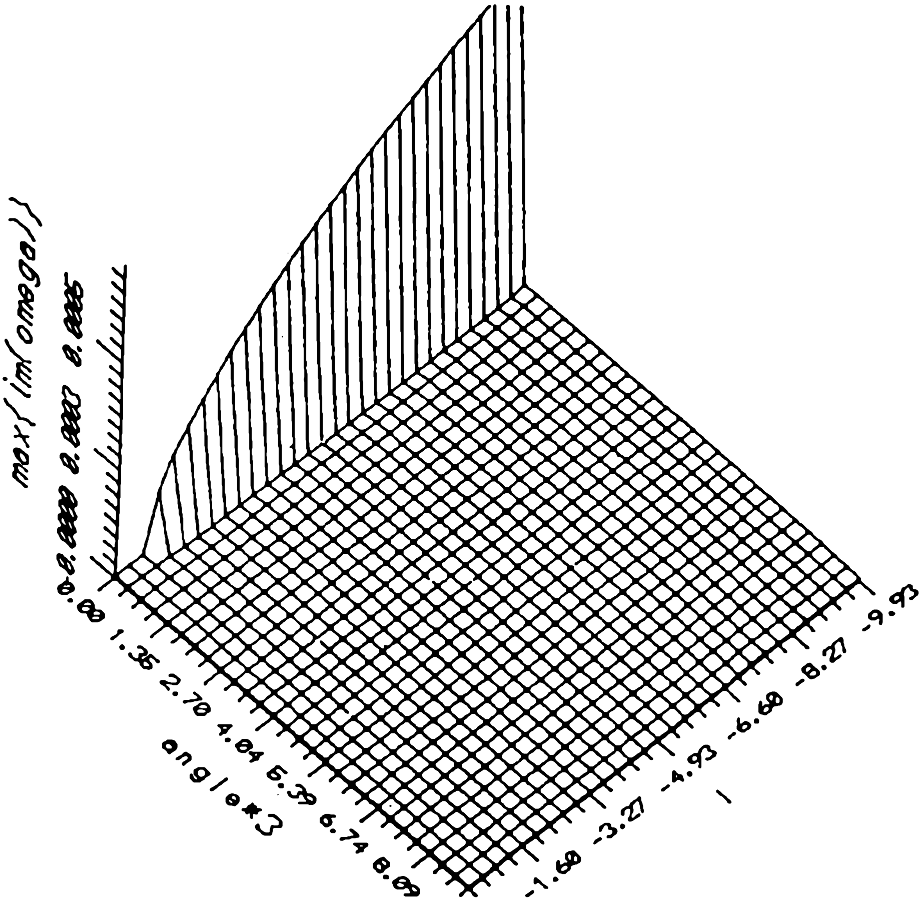

Fig. 5. As for Fig. 1, but for $a = 1$, $b = -3$ and $c = 1$. A case of periodic waves

observed, they are below an accuracy of the method. See Fig. 5. The qualitative nature of the stability does not depend on values of the wave amplitudes. For larger amplitudes waves are even slightly more stable in that the corresponding growth-rate is smaller.

THE 'SHOCK-WAVE' CASE

We have a range of periodic waves for $l_{min} < l < l_{max}$. For $l = l_{min}$ and $l = l_{max}$ there are the linear wave and shock-wave, respectively.

a) Small amplitude waves

In this case, the waves are most unstable to perpendicular disturbances and at the shock-wave limit. The growth-rate is about 0.1. However, the shock-wave is stable to two-dimensional perturbances. See Fig. 6.

b) Large amplitude waves

This is the case similar to the case of small amplitude waves. The corresponding growth-rate is a little bit smaller, however. See Fig. 7.

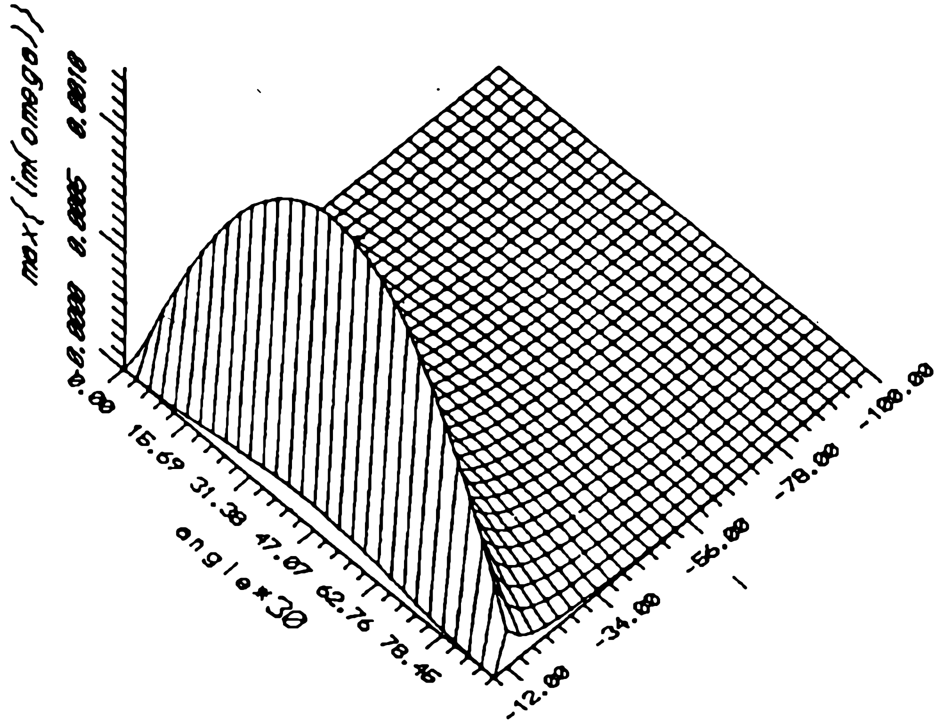

Fig. 6. As for Fig. 1, but for $a = -1$, $b = -1.5$ and $c = 2$. The 'shock-wave' case. The smallest and largest values of l correspond to a linear wave and the shock-wave, respectively. The small amplitude limit

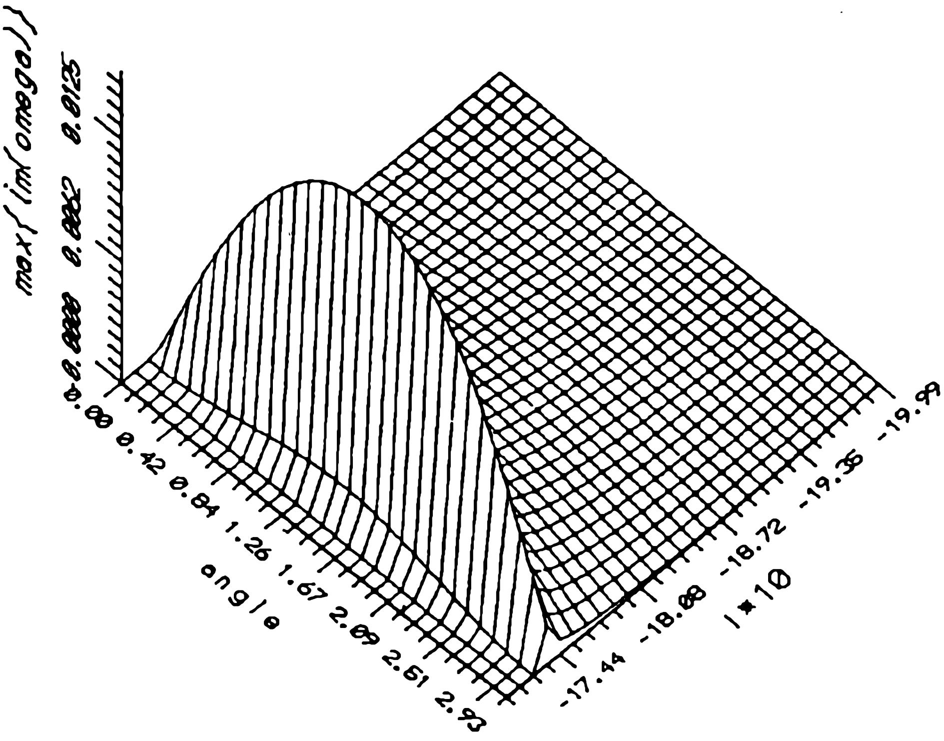

Fig. 7. As for Fig. 6, but for $b = -1.99$ and $c = 2$ and larger amplitude waves

In the summing-up it can be said that the Infeld-Rowlands method has been developed to study stability of stationary waves as solutions of the nonlinear exponential Schrödinger equation, with respect to small amplitude and long wave disturbances. The obtained results fully agree in small amplitude limit with those for the cubic nonlinear Schrödinger equation [11] and in the 'soliton' case with those discussed by Infeld and Ziemkiewicz [5].

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